Singularities in Fluid Dynamics and their Resolution

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Abstract Three types of singularity that can arise in fluid dynamical problems will be distinguished and discussed. These are: (i) singularities driven by boundary motion in conjunction with viscosity (e.g. corner singularities, or the Euler-disc finite-time singularity); (ii) free-surface (cusp) singularities associated with surface-tension and viscosity; (iii) interior point singularities of vorticity associated with intense vortex stretching. The singularities of types (i) and (ii) are now well known, and mechanisms by which the singularities may be resolved are clear. The question of existence of singularities of type (iii) is still open; current evidence for and against will be discussed.

1 Introduction

This paper is concerned with some examples of singularities, i.e unbounded local behaviour of the velocity field or its derivatives, in the flow of an incompressible fluid, and the manner in which such singularities must in practice be resolved. Singularities may be associated with the geometry of the fluid boundary or with some singular feature of the motion of the boundaries; they may arise spontaneously at a free surface as a result of viscous stresses and despite the smoothing action of surface tension; or they may conceivably occur at interior points of a fluid due to unbounded vortex stretching at high (or infinite) Reynolds number. In the last case, we are up against the unsolved and extremely challenging 'finite-time-singularity' problem for the Euler and/or Navier-Stokes equations. The question of existence of finite-time singularities is still open. Solution of this problem would have far-reaching

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R.L. Ricca (ed.), *Lectures on Topological Fluid Mechanics,*
Lecture Notes in Mathematics 1973, DOI: 10.1007/978-3-642-00837-5,

consequences for our understanding of the smallest-scale features of turbulent flow.

I have discussed some aspects of each of these problems previously (Moffatt 2001). In the present paper however, I shall place greater emphasis on the means by which each type of singularity may be resolved, and I shall indicate the nature of some recent advances.

2 Boundary-Driven Singularities

The simplest, and most prototypical, example of a boundary-driven singularity is given by the 'paint-scraper' problem of G.I. Taylor (1960) illustrated in figure 1. Fluid is contained in the corner between a fixed plate $\theta = \alpha$ and a plate $\theta = 0$ which moves parallel to itself with velocity U. Here, the singularity is imposed firstly through the geometrical singularity (the curvature of the fluid boundary being infinite at O), and secondly through the imposed discontinuity of boundary velocity at O.

Inertial forces are negligible in a region $r \ll \nu/|U|$ near the corner. Taylor's well-known similarity solution for the streamfunction $\psi(r,\theta)$ in this region takes the form $\psi(r,\theta) = -Urf(\theta)$, where

$$f(\theta) = \frac{(\alpha^2 - k\theta)\sin\theta - \theta\sin^2\alpha\cos\theta}{\alpha^2 - \sin^2\alpha}\,, \quad k = \tfrac{1}{2}(2\alpha - \sin 2\alpha)\,. \tag{1}$$

This yields a velocity field which is finite throughout the fluid domain. However, the stress field has a non-integrable $O(r^{-1})$ singularity as $r \to 0$. In particular, the pressure field has the form

$$p = p_0 - \frac{2\mu U}{r}\frac{\alpha\sin\theta + \sin\alpha\sin(\alpha-\theta)}{\alpha^2 - \sin^2\alpha}\,, \tag{2}$$

where μ is the dynamic viscosity of the fluid. This tends to $\pm\infty$ as $r \to 0$ according as $U <$ or > 0. In either case, the force F (per unit length) required to hold the scraper in position is infinite. This just indicates that there is something wrong with the proposed solution!

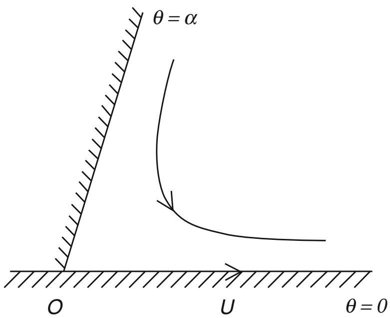

Fig. 1 Flow in a corner driven by tangential motion of one boundary. The pressure and stress are singular at the corner.

