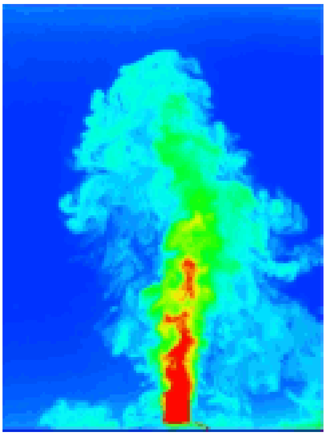

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Homogeneous turbulence: an introductory review

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Homogeneous turbulence: an introductory review

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A brief review of developments in the theory of homogeneous turbulence over the last 50 years is given, many of these developments stemming from lectures and discussions at the 1961 Marseille Colloquium *Mécanique de la Turbulence*. The following topics are discussed: Kolmogorov's 1961 lecture, intermittency and the finite-time singularity problem, the skewness factor paradox, Kraichnan's direct interaction approximation, helicity and the dynamo problem, two-dimensional turbulence, rapid distortion theory and the passive and active scalar problems.

Keywords: Kolmogorov; intermittency; skewness; helicity; rapid distortion

1. Marseille 1961

My credentials for presenting this opening review[1] are slender, relying as they do on the simple fact that I am one of the handful of surviving participants of the now legendary 1961 Marseille Colloque International du CNRS, Mécanique de la Turbulence (hereafter Marseille '61), which this present Colloquium seeks, if not to emulate, at least to commemorate. Allow me first to recall some features of that 1961 Colloquium that made it so special. I draw on the Proceedings [1] as a jog to my memory: a large red volume, perhaps less well-known hitherto than it should have been, but now scanned and accessible on the website <http://turbulence.ens.fr> of this present meeting.

The Colloquium was organised, and the Proceedings edited, by the late Alexandre Favre, Membre de l'Académie des Sciences, who deserves great credit for the memorable character of the event! The Scientific Committee consisted of G.K. Batchelor (my mentor and research supervisor at the time), J. Kampé de Fériet (the 'éminence grise' of French mechanics), L. Kovasznay (USA), H. Schlichting (Germany) and L. Sedov (USSR). In the event, Schlichting and Sedov were not present at the Colloquium, and Kampé de Fériet played a figurehead role, so the burden of scientific organisation fell on Batchelor, Kovasznay and Favre; they invited S. Corrsin, R.W. Stewart and H. Liepmann to act as Chairmen of various sessions – all eminent figures in the subsequent history of turbulence. But particularly memorable was the fact that three great father-figures were also present, namely Theodore von Kármán, G.I. (Sir Geoffrey) Taylor and A.N. Kolmogorov (who was accompanied by three colleagues from the Soviet Union, M.D. Millionschikov, A.M. Obukhov and A.M. Yaglom). This was a unique encounter, during that cold-war epoch, of great scientists from East and West, something that Batchelor had been particularly keen to engineer in

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difficult political circumstances. [The cold war was at its height: the Berlin blockade was in force from July to November that year, when the infamous Berlin Wall was constructed; the Soviet Union resumed nuclear testing on 1st September 1961, during the very week of Marseille'61, in response to a series of four atmospheric nuclear tests carried out by France in the Algerian Sahara between January 1960 and May 1961; and France itself was still, under General de Gaulle, in a state of political turmoil over the Algerian War of independence.]

It is noteworthy that neither Taylor nor von Kármán gave talks at the Colloquium, but they of course contributed to the discussions, and their presence lent great authority to the deliberations. Neither did Favre lecture, but his coordinating role as host was crucial. As for Millionschikov, he later rose to high office in the Soviet bureaucracy (a USSR postage stamp bearing his image was issued in 1974!), and his presence was sufficient to ensure that no member of the KGB was needed to 'keep an eye' on the Soviet delegation.

