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THE CORRELATION BETWEEN GRAVITATIONAL AND GEOMAGNETIC FIELDS CAUSED BY INTERACTION OF THE CORE FLUID MOTION WITH A BUMPY CORE–MANTLE INTERFACE

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The correlation discovered by Hide and Malin between the variable parts of the Earth's gravitational field and magnetic field (suitably displaced in longitude) was tentatively and qualitatively explained by them in terms of the influence on both fields of irregularities (or "surface bumps") at the core–mantle interface. In this paper, a quantitative analysis of this phenomenon is developed, through study of an idealised problem in which conducting fluid occupying the region $z < \eta(x)$ flows over the surface $z = \eta(x)$ in the presence of a magnetic field $(B_0,0,0)$, the whole system rotating with angular velocity $(0,0,\Omega)$. It is assumed that $|\eta'(x)| \ll 1$ so that perturbation methods are applicable. Determination of the magnetic potential in the "mantle" region $z > \eta(x)$ requires solution of the full hydromagnetic problem in the fluid. It is shown that three wave modes are excited, two of which (for values of the parameters of the problem of geophysical interest) have a boundary layer character. Phase interactions between these modes lead to a shift and a distortion of the magnetic pattern relative to the gravitational pattern. The correlation between the gravitational potential and the magnetic potential (shifted by a distance x_0) is determined on the plane $z = d$ ($d \gg |\eta|$) as a function of x_0/d and the curves obtained are qualitatively similar to that based on the observed data; the maximum correlation obtained varies between 0.67 and 1, depending on values of the parameters of the problem, and is about 0.72 for reasonable estimates of these parameters in the geophysical context.

1. Introduction

It has been suggested (Hide 1970; Hide and Malin 1970, 1972) that the observed correlation between the variable parts of the Earth's surface gravitational and geomagnetic fields may be explained on the assumption that bumps (or "inverted mountains") on the core–mantle interface are responsible directly or indirectly for the observed perturbations in both fields. Clearly any density jump at the interface will lead directly to gravitational perturbations vertically above the bumps. Moreover any flow of the electrically conducting fluid in the core region across the Earth's magnetic field and over the bumps will generate electric currents in the core and so magnetic field perturbations on the Earth's surface that are in some way determined by the bump structure or statistics.

An important feature of the observed correlation is that the gravitational field is correlated not with the geomagnetic field at the same location, but with the geomagnetic field shifted in longitude through an angle $\varphi_0(t)$ towards the east, where $\varphi_0(t)$ increases linearly with time at a rate of about 0.27°/year, a manifestation of the well-known "westward drift" of the field. The data for the year 1965 [see Fig. 2B on p. 76, copied from Hide and Malin (1970)] gave $\varphi_0 = 160°$, and the value of the correlation corresponding to this shift was 0.84. Backward extrapolation of the available data suggested that φ_0 was zero about 500–600 years ago, and it was tentatively concluded by Hide and Malin that some kind of "occurrence" at the core–mantle interface might

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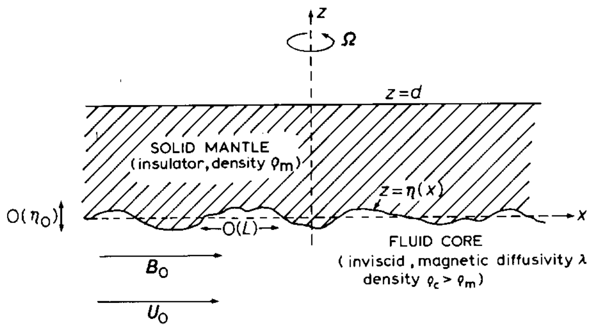

Fig. 1. The geometry of the model.

then have been responsible for the generation of the correlation that is now observed. Such an interpretation is perhaps open to question; and yet the correlation itself is a striking phenomenon, for which no alternative explanation has yet been offered.

