J Eng Math (2010) 68:263–268
DOI 10.1007/s10665-010-9390-z

Note on the suppression of transient shear-flow instability by a spanwise magnetic field

H. K. Moffatt

Received: 21 December 2009 / Accepted: 9 July 2010 / Published online: 28 July 2010

Abstract Shear flow is prone to transient instability in which perturbations having little or no variation in the streamwise direction can grow linearly for a long time if the Reynolds number is large. This behaviour is known to provide a trigger for the development of secondary instabilities and transition to turbulence. It is shown by a simple analysis of Kelvin modes that a spanwise magnetic field is efficient in suppressing this transient instability and therefore in inhibiting the transition to turbulence.

Keywords Magnetohydrodynamics · Transient instability · Transition to turbulence

1 Background

The inhibiting effect of a spanwise magnetic field on the process of transition to turbulence in channel flow of a conducting fluid has been recently investigated computationally by Krasnov et al. [1]. This effect occurs through suppression by the magnetic field of the transient growth of streamwise vortices, a process induced by the shearing flow.

My purpose here is to provide a simple analytical demonstration of this effect, by employing the technique that I used originally [2] to study the interaction of atmospheric turbulence with strong wind shear (essentially a 'rapid-distortion' approach). I showed in that paper that "the dominant contribution, both to the disturbance energy and to the Reynolds stress generated, comes ultimately from eddies having a cylindrical structure, the axis of the cylinders being parallel to the shear flow". These eddies were the basis of Townsend's approach to coherent structures (or 'big eddies' as he originally described them) in the second edition (1976) of his book "The Structure of Turbulent Shear Flow" [3, Sect. 3.12, Chap. 4], which, in this respect, differs significantly from the first edition (1956). These 'cylindrical eddies' are what came to be known as 'streamwise vortices' (see, for example, [4]), and the manner in which they emerge from an initially random field is a manifestation of 'transient instability', a phenomenon that became well-known through the work of Landahl [5]. Landahl wrote in the abstract of this paper: "It is shown that all

This paper is dedicated to Norman Riley, whose fluid dynamical insights I have admired, and whose friendship I have enjoyed, over many years.

H. K. Moffatt (✉)
DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK
e-mail: hkm2@cam.ac.uk

parallel inviscid shear flows of constant density are unstable to a wide class of initial infinitesimal three-dimensional disturbances in the sense that, according to linear theory, the kinetic energy of the disturbance will grow at least as fast as linearly in time. This can occur even when the disturbance velocities are bounded, because the streamwise length of the disturbed region grows linearly with time. This finding may have implications for the observed tendency of turbulent shear flows to develop a longitudinal streaky structure". Landahl's speculation in this last sentence has surely been vindicated with the passage of time.

The rapid-distortion approach has been greatly developed by Claude Cambon and his colleagues in Lyon (see [6, pp. 406–422] and references therein), with particular attention to the anisotropic structure induced by rotation and/or density stratification. The approach is also basic to the technique of determining 'optimal perturbations' for energy growth, as originally developed by Schmid and Henningson [7].

The effect of a magnetic field on homogeneous turbulence in the absence of shear is well-known [8,9] when the magnetic Reynolds number is small, this effect is to damp out Fourier components of the field having significant variation in the field direction, making the turbulence more and more invariant (statistically) in this direction. When both shear and magnetic field are present, these competing influences both induce anisotropy, but with different 'preferred directions'. This paper is concerned with the nature of the dynamical balance that emerges from this competition.

2 Idealised set-up

We consider an incompressible fluid of uniform density ρ, kinematic viscosity ν and magnetic diffusivity η, subjected to the combined action of a uniform shear flow

$$\mathbf{U} = (\alpha y, 0, 0) \tag{1}$$

and an applied spanwise magnetic field[1]

$$\mathbf{B}_0 = (0, 0, B_0). \tag{2}$$

The induced electromotive force is then

$$\mathbf{U} \times \mathbf{B}_0 = (0, -\alpha B_0 y, 0), \tag{3}$$

and this will tend to drive current in the y-direction. If however we assume that the fluid boundaries at '$y = \pm\infty$' are insulating, then a compensating electric field $\mathbf{E}_0 = -\nabla\Phi$ is established, with $\Phi = -\alpha B_0 y^2/2$, which reduces the mean current to zero.