The second session of Marseille '61 was devoted to *Energy Transfer in Homogeneous Turbulence*, and it is on aspects of homogeneous turbulence that I focus in this review. Homogeneous turbulence, i.e. turbulence whose statistical properties are invariant under translations of the frame of reference, is of course a fiction: no field of turbulence is strictly homogeneous, since the presence of fluid boundaries inevitably introduces a degree of inhomogeneity. Nevertheless, 'homogeneity' is a useful idealisation that can be approximately realised in wind tunnel 'grid-generated' turbulence, widely studied in the belief that this provides a legitimate description of at least the smaller scales of turbulence far from fluid boundaries. For turbulence that is strictly homogeneous, the abstract concept of the 'ensemble average' may be conveniently replaced by the more accessible concept of the 'space average'; ergodic theory tells us that these two averages are, under the condition of homogeneity, equivalent.

Turbulence that is homogeneous may also be 'isotropic', i.e. its statistical properties may be invariant under rotations, as well as translations, of the frame of reference. In such a field of turbulence, there is no 'preferred direction': all directions are statistically equivalent. It should be noted however that isotropic turbulence need not be invariant also under mirror reflexions, a distinction that becomes crucially important when magnetohydrodynamic effects are under consideration (see §6 below).

Homogeneous turbulence may be forced or unforced. Forcing can be achieved in numerical simulations, though not in laboratory experiments, by inclusion of a random body force $\mathbf{f}(\mathbf{x}, \mathbf{t})$, itself statistically homogeneous, in the Navier–Stokes equation governing the flow. If this body force is also statistically stationary in time, a statistically steady homogeneous turbulent state can result; in this case, the ensemble average is equivalent to a time average, which is both conceptually clear and computationally convenient.

Unforced homogeneous turbulence, on the other hand, necessarily decays in time, as a result of the non-linear cascade of energy to the very small scales of motion at which viscous dissipation of kinetic energy sets in. It is an essential tenet of Kolmogorov's [2] theory that the mean rate of cascade (and subsequent decay) of energy ϵ is independent of kinematic viscosity ν in the limit $\nu \to 0$.

2. K41 and Kolmogorov's 1961 lecture

This 1941 theory of Kolmogorov (K41) was then, as still even now, a central pillar of understanding of the mechanics of turbulence, and so it was particularly fitting that he should be present at Marseille '61. In fact, a highlight of the meeting, alluded to only in a footnote to the paper by T.H. Ellison (p. 115 of the Proceedings), was the display of slides by Stewart from the observations later published in *JFM* [3] of turbulent spectra in a tidal channel at Reynolds number (based on depth and mean velocity) up to 3×10^8. These experiments provided what Ellison described as 'by far the most convincing demonstration of the correctness of the Kolmogorov theory that has yet been made'. This was of exceptional importance for two reasons: first, because huge effort had been expended, particularly by Batchelor (and references therein) [4] in promoting the Kolmogorov theory, and much of the current understanding relied on the Kolmogorovian concept of the 'energy cascade' and the $k^{-5/3}$ energy spectrum to which this gave rise; and second, because this picture was being currently challenged by the alternative 'direct-interaction' theory of Kraichnan (1959) [5] which led to a significantly different ($k^{-3/2}$) spectrum. The tidal-channel observations reported by Stewart gave unambiguous support for the Kolmogorov scenario.

So far, so good! But then Kolmogorov was invited to give his lecture with the title 'Précisions sur la structure locale de la turbulence dans un fluide visqueux aux nombres de Reynolds élevés'. The paper is reproduced in both French and Russian in the Proceedings, and in English translation in *JFM* [6]. Kolmogorov delivered his lecture in French, and his only visual aid was the blackboard. What was so startling was that (following reference to parallel work of Obukhov and to comments that he attributed to Landau) Kolmogorov proceeded to point out fundamental inconsistencies in his own (1941) theory, in that this theory failed to take account of the spatial fluctuations in the rate of turbulent dissipation of energy ϵ, the key parameter of the theory. Kolmogorov offered arguments supporting a log-normal probability distribution for this quantity, and showed that the exponent in the energy spectrum would be slightly modified through this 'intermittency' effect. (Later work on the behaviour of higher-order structure functions of the form $S_n(r) = < (u(\mathbf{x}+\mathbf{r}) - u(\mathbf{x}))^n >$, for $n = 2, 3, 4, \ldots$, where u is, say, the velocity component parallel to the direction of the separation vector $\mathbf{r}$, has revealed that Kolmogorov's log-normal assumption itself is untenable ([7, 8]).)