The object of this paper is to explore the nature of the magnetohydrodynamic interaction between a bumpy interface and a flow over it influenced both by rotation and by the magnetic field of the Earth; and to derive the correlation at the Earth's surface between the resulting variable parts of the gravity and geomagnetic potential fields. In order to make progress, certain drastic idealisations are required. The model studied is depicted in Fig. 1. Firstly, we replace the Earth's spherical geometry by a Cartesian geometry with coordinate z vertically upwards. The Earth's surface is taken to be $z = d$, and the bumpy core–mantle interface to be $z = \eta(x)$, where, for all x, $|\eta(x)| \ll d$ and $|\eta'(x)| \ll 1$; $\eta(x)$ may be a stationary random function of x in which case we assume that $\langle\eta(x)\rangle = 0$, the angular brackets representing an average over x; alternatively, $\eta(x)$ may be an integrable function (having a Fourier transform) in which case we shall assume that the origin of z is chosen so that:

$$\int_{-\infty}^{\infty} \eta(x)\mathrm{d}x = 0 \tag{1}$$

(i.e. mountains are always compensated by valleys). The solid mantle $\eta(x) < z < d$ is assumed to be insulating and to have uniform density ρ_m; the core liquid $[z < \eta(x)]$ is assumed to be incompressible and inviscid and to have uniform density ρ_c and magnetic diffusivity λ. Both solid and liquid regions are assumed to have the same magnetic permeability μ_0.

It is supposed further that the whole system rotates with angular velocity $\boldsymbol{\Omega} = (0,0,\Omega)$, and that, relative to axes rotating with angular velocity $\boldsymbol{\Omega}$, the velocity field in the fluid satisfies the condition:

$$\boldsymbol{U} \approx (U_0,0,0) \qquad \text{as} \qquad z \to -\infty \tag{2}$$

Finally, we suppose that the whole space is permeated by a uniform magnetic field $\boldsymbol{B}_0 = (B_0,0,0)$, on which of course fluctuations will be superposed due to induction by the velocity field. A similar configuration of $\boldsymbol{B}_0$ and $\boldsymbol{\Omega}$ has been considered in the thermal convection context by Eltayeb (1972, 1975).

The assumption concerning the magnetic field requires some comment in relation to the geophysical situation. The field in the core of the Earth consists of a poloidal part $\boldsymbol{B}_P$ and a toroidal part $\boldsymbol{B}_T$, and it is generally believed that, throughout most of the core, $|\boldsymbol{B}_T| \gg |\boldsymbol{B}_P|$ [see for example Hide and Roberts (1961) and Roberts and Soward (1972)]. The field $\boldsymbol{B}_T$ is however associated with poloidal currents which cannot penetrate an insulating mantle; hence $\boldsymbol{B}_T$ must fall to zero at the core–mantle interface if the mantle is insulating (or to a small value if allowance is made for the weak conductivity of the mantle). A more realistic representation of the ambient magnetic field (than that proposed in the preceding paragraph) should therefore invoke a tangential component which decreases to zero at the interface $z = \eta(x)$, and should also include a weak normal component (representing the poloidal field). Such an attempt at verisimilitude would however lead to severe mathematical difficulties; and it is believed that the simpler model adopted here, although by no means perfect, does not drastically misrepresent the essential physics of the situation. The uniform field $\boldsymbol{B}_0$ may be imagined as an "applied" field extending throughout all space, but its effects will in fact be confined to a neighbourhood of the interface in the core region, as in the real spherical situation.