We now suppose that a weak random perturbation $\mathbf{u} = (u, v, w)$ is superposed on this shear flow on some scale $L = k_0^{-1}$, and we assume that the associated magnetic Reynolds number $R_m = |\mathbf{u}|L/\eta$ is small, as in any liquid metal in a laboratory context. Then the magnetic field is perturbed to $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ where $\mathbf{b}$ is determined in quasi-static manner by the reduced form of the induction equation of magnetohydrodynamics,

$$\eta\nabla^2\mathbf{b} = -(\mathbf{B}_0 \cdot \nabla)\mathbf{u}. \tag{4}$$

It follows that the rotational part of the Lorentz force is given by

$$(\mathbf{B}_0 \cdot \nabla)\mathbf{b} = -\eta^{-1}\nabla^{-2}(\mathbf{B}_0 \cdot \nabla)^2\mathbf{u}, \tag{5}$$

and so the linearised equation of motion, obtained from the Navier–Stokes equation in standard manner, is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{U} = -\frac{1}{\rho}\nabla\chi - \frac{1}{\rho\eta}\nabla^{-2}(\mathbf{B}_0 \cdot \nabla)^2\mathbf{u} + \nu\nabla^2\mathbf{u}, \tag{6}$$

where $\chi = p + \mathbf{B}^2/2$, the total pressure (fluid plus magnetic).

[1] The conventional permeability factor μ_0 is absorbed in the definition of magnetic field here and subsequently.

3 Development of Kelvin modes

Consider now a single Fourier component of $\mathbf{u}$ given at time $t = 0$ by

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \exp[\mathrm{i}\mathbf{k}_0 \cdot \mathbf{x}]. \tag{7}$$

Equation 6 admits solutions of the form

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(t) \exp[\mathrm{i}\mathbf{k}(t) \cdot \mathbf{x}], \tag{8}$$

where $\hat{\mathbf{u}}(0) = \mathbf{u}_0$ and $\mathbf{k}(0) = \mathbf{k}_0$. (We immediately drop the hat in the following.) Such modes are actually exact solutions of the nonlinear Navier–Stokes equations, as first recognised by Kelvin [10] in his pioneering study of the stability of shear flow; they are appropriately described as Kelvin modes. The incompressibility condition of course implies that

$$\mathbf{k}(t) \cdot \mathbf{u}(t) = 0, \tag{9}$$

so that

$$\dot{\mathbf{k}} \cdot \mathbf{u} + \dot{\mathbf{u}} \cdot \mathbf{k} = 0. \tag{10}$$

Substitution of (8) in (6) gives

$$\dot{\mathbf{u}} + \mathrm{i}(\dot{\mathbf{k}} \cdot \mathbf{x})\mathbf{u} + \alpha v(1, 0, 0) + \mathrm{i}\alpha y k_1 \mathbf{u} = \frac{1}{\mathrm{i}k\rho}\chi(t) - \frac{1}{\rho\eta k^2}(\mathbf{B}_0 \cdot \mathbf{k})^2\mathbf{u} - \nu k^2 \mathbf{u}. \tag{11}$$

Equation 11 must be satisfied identically at all $\mathbf{x}$; hence the coefficients of x, y and z must vanish, leading immediately to

$$\dot{\mathbf{k}} = (0, -\alpha k_1, 0), \tag{12}$$

so that

$$k_1 = \mathrm{cst}, \quad k_2 = k_{02} - \alpha t k_1, \quad k_3 = \mathrm{cst}. \tag{13}$$

When $k_1 \neq 0$, this simply describes progressive tilting of the wave-fronts (which are perpendicular to $\mathbf{k}(t)$) by the mean shear. This tilting is associated with linear increase in $|k_2(t)|$, and so, asymptotically, in the wave-number magnitude $k(t) = |\mathbf{k}(t)|$.

Equations 9 and 10 may be used to eliminate χ from (11), which now reduces to

$$\dot{\mathbf{u}} + \alpha v(1, 0, 0) = \frac{2\alpha k_1 \mathbf{k}}{k^2} v - \frac{B_0^2 k_3^2}{\rho\eta k^2}\mathbf{u} - \nu k^2 \mathbf{u}. \tag{14}$$