The resolution of the singularity is simple to describe in physical terms, but, so far as I know, no mathematical description is as yet available. Consider first the case $U < 0$, when $p \to +\infty$ at O. If F is large but finite, then, as recognised by Taylor, we must allow for a small gap between the two plates (figure 2), which is of the same order of magnitude as the thickness of the layer of paint spread on the plate in the paint-scraper context. If the length of the scraper (in the r-direction) is L, then the force F is of order $2\mu|U|\ln(L/\delta)$, and so δ has order of magnitude

$$\delta \sim L\exp(-cF/\mu|U|)\,, \tag{3}$$

where c is a constant of order unity.

If $U > 0$, so that according to (2), $p \to -\infty$ at O, then cavitation must presumably occur in a δ-neighbourhood of O (figure 3). Within the cavity, the pressure p equals the vapour pressure p_v, and from (2), δ is given in order of magnitude by

$$\delta \sim (p_0 - p_v)/\mu U\,, \tag{4}$$

p_0 being interpreted as the pressure far from the corner. The force F is then rendered finite:

$$F \sim 2\mu U\ln(L\mu U/(p_0 - p_v))\,. \tag{5}$$

In both cases, verification of this description requires solution of a difficult mixed boundary-value problem involving determination of the shape of the free surface on which the tangential stress is zero and the normal stress is constant.

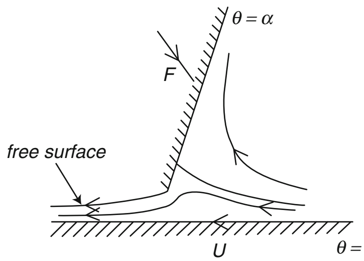

Fig. 2 Stress resolution when $U < 0$: for finite force applied to the stationary plate, a small gap is necessarily present, and a small amount of fluid leaks through the gap.

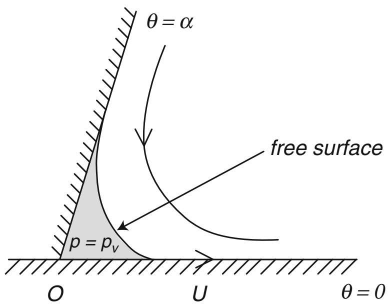

Fig. 3 Stress resolution when $U > 0$: the liquid in the corner must cavitate, the shape of the free surface being determined by the condition that the vapour pressure is constant in the cavity.

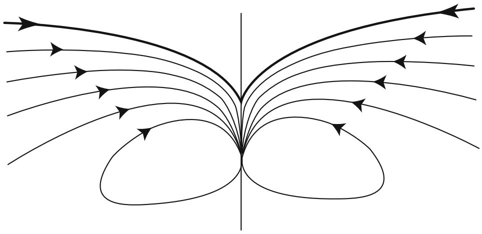

Fig. 4 Cusp formation on the free surface Γ of a viscous fluid, the flow being induced by a vortex dipole at depth d below the undisturbed position of the free surface. When the effects of surface tension are weak, the cusp forms at depth $2d/3$. [From Jeong & Moffatt 1992.]

3 Cusp Singularities at a Free Surface

If a viscous fluid has a free surface, and an internally-driven flow leads to strong convergence of the flow on this free surface towards a line (or curve) on it, then there is a tendency to form an inward-dipping cusp-type singularity located along this line or curve. The prototype flow here is that studied by Jeong & Moffatt (1992), in which the flow is induced by a vortex dipole of strength α placed at depth d below the undisturbed free-surface level (figure 4). The exact solution for the Stokes flow for this geometry, and with surface tension γ at the free surface, exhibits a remarkable property: the radius of curvature R of the free surface at the plane of symmetry is given by