In the insulating region $z > \eta(x)$, the gravitational field $\boldsymbol{g}(\boldsymbol{x})$ and magnetic field $\boldsymbol{B}(\boldsymbol{x})$ can be represented in terms of scalar potentials $\Phi(x,z)$ and $\Psi(x,z)$:

$$\boldsymbol{g} = -\nabla\Phi, \qquad (\mu_0\rho_m)^{-1/2}(\boldsymbol{B} - \boldsymbol{B}_0) = -\nabla\Psi \tag{3}$$

We shall seek to determine the steady form of Φ and Ψ, and hence to evaluate the cross-correlation:

$$R(x_0) = \frac{[\Phi(x,d), \Psi(x - x_0,d)]}{[(\Phi,\Phi)(\Psi,\Psi)]^{1/2}} \tag{4}$$

where $(\cdot,\cdot)$ is an appropriately defined scalar product. It will be clear that, in restricting attention to the steady situation, we can only derive a *steady* correlation function $R(x_0)$, and that the westward drift of the Ψ-field relative to the Φ-field cannot be obtained from this primitive model. Nevertheless, it seems reasonable to concentrate at this stage on gaining an understanding of the steady situation, and on isolating those parameters of the problem on which the function $R(x_0)$ (and in particular its maximum value) depends.

2. General considerations concerning the correlation function $R(x_0)$

Let $\phi(x) = \Phi(x,d)$ and $\psi(x) = \Psi(x,d)$ and suppose that ϕ and ψ have Fourier representations:

$$\phi(x) = \int_{-\infty}^{\infty} \hat{\phi}(k) \exp(ikx)dk, \qquad \psi(x) = \int_{-\infty}^{\infty} \hat{\psi}(k) \exp(ikx)dk \tag{5}$$

where $\hat{\phi}$ and $\hat{\psi}$ may be generalised functions (Lighthill, 1959) (to include the possibilities that the spectra may be discrete, or that ϕ and ψ may be stationary random functions of x). These satisfy the reality conditions:

$$\hat{\phi}(-k) = \hat{\phi}^\star(k), \qquad \hat{\psi}(-k) = \hat{\psi}^\star(k) \tag{6}$$

where the star indicates a complex conjugate, and we may therefore rewrite eq. 5 in the form:

$$\phi(x) = 2\,\mathrm{Re} \int_0^{\infty} \hat{\phi}(k) \exp(ikx)dk, \qquad \psi(x) = 2\,\mathrm{Re} \int_0^{\infty} \hat{\psi}(k) \exp(ikx)dk \tag{7}$$

It is convenient to be able to regard the wavenumber k as always positive, and we shall therefore use the form (7) throughout.

If ϕ and ψ have well-behaved Fourier transforms, then the natural definition of the scalar product is:

$$[\phi(x), \psi(x - x_0)] = \int_{-\infty}^{\infty} \phi(x)\psi(x - x_0)dx = 4\pi\mathrm{Re}\int_0^{\infty} \hat{\phi}^\star(k)\hat{\psi}(k) \exp(-ikx_0)dk \tag{8}$$

and the correlation function (4) becomes:

$$R(x_0) = \frac{\mathrm{Re}\int_0^\infty \hat{\phi}^\star(k)\hat{\psi}(k) \exp(-ikx_0)dk}{[\int_0^\infty |\hat{\phi}(k)|^2 dk \int_0^\infty |\hat{\psi}(k)|^2 dk]^{1/2}} \tag{9}$$

This expression can also be adopted as the definition of the correlation function for the case of stationary random fields, or fields with discrete spectra.

If ϕ and ψ are sinusoidal with the same wavelength $2\pi/k_1$, so that:

$$\hat{\phi}(k) = \phi_1\delta(k - k_1), \qquad \hat{\psi}(k) = \psi_1\delta(k - k_1) \tag{10}$$

then eq. 9 becomes:

$$R(x_0) = \mathrm{Re}[\phi_1^{\star}\psi_1 \exp(-\mathrm{i}k_1x_0)]/|\phi_1||\psi_1| = \cos(k_1x_0 - \arg\phi_1/\psi_1) \tag{11}$$

and it follows that, as would be expected from the most elementary considerations, $R(x_0) = 1$ when $k_1x_0 = \arg(\phi_1/\psi_1)$; i.e. there is perfect correlation between the two sine curves if one is shifted relative to the other so that their maxima coincide.