This lecture, Kolmogorov's final contribution to turbulence, was, to put it mildly, a bit of a bombshell! There had been hints previously that all was not well with the Kolmogorov picture; in particular, as described in Batchelor's [4] monograph, the dimensionless flatness factor of the velocity derivative

$$F = \frac{\left\langle (\partial u/\partial x)^4 \right\rangle}{\left\langle (\partial u/\partial x)^2 \right\rangle^2}, \tag{1}$$

which, on the Kolmogorov theory should be a universal constant in the limit of large Reynolds number Re, in fact showed a worrying experimental increase with increasing Re, with no tendency to settle down to a limiting constant value. The implication was that, as Re increases, the vorticity of the turbulence tends to become more and more concentrated in 'fine structures' (vortex tubes and/or sheets being the most natural candidates), a picture that was hard to reconcile with the idea of an energy cascade from the energy-containing scale l_0 through a geometric sequence of length scales like $2^{-n} l_0$. Kolmogorov's revised theory, taking account of the intermittency of $\epsilon(\mathbf{x}, t)$, was however potentially capable of explaining the increase of F with Re, and even more so, of the behaviour of higher-order statistical properties of the small-scale features of the turbulence (including in particular the vorticity field).

3. Intermittency and the finite-time singularity problem

Kolmogorov's paper marked the onset of a widespread effort in the turbulence research community to understand the phenomenon of intermittency, for which many competing models have been proposed. This has been much stimulated on the one hand by the detection of concentrated vortices within a field of turbulence [9] and on the other by direct numerical simulations (DNS), which display turbulence as a random superposition of concentrations of vortex tubes on all scales from the Taylor microscale $\sim l_0 Re^{-1/2}$ down to the Kolmogorov scale $\sim l_0 Re^{-3/4}$ (e.g. [10, 11]). The cascade picture has now given way to a picture in which the progressive degradation of energy from large to small scales is simply a consequence of vortex tube stretching, each vortex tube being locally stretched (with associated decrease of cross-section, i.e. decrease of scale) by the straining flow induced by all the other vortices of the field of turbulence.

This revised picture has in turn re-stimulated interest in the fundamental problems of regularity associated with the Euler equations and the Navier–Stokes equations at very high Re as first addressed by Leray [12]: given smooth initial conditions for the vorticity field, does this field in general remain smooth for all time, or alternatively, does there exist any smooth initial condition of finite energy from which a singularity of vorticity emerges within a finite time? The question is of course fascinating in its own right, and quite independently of the general problem of turbulence. The natural initial condition to consider is a vortex-tube tangle, each tube containing axial vorticity of Gaussian cross-section. Is it not amazing that we still do not know whether such tangles can lead to a singularity or not! DNS is as yet inconclusive, in no small part because numerical techniques inevitably break down as a singularity is approached. If a singularity can indeed form, then some kind of self-similar structure (or structure within a structure) of scale decreasing to zero at the singularity time is to be expected, but such description remains elusive.

The problem is important for turbulence, because it has an obvious bearing on the structure of turbulence at the smallest scales. Considering for the moment the inviscid limit $\nu \to 0$, the Kolmogorov spectrum extends to wavenumber $k = \infty$ (i.e. zero length-scale). The time-scale for energy transfer from scale k^{-1} to scale zero, as determined dimensionally from ϵ and k, is then $t_k = \epsilon^{-1/3} k^{-2/3}$. This suggests that in the Euler limit, singularities must develop at finite time. Of course, a divergent enstrophy spectrum $\sim k^{1/3}$ suggests the same. But no explicit solution of the Euler equations has as yet revealed the singular asymptotic structure of the vorticity field that such arguments might suggest.

DNS has played an increasingly prominent role in turbulence research ever since the pioneering work of Orszag and Patterson [13] (see, for example, the recent review by Ishihara et al. [14]), and its importance cannot be over-estimated. It remains fair to say however that advances in understanding require a symbiotic interaction between DNS and theoretical analysis: each activity feeds on, and at the same time provides guidance for, the other; now only in this way can further real progress be made.