$$\frac{R}{d} = \frac{256}{3} \exp(-32\pi Ca) \,, \tag{6}$$

where $Ca = \mu\alpha/d^2\gamma$ is the capillary number of the flow. If we adopt the 'level-playing-field' assumption $Ca = 1$ (i.e. viscous and surface-tension effects are *a priori* given equal weight) then (6) gives $R/d \sim 10^{-42}$ (!), so that R is many orders of magnitude below the scale at which the continuum approximation is valid. From a mathematical point of view, the solution is unimpeachable; from a physical point of view (as recognised by Jeong & Moffatt 1992), it is of course completely untenable.

One possible resolution of the (physical) singularity has, in this case, been found by Eggers (2001), again through consideration of pressure effects – this time, pressure in the thin layer of air that is drawn into the cusp region (figure 5). This pressure can be determined in the first instance through the lubrication (thin-film) approximation, and the resulting modification of the viscous flow near the cusp can then be calculated. Eggers shows that the pressure field induces a runaway effect, in which air is drawn in a thin layer down into the interior of the viscous liquid (with the possibility of subsequent break-up and entrainment of bubbles into the fluid interior).

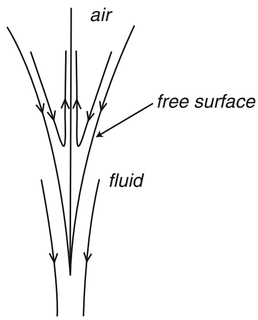

Fig. 5 Resolution of the cusp singularity: air is drawn into the long thin region in the immediate neighbourhood of the cusp, and the air pressure gradient causes deformation and instability of the free surface, leading to engulfment of air bubbles into the viscous fluid.

This mechanism, whereby one fluid may be drawn into the interior of another more viscous fluid, is of great potential practical importance in relation to mixing processes in chemical engineering, and merits further systematic study, both experimental and theoretical. It is noteworthy that the change of surface topology that is a characteristic feature of the mixing of two fluids (e.g. oil and vinegar) can be initiated only through the formation of surface singularities, of which the cusp appears to be the naturally occurring prototype.

4 A Simple Finite-Time Singularity: the Euler Disk

Euler's disk is a toy (Bendik 2000) which exhibits a finite-time singularity in spectacular manner. It is a heavy polished steel disc which can be set in rolling motion on its slightly bevelled edge. Classical rigid-body dynamics shows that, if the point of rolling contact P describes a circle with constant angular velocity Ω, then

$$\Omega^2 \sin\alpha = 4g/a\,, \tag{7}$$

where a is the radius of the disk, and α the angle between its plane and the horizontal surface on which it is assumed to roll.

Dissipative effects (e.g. vibration of the supporting structure, rolling friction, viscosity in the surrounding air, ...) induce a slow decrease of α towards zero. If

$$\dot{\alpha}/\alpha \ll \Omega\,, \tag{8}$$

then the 'balance condition' (7) persists as an adiabatic constraint, and Ω tends to infinity as α tends to zero.

The energy (potential plus kinetic) of the disk is given by $E = \frac{3}{2}Mga\sin\alpha$, where M is the mass of the disk. The rate of dissipation of energy Φ can (in principle) be calculated as a function of E, and the equation $dE/dt = -\Phi(E)$ may then be integrated. If $\Phi(E) \sim E^\lambda$ where $\lambda < 1$, then $E \sim (t^* - t)^\beta$, where $\beta = (1-\lambda)^{-1}$. Then from (7), $\Omega \sim (t^* - t)^{-\beta/2}$, indicating a singularity of Ω (and with it, a singularity of the vorticity at every point in the thin layer between the disk and the table) as t tends to t^*, a time determined implicitly by the initial conditions.