This simple property obviously does not extend to the case when $\phi(x)$ and $\psi(x)$ contain two or more spectral ingredients, since clearly the "shift" $x(k_1)$ that would maximise the correlation if the wave number k_1 alone contributed need not be the same as the shift $x(k_2)$ at another wavenumber k_2. To be specific, let:

$$\hat{\phi}(k) = \phi_1\delta(k - k_1) + \phi_2\delta(k - k_2), \qquad \hat{\psi}(k) = \psi_1\delta(k - k_1) + \psi_2\delta(k - k_2) \tag{12}$$

then it is readily verified that:

$$R(x_0) = \frac{|\phi_1||\psi_1|\cos(k_1x_0 - \arg\phi_1/\psi_1) + |\phi_2||\psi_2|\cos(k_2x_0 - \arg\phi_2/\psi_2)}{(|\phi_1|^2 + |\phi_2|^2)^{1/2}(|\psi_1|^2 + |\psi_2|^2)^{1/2}}$$

$$\leqslant \frac{|\phi_1||\psi_1| + |\phi_2||\psi_2|}{(|\phi_1|^2 + |\phi_2|^2)^{1/2}(|\psi_1|^2 + |\psi_2|^2)^{1/2}} = \cos\xi \tag{13}$$

where ξ $(0 \leqslant \xi \leqslant \pi/2)$ is the angle between the two-dimensional vectors $(|\phi_1|, |\phi_2|)$ and $(|\psi_1|, |\psi_2|)$. Hence, $R(x_0) < 1$ for all x_0 unless $\arg(\phi_1/\psi_1) = \arg(\phi_2/\psi_2)$ and $|\phi_1|/|\phi_2| = |\psi_1|/|\psi_2|$, i.e. unless $\hat{\phi}(k) = C\hat{\psi}(k)$ where C is some complex constant.

It is clear from these considerations that the correlation between $\phi(x)$ and $\psi(x - x_0)$ will be less than perfect (for *every* x_0) if and only if the Fourier components of the magnetic field corresponding to Fourier components of the surface function $\eta(x)$ have a phase shift that depends non-trivially on the wave number k. A phase-shift does indeed exist due to asymmetric convection by the mean velocity U_0; the actual extent of the shift, and the consequences for the correlation function, can be determined only by means of the detailed analysis of the following sections.

3. Potential fields in the mantle

It is a trivial matter to obtain $\Phi(x,z)$ from the equation $\nabla^2\Phi = 0$ and the appropriate linearised boundary conditions across the density jump at $z = \eta(x)$:

$$[\Phi] = 0, \qquad [\partial\Phi/\partial z] = -4\pi G_1(\rho_c - \rho_m)\eta(x) \tag{14}$$

where the square brackets denote the discontinuity from $z = 0^-$ to $z = 0^+$, and G_1 is the gravitational constant. If $\hat{\eta}(k)$ is the Fourier transform of $\eta(x)$, then the solution to this potential problem is:

$$\Phi(x,z) = 2G\,\mathrm{Re}\int_0^\infty k^{-1}\hat{\eta}(k)\exp(-k|z|)\exp(\mathrm{i}kx)\,\mathrm{d}k \tag{15}$$

where $G = 2\pi G_1(\rho_c - \rho_m)$. In particular (cf. eq. 7):

$$\hat{\phi}(k) = Gk^{-1}\hat{\eta}(k)\,\mathrm{e}^{-kd} \tag{16}$$

Similarly, the field $\Psi(x,z)$ for $z > \eta(x)$ is a potential field, and has the structure:

$$\Psi(x,z) = 2\,\mathrm{Re} \int_0^\infty \hat{\psi}_0(k) \exp(-kz) \exp(ikx)\, dk \tag{17}$$

and correspondingly:

$$\hat{\psi}(k) = \hat{\psi}_0(k) \exp(-kd) \tag{18}$$

The function $\hat{\psi}_0(k)$ can be determined only through matching the mantle field with the core field, and this necessarily involves the hydromagnetics of the core region.