Kelvin noted that the viscous term simply contributes a factor

$$\exp\left(-\nu \int_0^t k^2 \mathrm{d}t\right) = \exp\left[-\nu\left(k_0^2 t - \alpha k_{02} k_1 t^2 + \frac{1}{3}\alpha^2 k_1^2 t^3\right)\right] \tag{15}$$

to the evolution of the amplitude $\mathbf{u}(t)$. Ultimately, the t^3-term in the exponent dominates (provided $k_1 \neq 0$) and leads to decay on an $O(\mathrm{Re}^{1/3})$ time-scale (closely related to the time-scale found by Rhines and Young [11] in their study of passive scalar mixing in flows with closed streamlines; a similar effect was simultaneously found by Moffatt and Kamkar [12] in the context of magnetic flux expulsion). Kelvin used this exponential decay to demonstrate the asymptotic linear stability of uniform shear flow (and it is appropriate therefore to describe the effect as 'Kelvin damping'); he did not recognise, however, that the disturbance amplitude can grow by an arbitrarily large amount (for sufficiently large Reynolds number) before this ultimate decay sets in; this is the essence of transient instability.

In similar vein, the effect of the magnetic field is evidently to contribute a further decay factor. Assuming for simplicity that $k_{02} = 0$, this factor is

$$F = \exp\left(-\frac{B_0^2 k_3^2}{\rho\eta}\int_0^t \frac{1}{k^2}\mathrm{d}t\right) = \exp\left(-\frac{\lambda k_3^2}{k_1 l}\tan^{-1} T\right), \tag{16}$$

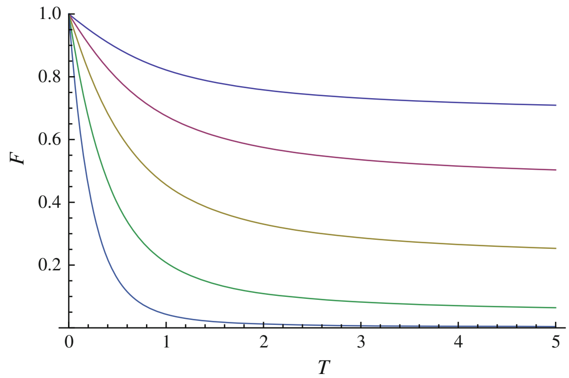

Fig. 1 The magnetic damping factor F (Eq. 16) as a function of the scaled time variable T, for $a = (\lambda/k_1)(k_3^2/l) =$ 0.25, 0.5, 1, 2, 4 (values increasing downwards)

where $T = k_1\alpha t/l$, $l^2 = k_1^2 + k_3^2$, and $\lambda = B_0^2/\rho\eta\alpha$ (a dimensionless magnetic interaction parameter). For $|k_1\alpha|t \gg l$ (i.e., $|T| \gg 1$), this converges to

$$\exp\left(-\frac{\pi\lambda k_3^2}{2|k_1|l}\right), \tag{17}$$

indicating a strong cumulative damping effect when $|k_1|$ is small; and as shown in [2], it is only when $|k_1|$ is small that the transient instability is in evidence. However, this damping, unlike exponential damping, is finite: the exponent in the integral converges because of the increase of k^2 with t (asymptotically like t^2, attributable directly to the tilting of the wave-fronts, which intensifies the effect of Joule dissipation). Figure 1 shows the magnetic damping factor F as a function of T for five values of the parameter $a = (\lambda/k_1)(k_3^2/l)$. It is interesting to note how, for fixed λ, the damping gets stronger as k_1 decreases. This magnetic damping effect is further evident in the following computations.

4 Numerical evidence for magnetic damping

The evolution curves are easily computed for any chosen values of the parameters λ, ν, $\mathbf{k}_0$, and $\mathbf{u}_0$ (consistent with $\mathbf{k}_0 \cdot \mathbf{u}_0 = 0$). Here we shall assume that ν is so small that the viscous effect can be neglected over the time-scale considered (although over very long times, the Kelvin damping must obviously prevail). With $\tau = \alpha t$, the first two components of (14) are then:

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \left(2\frac{k_1^2}{k^2} - 1\right)v - \lambda\frac{k_3^2}{k^2}u \tag{18}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = 2\frac{k_1 k_2(\tau)}{k^2}v - \lambda\frac{k_3^2}{k^2}v. \tag{19}$$

Note that these equations do not involve $w(\tau)$ which may be most easily determined from the incompressibility condition $\mathbf{k}\cdot\mathbf{u} \equiv k_1 u + k_2(\tau)v + k_3 w = 0$. Equations 18 and 19 may be easily integrated analytically as in [2] (but now incorporating the factor (16)). Here, however, we simply present some sample results obtained using *Mathematica*. Figure 2 shows the time-evolution of $u(\tau)$, $v(\tau)$, $w(\tau)$ for a mode with initial wave-vector $\mathbf{k}_0 = (0.1, 1, 1)$ and $\mathbf{u}_0 = (1, 2, -2.1)$ (so that $\mathbf{k}_0 \cdot \mathbf{u}_0 = 0$), and for a succession of increasing values of λ.