The value of λ depends on which dissipative mechanism is dominant. In the original theory of Moffatt (2000b), only viscous dissipation in the (ultimately) thin layer of air between the disk and the table was taken into account. Lubrication theory then led to a value $\lambda = -2$ (so $\beta = 1/3$), and gave a time t^* of order 100s (using the known parameters of the toy Euler disk) consistent with crude experiment. I have frequently been asked what happens if the disk is rolled in a vacuum, so that air viscosity as a dissipative mechanism is removed. The answer is that, unless the vacuum is extreme (so that the mean-free-path of air molecules is of the same order as the gap width αa), there is little change in the conclusion; this is because the viscosity μ of air is fairly insensitive to reduction of pressure (the mean-free-path goes up as the collision rate of molecules goes down). This is not to say that other dissipative mechanisms (notably solid rolling friction) are unimportant; however, air viscosity is the one mechanism for which a semi-quantitative description has as yet been provided.

An improved description of the influence of air viscosity, that takes account of the formation of Stokes layers on the disk and on the table, has been given by Bildsten (2002). This gives $\lambda = -5/4$, $\beta = 4/9$. The Stokes layers eventually overlap, at which stage the lubrication theory referred to above becomes applicable.

We must now ask how the singularity of Ω is to be resolved. This may be approached in two different ways, each leading to the same conclusion. If $\alpha = k(t^* - t)^{\beta}$, where $0 < \beta < 1$, then using (7), the adiabatic approximation (8) breaks down when $(t^* - t)^{\beta-2} \sim g/ka$. In the lubrication approximation of Moffatt (2000b), this gave $(t^* - t) \sim 10^{-2}$s, so that the singularity is averted in (literally) the last split second!

The second approach is more physical. The normal reaction at P is $N = M(g + a\ddot{\alpha})$, and the rolling condition of course requires that $N > 0$. With the above time-dependence of α, N goes to zero when $(t^* - t)^{\beta-2} \sim g/ka$, just when the adiabatic approximation breaks down. It is interesting to note that this coincidence holds independently of the dissipative mechanism (which only serves to determine the precise value of β). Thus, it would appear that in this final split second, the rolling condition (on which (7) is based) is no longer satisfied: either there is slipping at the point P, or (more probably) there is lift-off (momentary loss of contact between disk and table). A different dynamical regime is then applicable just before the disk comes finally to rest.

5 Finite-Time Singularities at Interior Points

The question of regularity of solutions of the Navier-Stokes equations continues to attract intense interest: do solutions remain smooth for all time?

Alternatively, are there any initial conditions of finite energy for which the solution exhibits a singularity at finite time? Such a singularity may conceivably occur through intense vortex stretching. But then one might normally expect the scale of the vortex core to decrease to the point at which viscous diffusion places a brake on the intensification process.

Surprisingly however, it is not necessarily the case that viscosity inevitably wins over vortex stretching, as the following simple example (from Moffatt 2000a) demonstrates. Consider the action of the time-dependent uniform strain field $\mathbf{U}$, whose components in cylindrical polar coordinates (r, θ, z) are

$$\mathbf{U} = \left(-\tfrac{1}{2}\gamma(t)r,\ 0,\ \gamma(t)z\right) , \tag{9}$$

on a diffusing vortex tube in which the vorticity is

$$\boldsymbol{\omega} = (0, 0, \omega(r,t)) . \tag{10}$$

The strain field $\mathbf{U}$ has infinite energy since the integral of $\mathbf{U}^2$ obviously diverges at infinity. Nevertheless, we may usefully consider its local effect (near $r = 0, z = 0$). The vorticity equation reduces to

$$\frac{\partial \omega}{\partial t} = \frac{\gamma(t)}{2r}\frac{\partial}{\partial r}(r^2\omega) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right) , \tag{11}$$

and we note that if γ is steady, and equal to γ_s say, then we have the familiar steady solution known as the Burgers (1948) vortex:

$$\omega = \frac{\Gamma}{\pi\delta_0^2}\exp\left(\frac{-r^2}{\delta_0^2}\right), \tag{12}$$

where $\delta_0 = 2(\nu/\gamma_s)^{1/2}$ is the 'core radius'.