4. Hydromagnetics of the core region

In the core region, it is convenient to define the Alfvén velocities:

$$\boldsymbol{H}_0 = (\mu_0 \rho_c)^{-1/2} \boldsymbol{B}_0, \quad \boldsymbol{H} = (\mu_0 \rho_c)^{-1/2} \boldsymbol{B} \tag{19}$$

The equations for $\boldsymbol{U}$ and $\boldsymbol{H}$ are then:

$$\partial \boldsymbol{U}/\partial t + \boldsymbol{U} \cdot \nabla \boldsymbol{U} + 2\boldsymbol{\Omega} \wedge \boldsymbol{U} = -\nabla P + \boldsymbol{H} \cdot \nabla \boldsymbol{H} \tag{20}$$

$$\partial \boldsymbol{H}/\partial t + \boldsymbol{U} \cdot \nabla \boldsymbol{H} = \boldsymbol{H} \cdot \nabla \boldsymbol{U} + \lambda \nabla^2 \boldsymbol{H} \tag{21}$$

and:

$$\nabla \cdot \boldsymbol{U} = \nabla \cdot \boldsymbol{H} = 0 \tag{22}$$

Here P is a reduced pressure including magnetic and centrifugal ingredients. We shall suppose that the Rossby number $R_0 = U_0/\Omega L$ (where L is the horizontal scale of the bumps on the surface) is small (see in Table I the estimates in the geophysical context), and we shall neglect the inertia term $\boldsymbol{U} \cdot \nabla \boldsymbol{U}$ in eq. 20 as compared with the Coriolis term $2\boldsymbol{\Omega} \wedge \boldsymbol{U}$. This well-known approximation (the "magnetostrophic" approximation) is known to have

TABLE I

Estimates of the various dimensional quantities in the geophysical context and resulting values of the various dimensionless combinations that appear in the course of the analysis

Estimates *	Resulting values		
$d = 2{,}900$ km	$\beta = L/d$		$\lesssim 0.34$
$L \lesssim 1{,}000$ km	$A = U_0/H_0$		$\approx 2.8 \cdot 10^{-4}$
$\eta_0 \lesssim 5$ km	$\epsilon = A^{1/2}$		$\approx 1.7 \cdot 10^{-2}$
$\Omega = 7.3 \cdot 10^{-5}$ s^{-1}	$Q = \Omega\lambda/H_0^2$		$\approx 1.7 \cdot 10^{-3}$
$U_0 \approx 10^{-4}$ m/s	$q = Q/\epsilon^{4/3}$	$(\rho = \frac{4}{3})$	≈ 0.40
$B_0 \approx 0.04$ Wb/m^2	$S = H_0/\Omega d$		$\approx 1.7 \cdot 10^{-3}$
$H_0 \approx 0.36$ m/s	$s = S/\epsilon^{4/3}$	$(\mu = \frac{4}{3})$	≈ 0.40
$\rho_m = 5.6 \cdot 10^3$ kg/m^3	$K_0 = AS^{-1}Q^{-1/2}$	$= \Omega^{1/2}U_0 d/\lambda^{1/2}H_0$	≈ 4.0
$\rho_c = 9.6 \cdot 10^3$ kg/m^3	$R_m = AQ^{-1}S^{-1}$	$= U_0 d/\lambda$	≈ 96
$\lambda \approx 3$ m^2/s	$\beta R_0 = AS$	$= U_0/\Omega d$	$\approx 4.7 \cdot 10^{-7}$

The boundary layer thickness $Q^{1/2}L$ is, on the basis of these figures, of order 41 km, safely larger than the bump amplitude η_0.
* After Hide (1970) and Roberts and Soward (1972).

the effect of filtering out high-frequency $[0(\Omega)]$ inertial-type waves from the system (20)–(21). The approximation is not essential in the present context (Dillon, 1975); but it greatly simplifies the analysis, and it may be verified retrospectively that under the asymptotic procedures to be adopted in Section 6, the effects of inertia are totally negligible.