4. The paradox of the skewness factor

A related puzzle concerns the skewness factor of the velocity derivative

$$S = \frac{\langle (\partial u/\partial x)^3 \rangle}{\langle (\partial u/\partial x)^2 \rangle^{3/2}}, \tag{2}$$

which again, on K41 theory, should be a universal constant in the limit $Re \to \infty$. Batchelor said in his Marseille '61 lecture that 'the available measurements ... suggest that it is still decreasing (in magnitude) at the highest Reynolds numbers of the measurements, and we cannot yet be sure that the asymptotic value is non-zero'. This remains a crucial issue, because in 3D isotropic turbulence the vorticity $\boldsymbol{\omega}$ satisfies

$$\left\langle |\boldsymbol{\omega}|^3 \right\rangle \sim |S| \left\langle \boldsymbol{\omega}^2 \right\rangle^{3/2} , \tag{3}$$

and the enstrophy equation takes the form

$$\frac{\partial \left\langle \boldsymbol{\omega}^2 \right\rangle}{\partial t} = C \left\langle \boldsymbol{\omega}^2 \right\rangle^{3/2} - 2\nu \left\langle (\nabla \times \boldsymbol{\omega})^{\mathbf{2}} \right\rangle , \tag{4}$$

where C is a dimensionless positive constant proportional to $|S|$. In the formal limit $\nu = 0$, this equation implies blow-up of enstrophy in a finite time, and again, this would appear to imply the appearance of singularities in the vorticity distribution, at least in the Euler limit. So, what is the latest information on the skewness factor in the high Re limit? I do not know that we are any nearer to a definite answer to this central question than we were in 1961.

5. Kraichnan's direct interaction approximation

Bob Kraichnan also gave a memorable lecture at Marseille '61 in which he compared various approaches to the closure problem of turbulence, including the quasi-normal (or zero-fourth-cumulants) approximation of Millionschikov [15] and his own direct-interaction approximation published two years earlier [5]. Kraichnan simply stood at the microphone and spoke for 40 minutes with extraordinary lucidity, and without the help of a single visual aid, even chalk on blackboard! His lecture was followed by a critique of 'Kraichnan's theory of turbulence' by Ian Proudman, a younger colleague of Batchelor, who had published an extensive treatment of the quasi-normal theory with W.H. (Bill) Reid some years earlier ([16]; see also [17])[2]. Proudman's critique was savage by any standard, his conclusion being: 'I do not hold out much hope for useful results from the theory [of Kraichnan]'. Kraichnan was understandably perturbed by this unfavourable assessment.

In some respects, Proudman's pessimism was justified. As mentioned earlier, the experimental evidence did not support a $k^{-3/2}$ energy spectrum, which was a key prediction of the theory. However, in his response (published as an Appendix to his own paper in the Proceedings), Kraichnan pointed out a key 'realisability' property of his theory, namely that it guaranteed that the energy spectrum would remain positive for all time. He referred to the paper by Ogura (presented at the IUGG-IUTAM Symposium that followed immediately on Marseille '61, and later published as Ogura [19]), which provided numerical evidence that the quasi-normal closure (which incidentally leads to an enstrophy equation of the form (4)) leads to the inadmissible development of a negative energy spectrum in the energy-containing range of wave-numbers. In this respect therefore, the direct interaction approximation was clearly superior (and this because it was an exact closure for a physically realisable dynamical system, although unfortunately not the Navier–Stokes system).

Kraichnan was subsequently able to modify his theory in such a way as to yield a $k^{-5/3}$ spectrum (his Lagrangian-History Direct Interaction Approximation, or LHDIA). I spent four months in Stan Corrsin's group at Johns Hopkins University (September–December 1965), and I recall that we met for a weekly sandwich-lunch discussion session, to work our way through this theory, but such was its complexity that we were not much the wiser by the end of the term. The theory [20] takes account of the 'sweeping' of small eddies by large eddies, in a mixed Lagrangian–Eulerian framework. In effect, this forces the energy transfer to be local in wave-number space, and this in turn leads on dimensional grounds to the Kolmogorov spectrum $E(k) = C\epsilon^{2/3}k^{-5/3}$. The reward for all the hard work is that the universal constant C is determined by the theory; the downside is that, although $k^{-5/3}$ appears fairly robust, the 'constant' C varies by about $\pm 30\%$ in different experimental contexts (Monin and Yaglom [21], Chap. 8).