Suppose now that, through some as yet unspecified mechanism, $\gamma(t)$ is given by

$$\gamma(t) = c(t^* - t)^{-1} , \quad (c > 0) , \tag{13}$$

and that $\omega(r, 0)$ is given by the Gaussian formula (12). Then it is easily verified that the unique solution of (11) is

$$\omega(r,t) = \frac{\Gamma}{\nu(t^* - t)}f(\eta), \quad \eta = \frac{r}{(\nu(t^* - t))^{1/2}} , \tag{14}$$

and

$$f(\eta) = \frac{c-1}{4\pi}e^{-\frac{1}{4}(c-1)\eta^2} , \tag{15}$$

If $c > 1$ and $0 < t < t^*$, then this describes an approach to a singularity of vorticity on $r = 0$ as $t \to t^*$. If $0 < c < 1$, and $t^* < 0 < t - t^*$, then it describes the more familiar process of diffusion of a vortex starting from a singularity at $t = t^*$, this diffusion being attenuated by the action of the strain field.

The situation when $c = 1$ is peculiar. For $\nu > 0$, (15) gives $f(\eta) = 0$. However, if we go to the inviscid limit $\nu \to 0$, then with $t^* > 0$ and taking $c - 1 = 4\nu t^*/\delta_0^2$, (i.e. $c \to 1$ in the limit), then we have the limiting solution

$$\omega(r,t) = \frac{t^*}{t^* - t}\frac{\Gamma}{\pi\delta_0^2}\exp\left(-\frac{r^2}{\delta_0^2}\frac{t^*}{t^* - t}\right), \tag{16}$$

exhibiting singular behaviour as $t \to t^*$.

The self-similar form (14) is a special case of the Leray (1934) transformation of the Navier-Stokes equation, which may be written

$$\mathbf{u}(\mathbf{x},t) = \left(\frac{\Gamma}{t^* - t}\right)^{\frac{1}{2}}\mathbf{U}(\mathbf{X}), \qquad \mathbf{X} = \mathbf{x}(\Gamma(t^* - t))^{-\frac{1}{2}}, \tag{17}$$

where Γ is a constant with the dimensions of circulation. Under this transformation, the vorticity takes the form

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = \frac{1}{(t^* - t)}\boldsymbol{\Omega}(\mathbf{X}), \qquad \boldsymbol{\Omega} = \nabla_{\mathbf{X}} \wedge \mathbf{U}, \tag{18}$$

and the vorticity equation reduces to

$$-\nabla \wedge (\mathbf{U} + \tfrac{1}{2}\mathbf{X}) \wedge \boldsymbol{\Omega} = \epsilon \nabla^2 \boldsymbol{\Omega}, \tag{19}$$

where ∇ now represents $\partial/\partial\mathbf{X}$, and $\epsilon = \nu/\Gamma$. We must obviously interpret this equation as describing flow in an inner region in a neighbourhood of the point $\mathbf{x} = \mathbf{0}$, with scale collapsing like $(t - t^*)^{1/2}$. An inner solution must match to an outer solution which is non-singular as $t \to t^*$. The only reasonable possibility (Moffatt 2000a) is

$$\boldsymbol{\Omega}(\mathbf{X}) \sim |\mathbf{X}|^{-2} \text{ as } |\mathbf{X}| \to \infty \tag{20}$$

$$\boldsymbol{\omega}(\mathbf{x},t) \sim \Gamma|\mathbf{x}|^{-2} \text{ as } |\mathbf{x}| \to 0\,. \tag{21}$$

Now, following the approach of Nečas, Ružička & Šerák (1996), it has been rigorously shown by Tsai (1998) that, when $\epsilon > 0$, (19) has no non-trivial smooth solution satisfying an outer condition of the form (20). In counterbalance to this negative result, Pelz (1997, 2001) has provided strong numerical evidence (using vortex filament techniques) that, when $\epsilon = 0$, there are certain highly symmetric vorticity distributions which appear to collapse towards a singularity, following the Leray scaling (17). This singularity involves decrease of all scales (including the scale of vortex cores) in an inner region, like $(t^* - t)^{1/2}$. (See also Kerr 1997 for direct numerical simulation of Euler flows involving interacting vortices.)