If we now put $\boldsymbol{U} = \boldsymbol{U}_0 + \boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h}(\boldsymbol{x})$ and linearise (20)–(21) (with inertia terms in eq. 20 dropped) we obtain the equations:

$$2\boldsymbol{\Omega} \wedge \boldsymbol{u} = -\nabla p + \boldsymbol{H}_0 \cdot \nabla \boldsymbol{h} \tag{23}$$

$$\boldsymbol{U}_0 \cdot \nabla \boldsymbol{h} = \boldsymbol{H}_0 \cdot \nabla \boldsymbol{u} + \lambda \nabla^2 \boldsymbol{h} \tag{24}$$

and:

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{h} = 0 \tag{25}$$

where p is the perturbation in P. These equations admit solutions of the form:

$$(\boldsymbol{u}, \boldsymbol{h}, p) = 2\mathrm{Re} \int_0^\infty (\hat{\boldsymbol{u}}, \hat{\boldsymbol{h}}, \hat{p}) \exp(\mathrm{i}\boldsymbol{m} \cdot \boldsymbol{x})\,\mathrm{d}k \tag{26}$$

where $\boldsymbol{m} = k(1,0,\gamma)$ and possible values of γ are to be determined; clearly these must satisfy $\mathrm{Im}\gamma < 0$ since the perturbations must vanish as $z \to -\infty$. Substitution in eqs. 23–25 gives:

$$2\boldsymbol{\Omega} \wedge \hat{\boldsymbol{u}} = -\mathrm{i}\boldsymbol{m}\hat{p} + \mathrm{i}H_0 k \hat{\boldsymbol{h}} \tag{27}$$

$$\mathrm{i}U_0 k \hat{\boldsymbol{h}} = \mathrm{i}H_0 k \hat{\boldsymbol{u}} - \lambda k^2 (1 + \gamma^2) \hat{\boldsymbol{h}} \tag{28}$$

and:

$$\boldsymbol{m} \cdot \hat{\boldsymbol{u}} = \boldsymbol{m} \cdot \hat{\boldsymbol{h}} = 0 \tag{29}$$

In terms of the dimensionless numbers:

$$A = U_0/H_0, \qquad Q = 2\Omega\lambda/H_0^2 \tag{30}$$

and the dimensionless parameters:

$$\kappa = H_0 k/\Omega, \qquad \sigma = 2[Q(1 + \gamma^2) + 2\mathrm{i}A\kappa^{-1}]^{-1} \tag{31}$$

eq. 28 becomes:

$$\hat{\boldsymbol{h}} = (\mathrm{i}\sigma/\kappa)\hat{\boldsymbol{u}} \tag{32}$$

and eq. 27 becomes:

$$\Omega\sigma\hat{\boldsymbol{u}} + 2\boldsymbol{\Omega} \wedge \hat{\boldsymbol{u}} = -\mathrm{i}\boldsymbol{m}\hat{p} \tag{33}$$

Taking the cross-product (twice) with $\boldsymbol{m}$ gives:

$$\sigma \boldsymbol{m} \wedge \hat{\boldsymbol{u}} - 2k\gamma\hat{\boldsymbol{u}} = 0, \qquad -\sigma m^2 \hat{\boldsymbol{u}} - 2k\gamma \boldsymbol{m} \wedge \hat{\boldsymbol{u}} = 0 \tag{34}$$

and elimination of $\hat{\boldsymbol{u}}$ and $\boldsymbol{m} \wedge \hat{\boldsymbol{u}}$ now gives:

$$\sigma^2 = -4\gamma^2(1 + \gamma^2)^{-1} \tag{35}$$

From eqs. 31 and 35, we obtain a cubic equation for γ^2:

$$(1 + \gamma^2) + \gamma^2 [Q(1 + \gamma^2) + 2\mathrm{i}A\kappa^{-1}]^2 = 0 \tag{36}$$

Let the roots of this cubic be γ_n^2 ($n = 1, 2, 3$), with $\mathrm{Im}\gamma_n < 0$, and let the corresponding values of σ (given by eq. 31) be σ_n. Evidently γ_n and σ_n are functions of κ, Q and A.