6. Helicity and the dynamo problem

One important aspect of homogeneous turbulence is the manner in which it acts upon a convected scalar field (like temperature) or vector field (like magnetic field), in conjunction with the effects of molecular diffusion. The passive scalar problem is to be covered in another section of this meeting, so I shall say no more here (except to recall that Philip Saffman gave a remarkable lecture at Marseille '61 on the counter-intuitive interaction of molecular and turbulent diffusion of a scalar released from a fixed point in a turbulent flow – Saffman [22]). But I would like to comment on the vector field problem, because many of the advances in our understanding of the dynamics of turbulence have come from parallel considerations concerning MHD turbulence.

The emphasis in 1961 was on the increasing degradation of energy in going from large to small scales, i.e. the production of chaos (the small-scale turbulence) from order (the large-scale shearing or straining motion). A dramatic change came from an unexpected quarter in the paper of Steenbeck et al. [23][3]. These authors recognised that a large-scale well-ordered magnetic field could arise more or less spontaneously as a result of coherent interaction of small-scale random motion and the small-scale random field fluctuations to which this motion gives rise; in other words, order could arise out of chaos. I see this now as the most far-reaching and important discovery in the context of turbulence of this last half-century; but I have to admit to a certain degree of prejudice, because I came into this topic myself quite independently from a different direction three years later. I had been intrigued by the paper of Woltjer [26] who had shown that under certain circumstances, the quantity

$$H_M = \int \mathbf{A} \cdot \mathbf{B} \, dV \tag{5}$$

(where $\mathbf{A}$ is a vector potential for $\mathbf{B}$) is an invariant in a perfectly conducting fluid. The key was to recognise that the reason for this invariance is simply that the lines of force are transported with the fluid, and their topology is therefore conserved. It was a small step to realise that the same type of invariant must exist for the vorticity field in an ideal fluid in which vortex lines are 'frozen' in the flow. This 'inviscid invariant' is the helicity

$$H = \int \mathbf{u} \cdot \boldsymbol{\omega} \, dV, \tag{6}$$

since $\mathbf{u}$ is, in effect, a vector potential for $\boldsymbol{\omega}$. This new invariant [27], [28] is a consequence of Kelvin's [29] theorem on the conservation of circulation; but it had lain undiscovered for close on 100 years since then!

It might be thought that helicity conservation, in conjunction with energy conservation, should have a profound effect on the cascade process through the inertial range (where viscous effects are negligible). Indeed, if we think of this energy cascade in terms of vortex tube stretching, then conservation of helicity does enter the picture, implying as it does, that vortex tube topology is conserved. The effect is less obvious in spectral terms; the fact that the helicity spectrum is not positive-definite means that one cannot argue for a simple cascade of helicity from large to small scales: helicity can be created at large scales at the expense of the creation of equal and opposite helicity at small scales!

The presence of non-zero helicity does however have a profound effect on the generation of a large-scale magnetic field (the dynamo effect) as Steenbeck et al. had predicted. Even at low magnetic Reynolds number Re_m, magnetic field will grow under the effect of homogeneous turbulence provided only that the helicity is non-zero (providing an 'alpha effect'), and the space available for the growth of the field is large enough. This space requirement is easily satisfied in planetary and astrophysical contexts! There is also an enhanced diffusivity due to the turbulence (the 'beta effect'), but the alpha effect dominates on a large enough scale [30].

In his Marseille '61 lecture, 'Turbulence in compressible and electrically conductive media', Kovasznay anticipated this turbulent diffusivity effect. He also emphasised the need for a 'clear, simple and meaningful experiment' in order 'to obtain educated guesses about relative importance of different approaches'. Such an experiment has been brilliantly conceived and realised by the French team working at Paris (ENS and Saclay), Lyon and Cadarache (Monchaux [31]) have demonstrated dynamo action in a 'washing machine' in which liquid sodium is stirred into strongly helical turbulent motion by counter-rotating propellers. Both mean flow and turbulent dynamo effects are present in this experiment, which has for the first time demonstrated that the process responsible for generation of the Earth's magnetic field can indeed be simulated in the laboratory.