Note first the relatively long phase of linear growth of $u(\tau)$ when $\lambda = 0$. This is because k_1 is relatively small; the behaviour is entirely consistent with the description given in [2]. In fact this transient phase of linear growth persists for a time of order $k_0/\alpha|k_1|$. If $|k_1|$ is reduced, then the linear phase lasts longer, and the amplitude $|u|$ grows proportionately larger. However, when $\lambda > 0$, Joule damping occurs on the same time-scale, and the effect of the magnetic field is clear: the linear growth is suppressed, and quite dramatically so when λ is increased to values of order 0.1 or greater. It does not take much magnetic field to suppress the transient instability.

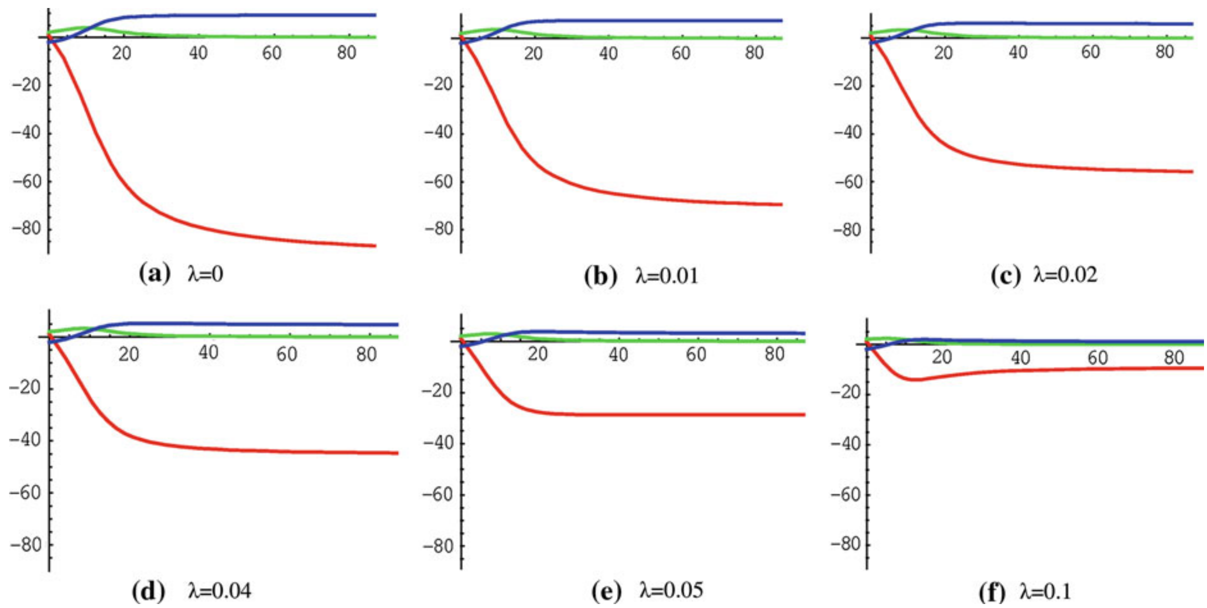

Fig. 2 Evolution of $u(\tau)$ (lower curve, *red* in online version), $v(\tau)$ (curve tending to zero for large τ, *green* in online version), $u(\tau)$ (upper curve, *blue* in online version), with $k_1 = 0.1,\ k_{02} = 1,\ k_3 = 1$, and initial conditions $u(0) = 1,\ v(0) = 2,\ w(0) = -2.1$, for six values of the magnetic interaction parameter λ. The transient growth of $u(\tau)$ up to about $\tau = 30$ is evident in (**a**); this transient growth is increasingly suppressed with increase of λ