Eq. 33, together with $\boldsymbol{m} \cdot \hat{\boldsymbol{u}} = 0$ now determine the ratios and relative phases of the velocity components in each mode; for $n = 1$, 2 and 3, we have:

$$\hat{\boldsymbol{u}}_n = a_n(k)(1, -2\sigma_n^{-1}, -\gamma_n^{-1}), \qquad \hat{\boldsymbol{h}}_n = (\mathrm{i}\sigma_n/\kappa)\hat{\boldsymbol{u}}_n \tag{37}$$

where $a_n(k)$ are amplitudes as yet undetermined. The general solution of eqs. 23–25 vanishing at $z = -\infty$ then takes the form:

$$\boldsymbol{u} = 2\,\mathrm{Re} \sum_{n=1}^{3} \int_0^\infty a_n(k)(1, -2\sigma_n^{-1}, -\gamma_n^{-1}) \exp[\mathrm{i}k(x + \gamma_n z)]\,\mathrm{d}k \tag{38}$$

$$\boldsymbol{h} = 2\,\mathrm{Re} \sum_{n=1}^{3} \int_0^\infty \frac{\mathrm{i}}{\kappa}\, a_n(k)(\sigma_n, -2, -\sigma_n\gamma_n^{-1}) \exp[\mathrm{i}k(x + \gamma_n z)]\,\mathrm{d}k \tag{39}$$

5. Application of the boundary conditions at the core–mantle interface

The amplitudes $a_n(k)$ and $\hat{\psi}_0(k)$ must now be determined in terms of $\hat{\eta}(k)$ through the boundary conditions at $z = \eta(x)$. These conditions are that the normal component of $\boldsymbol{U}$ must vanish and all three components of $\boldsymbol{B}$ must be continuous. The linearised form of these conditions is:

$$u_z = U_0 \partial\eta/\partial x, \qquad \boldsymbol{h} = -(\rho_\mathrm{m}/\rho_\mathrm{c})^{1/2}\nabla\Psi \qquad \text{at} \qquad z = 0 \tag{40}$$

With $\boldsymbol{u}, \boldsymbol{h}$ and Ψ given by eqs. 38, 39 and 17 respectively, these conditions may be expressed in matrix notation:

$$\begin{pmatrix} \gamma_1^{-1} & \gamma_2^{-1} & \gamma_3^{-1} & 0 \\ \sigma_1 & \sigma_2 & \sigma_3 & 1 \\ 1 & 1 & 1 & 0 \\ \sigma_1/\gamma_1 & \sigma_2/\gamma_2 & \sigma_3/\gamma_3 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ r\kappa^2\hat{\psi}_0 \end{pmatrix} = \begin{pmatrix} -\mathrm{i}kU_0\hat{\eta} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{41}$$

where $r = (\rho_\mathrm{m}/\rho_\mathrm{c})^{1/2}\Omega H_0$. The determinant Δ of the 4×4 matrix may be evaluated in the form:

$$\Delta = \gamma_1^{-1}\Delta_1 + \gamma_2^{-1}\Delta_2 + \gamma_3^{-1}\Delta_3 \tag{42}$$

where:

$$\Delta_1 = \frac{\sigma_3}{\gamma_3} - \frac{\sigma_2}{\gamma_2} + \mathrm{i}(\sigma_3 - \sigma_2), \qquad \Delta_2 = \frac{\sigma_1}{\gamma_1} - \frac{\sigma_3}{\gamma_3} + \mathrm{i}(\sigma_1 - \sigma_3), \qquad \Delta_3 = \frac{\sigma_2}{\gamma_2} - \frac{\sigma_1}{\gamma_1} + \mathrm{i}(\sigma_2 - \sigma_1) \tag{43}$$

and inversion of (41) gives the amplitudes a_n in the form

$$a_n = -(\Delta_n/\Delta)\mathrm{i}kU_0\hat{\eta}, \qquad (n = 1, 2, 3) \tag{44}$$