7. Two-dimensional turbulence

Two-dimensional turbulence was briefly discussed by Batchelor in the closing sections of his 1953 monograph, where he pointed out an important consequence of (inviscid) enstrophy conservation: the inevitability of an inverse energy cascade (although he did not use this terminology). I believe his interest in this problem was rekindled through discussions with Kraichnan at Marseille '61. At any rate, soon after, he put his research student Roger Bray to work on a numerical simulation of 2D turbulence [32]. More or less simultaneously, Kraichnan developed his ideas on the subject. Kraichnan [33] and Batchelor [34] gave his version of the enstrophy cascade, with reference to Bray's numerical results. The inverse energy cascade can also be regarded as a manifestation of the emergence of 'order out of chaos' as later clarified by McWilliams [35] in his paper 'The emergence of isolated coherent vortices in turbulent flow'. The importance of nearly 2D turbulence in the context of atmospheric dynamics has led to intense activity in this field. Density stratification, strong rotation with associated Coriolis effects, and strong 'applied' magnetic field can all (separately or possibly in conjunction) lead to 'two-dimensionalisation' of a field of turbulence, although the residual effect of weak three-dimensionality can remain of crucial importance. Such turbulence is strongly anisotropic, but nevertheless can remain homogeneous; it is a subject of continuing research interest in which real progress may be anticipated.

8. Rapid distortion theory

Homogeneous turbulence that is subjected to uniform strain, whether rotational or irrotational, remains homogeneous, although increasingly anisotropic as the effect of the strain progresses. If this straining is sufficiently rapid, then its effect on the turbulence may be treated in at least a first approximation by a linearised analysis in which nonlinear interaction of the turbulence with itself is neglected. This is what is known as rapid distortion theory (or RDT). Like so many aspects of turbulence, RDT goes back to a paper of G.I.Taylor [36] who analysed the damping of turbulent fluctuations in flow through a contraction in a wind tunnel.

In his 1956 book, *The Structure of Turbulent Shear Flow* [37], Alan Townsend had used a version of RDT to determine the structure of what he described as the 'big eddies' in free turbulent flows. He assumed that only the irrotational part of the strain was relevant and allowed this strain to progress only until a degree of anisotropy consistent with experimental measurements was attained. In his Marseille '61 lecture on 'Free turbulent flows', Hans Liepmann made passing reference to this approach, but regarded the resulting models of eddy structure as but 'a groping for an eventual representation of a stochastic rotational field'.

I could not understand Townsend's reason for focusing on just the irrotational ingredient of the strain field, and in a later investigation I found that the effect of plane shear on turbulence was indeed quite different [38]: as is now well recognised, plane shearing is the mechanism by which streamwise vortices become prominent in the flow. S.J. Kline discovered the related 'streaks' in a turbulent boundary layer at about the same time [39]. 'Coherent structures' became a growth industry during the 1970s, a period that marked a return towards mechanistic approaches based on vortex interactions in **x**-space, in parallel with the purely statistical approach that found its most natural expression in **k**-space.

In the second edition of his book (1976, §3.12 *et seq.*), Townsend adopted the plane-shear (as opposed to the irrotational-strain) description in his approach to the energy-containing eddies, and concluded: 'Generally, the measurements of the behaviour of turbulent fluid under either irrotational distortion or plane shearing flow show that the larger eddies, possessing most but not all of the energy, are persistent and coherent structures which interact with the mean flow in the "rapid-distortion" manner, and which lose energy and interact with the smaller eddies by processes that can be modelled by an effective viscosity'. The fact that linearised RDT is more successful than we have any right to expect provides one ray of light in the continuing search for a complete theoretical description of turbulence.