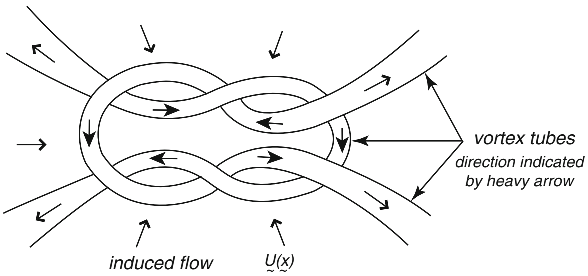

Fig. 6 A knotted vortex configuration that could conceivably give rise to a finite time singularity for the Euler equations. The vortices are conceived as inducing an inward flow contracting the knot, which must be compensated by an outward flow in the vortex tubes draining fluid from the vortices, and leading to a tightening of the knot.

Each vortex in the Pelz assemblage is subject to the strain field induced by the total vorticity distribution, and this strain increases in proportion to $(t^* - t)^{-1}$ (cf. (13)) essentially because, in the inner zone, all of the vortices approach each other, while conserving individual circulations. If this strain increases like $c(t^* - t)^{-1}$, where $c > 1$, then, as shown by the exact solution (14) of the Navier-Stokes equations, this singular behaviour should persist when ϵ is small but nonzero; the Reynolds number based on the length and velocity scales in the inner zone is then just ϵ^{-1}, which is large and which remains constant. But this description is in contradiction with Tsai's result. We escape from this contradiction only if, in the Pelz scenario, the vortices arrange themselves asymptotically in the inner zone so that the maximal strain rate acting on each vortex is $c(t - t^*)^{-1}$ with precisely $c = 1$. In this case, the singularity does not survive the transition from $\epsilon = 0$ to $\epsilon > 0$.

Figure 6 shows a hypothetical vorticity distribution (a reef knot) which might be considered as a possible candidate for an Euler singularity. Note the conical spread of vorticity (corresponding to the behaviour (20)) on leaving the 'inner zone'. The quasi-steady vorticity equation includes a term involving the outward sweeping of vorticity by the spherically symmetric 'velocity field' $\frac{1}{2}\mathbf{X}$ (with divergence 3/2); this outward sweeping must be compensated by equal and opposite inward sweeping by the induced velocity field $\mathbf{U}$, which must therefore be inwards wherever $\boldsymbol{\Omega}$ has a nonzero nonradial component, and which, being divergenceless, must be outwards only where $\boldsymbol{\Omega}$ is exactly radial. It is difficult to see how the transition from inward to outward flow can be compatible with these requirements.

Similar considerations must apply to the Pelz configuration (or to any Euler flow exhibiting collapse to a singularity with Leray scaling): inflow across the vorticity field must be compensated by outflow only in regions where the vorticity is radial. It remains to be shown whether this can actually be achieved.

We should note also that Tsai's (1998) 'anti-singularity' result, described above, does not cover the possibility of *non-self-similar singularities* of the Navier-Stokes equations. A simpler system which provides an illuminating illustration of possible behaviour is provided by the equation

$$\frac{\partial \phi}{\partial t} = \phi^2 + \epsilon \frac{\partial^2 \phi}{\partial t^2} \,. \tag{22}$$

This equation is one of a class studied by Budd et al (2002); it clearly admits self-similar singular behaviour when $\epsilon = 0$. When $\epsilon \neq 0$, the singularity survives (at a shifted singularity time), but it is no longer self-similar in character.

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