5 Singular behaviour when $k_1 = 0$

When $k_1 \to 0$, the behaviour is singular in two respects. First, the wave-fronts are parallel to the (x, z) plane, and do not tilt when $k_1 = 0$. This means that, when $\lambda = 0$, the transient instability in which $u(t)$ increases linearly with t continues for so long as the linearisation remains valid and viscous effects remain negligible. This behaviour is simply due to transport in the y-direction of the x-momentum of the mean shear flow. Secondly, the Joule damping coefficient $\lambda k_3^2/k^2 = \sigma$, say, in (18) and (19) now remains constant rather than decreasing in time, so the damping is exponential, and all modes for which $k_3 \neq 0$ decay exponentially in time, with decay time $t_\eta = (\sigma\alpha)^{-1} = \rho\eta k^2/B_0^2 k_3^2$. Explicitly, the solution for $u(\tau)$ when $k_1 = 0$ is

$$u(\tau) = (u(0) - \tau v(0))\, \mathrm{e}^{-\sigma\tau}. \tag{20}$$

When $\sigma \ll 1$, $u(\tau)$ is linear until $\tau = O(\sigma^{-1})$, and at this stage,

$$u(\tau) \sim u(0) - v(0)/\sigma. \tag{21}$$

The transient effect is still large, but ultimately exponential decay prevails. The time-scale of viscous decay is $t_\nu = (\nu k^2)^{-1}$, so the viscous effect is indeed negligible provided the modified Hartmann number

$$M = \left(\frac{t_\nu}{t_\eta}\right)^{1/2} = \frac{B_0 k_3}{(\rho\eta\nu)^{1/2} k^2} \tag{22}$$

is sufficiently large.

6 Discussion

We have focussed attention on the evolution of single Fourier components of the initial perturbation field, but it is of course a straightforward matter to consider the evolution of the statistical properties of a random field (in particular of its spectrum tensor) by simple superposition of Fourier components. Nonlinear effects (discussed qualitatively in [2]) arise through the interaction of the resulting Kelvin modes, and this is where the great difficulty of the full turbulent-shear-flow problem arises. Nevertheless the linear treatment offered by rapid distortion theory provides a convenient and illuminating starting point for such investigations, although its limitations are apparent.

The restriction to a spanwise magnetic field can be easily relaxed to include streamwise (x) and cross-stream (y) field components. The restriction to low magnetic Reynolds number may also be relaxed. Also, other effects (e.g. of rotation or of uniform density stratification) can be easily incorporated in this type of analysis. It is then natural to consider Kelvin modes of nonzero helicity, and the '$\alpha\omega$' dynamo action that such modes (in conjunction with the mean shear) may excite—a fertile field for future investigation!

Acknowledgment I thank André Thess, who restimulated my interest in this problem through his lecture at the Conference on *Instability and Turbulence in MHD Flows* (University of Warwick, June 2006).

References

1. Krasnov D, Rossi M, Zikanov O, Boeck T (2008) Optimal growth and transition to turbulence in channel flow with spanwise magnetic field. J Fluid Mech 596: 73–101
2. Moffatt HK (1967) Interaction of turbulence with strong wind shear. In: Yaglom AM, Tatarski VI (eds) Atmospheric turbulence and radio wave propagation. Nauka, Moscow, pp 139–156, no 10. www.moffatt.tc
3. Townsend AA (1976) The structure of turbulent shear flow, 2nd edn. Cambridge University Press, Cambridge
4. Blackwelder RF, Eckelmann H (1979) Streamwise vortices associated with the bursting phenomenon. J Fluid Mech 94:577–594
5. Landahl MT (1980) A note on an algebraic instability of inviscid parallel shear flows. J Fluid Mech 28:571–592
6. Sagaut P, Cambon C (2008) Homogeneous turbulence dynamics. Cambridge University Press, Cambridge
7. Schmid PJ, Henningson DS (1994) Optimal energy density growth in Hagen-Poiseuille flow. J Fluid Mech 277:197–225
8. Moffatt HK (1967) On the suppression of turbulence by a uniform magnetic field. J Fluid Mech 28:571–592
9. Moreau R, Sommeria J (1982) Why, how, and when, MHD turbulence becomes two-dimensional. J Fluid Mech 118:507–518
10. Lord Kelvin (W. Thomson) (1887) Stability of fluid motion: rectilinear motion of viscous fluid between two parallel plates. Philos Mag 24(5):188–196
11. Rhines PB, Young WR (1983) How rapidly is a passive scalar mixed within closed streamlines? J Fluid Mech 133:133–145
12. Moffatt HK, Kamkar H (1983) The time-scale associated with flux expulsion. In: Soward AM (eds) Stellar and planetary magnetism. Gordon and Breach, New York, pp 91–98, no 45. www.moffatt.tc