Additional effects such as uniform density stratification, Coriolis forces due to rotation, and Lorentz forces due to an applied uniform magnetic field can all act in conjunction with uniform distortion to induce anisotropy in a homogeneous field of turbulence, without inducing inhomogeneity. Much work in this area has been described in the book *Homogeneous Turbulence Dynamics* by Sagaut and Cambon [40], which testifies to the continuing vitality of RDT, particularly in geo- and astro-physical contexts.

9. The active scalar field problem

If weak variations of a scalar contaminant are present in a field of turbulence, then this scalar field is subject to convection and molecular diffusion, but does not react back dynamically upon the convecting velocity field; this constitutes the 'passive scalar field problem'.

There are many circumstances however in which the contaminant *may* react back upon the velocity field. For example, if the 'contaminant' is temperature $\theta(\mathbf{x}, t)$, then buoyancy forces come into play, and the velocity field $\mathbf{u}(\mathbf{x}, t)$ is itself influenced to a greater or lesser extent by the θ-field.

In an extreme situation, we may suppose that $\mathbf{u}$ is actually 'driven' by the θ-field. This was the situation considered by Batchelor et al. [41], in their paper *Homogeneous buoyancy-generated turbulence* (Batchelor's last significant contribution to turbulence). The non-linearity $\mathbf{u} \cdot \nabla\theta$ associated with convection of θ is now just as important as the nonlinear contribution $\mathbf{u} \cdot \nabla\mathbf{u}$ to the Navier–Stokes equations.

If we consider turbulence in a rotating system at small Rossby number (i.e. dominant Coriolis effect, as in the Earth's liquid core), then the non-linear acceleration $\mathbf{u} \cdot \nabla \mathbf{u}$ is negligible, and we have a form of turbulence driven by buoyancy forces in which the sole nonlinearity is represented by the convective term $\mathbf{u} \cdot \nabla \theta$ in the advection–diffusion equation, a state of affairs that may be described as 'geostrophic turbulence'. If a linearised Lorentz force is also present, then we have 'magnetostrophic turbulence' which again (starting from a very weak field) is potentially capable of dynamo action [42]. If the statistics of the θ-distribution are assumed known, then the statistics of the velocity and magnetic fields can be deduced by linearised analysis similar to RDT. The essential problem that remains is to determine the statistical evolution of the θ-field, given the source of temperature variations, a problem whose regularity properties have recently been investigated by Friedlander [43]. This type of turbulence problem is far removed from conventional laboratory turbulence; but it contains the same essential ingredients: non-linear spreading of energy in **k**-space, and dissipation by molecular diffusivity at the smallest scales. Such problems continue to present a challenge for the future.

10. Conclusion

The Marseille 1961 Turbulence Colloquium was remarkably wide-ranging in the issues that it explored, and the discussions at the Colloquium sowed the seeds for a number of important later research developments. Computers were still at a relatively primitive stage of development, but even then the numerical work of Ogura [19] that demonstrated a fundamental flaw of the quasi-normal approximation gave a hint of the central role that computation was to play in the future. The search for a fundamental theory of turbulence was very much alive then; it is still alive today, though perhaps in a slightly battered state, since so many approaches have been tried and discarded along the way. The fact that order can emerge from chaos in MHD turbulence (through the spontaneous growth of a large-scale magnetic field) is, in my view, one of the most striking discoveries of the last 50 years, as is the identification of concentrated vortices as perhaps the key element of turbulent flow. The interaction of these vortices remains a pressing problem, both for theory and numerics, and the underlying problem of regularity of the Navier–Stokes equations (otherwise known as the finite-time singularity problem) remains a central challenge for mathematically oriented fluid dynamicists, in essential partnership with experts in scientific computing. Let us hope the next 50 years will reveal the answer to this problem, one way or another!

Notes

1. Lecture delivered at the Turbulence Colloquium, Marseille 2011, commemorating Mécanique de la Turbulence, Marseille 1961.
2. A new 'cross-independence' closure theory has recently been proposed by Tatsumi [18], 54 years after the same author's first paper on the decay of isotropic turbulence, a remarkable span of sustained investigation!
3. This paper and subsequent papers by the same authors were published in German, and were not very well known in the West until translated into English by Roberts and Stix [24], and subsequent publication of the book of Krause and Rädler [25].

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