Finally, $\hat{\psi}_0$ is given by:

$$\hat{\psi}_0 = -\Omega r^{-1}\tau(\kappa; Q, A)\hat{\eta}(k) \tag{45}$$

where $\tau = \mathrm{i}A\Delta_0/\kappa\Delta$, and:

$$\Delta_0 = \frac{\sigma_1}{\gamma_1}(\sigma_3 - \sigma_2) + \frac{\sigma_2}{\gamma_2}(\sigma_1 - \sigma_3) + \frac{\sigma_3}{\gamma_3}(\sigma_2 - \sigma_1) \tag{46}$$

6. Asymptotic evaluation of the correlation $R(x_0)$

On dimensional grounds, it is appropriate to express the Fourier transform of $\eta(x)$ in the form:

$$\hat{\eta}(k) = \eta_0 L \zeta(kL) \tag{47}$$

where, by virtue of the condition (1), $\zeta(0) = 0$; here, η_0 is a measure of the amplitude of the surface irregularities. The bump spectrum is given by:

$$|\hat{\eta}(k)|^2 = \eta_0^2 L^2 |\zeta(kL)|^2 = \eta_0^2 L^2 F(kL) \tag{48}$$

say, and in general $F(\xi) = 0(\xi^2)$ as $\xi \to 0$. If the bumps are symmetric about $x = 0$, then $\zeta(\xi) = 0(\xi^2)$ and $F(\xi) = 0(\xi^4)$ as $\xi \to 0$. A simple "prototype" symmetric bump is that given by:

$$\eta(x) = 4\pi^{1/2}\eta_0(1 - 2x^2/L^2)\exp(-x^2/L^2) = 2\pi^{1/2}\eta_0 L^2 d^2 \exp(-x^2/L^2)/\mathrm{d}x^2 \tag{49}$$

for which:

$$\zeta(\xi) = \xi^2 \exp(-\tfrac{1}{4}\xi^2), \qquad F(\xi) = \xi^4 \exp(-\tfrac{1}{2}\xi^2) \tag{50}$$

Substitution of eqs. 16, 18, 45 and 48 in eq. 9 now gives $R(x_0)$ as a function of the dimensionless variable $X = x_0/d$ and the parameters A, Q and

$$\beta = L/d, \qquad S = H_0/\Omega d \tag{51}$$

in the form:

$$R(X; \beta, Q, A, S) = I(X; \beta, Q, A, S)/[I_G(\beta) I_M(\beta, Q, A, S)]^{1/2} \tag{52}$$

where:

$$I = \mathrm{Re} \int_0^\infty K^{-1}\, \tau(SK; Q, A) F(\beta K) \exp(-2K + iKX)\mathrm{d}K \tag{53}$$

$$I_G = \int_0^\infty K^{-2} F(\beta K) \exp(-2K)\mathrm{d}K \tag{54}$$

and:

$$I_M = \int_0^\infty |\tau(SK; Q, A)|^2 F(\beta K) \exp(-2K)\mathrm{d}K \tag{55}$$

The gravitational integral I_G may be simply evaluated for specific choices of $F(\xi)$, e.g. that given by eq. 50. Similarly, I and I_M could in principle be evaluated numerically for specific values of X, β, Q, A and S. However, in the geophysical context, Q, A and S are all small (see Table I), and asymptotic evaluation of I and I_M, taking advantage of this fact, is possible.

Guided by the estimates of Table I, we let:

$$A = \epsilon^2, \qquad Q = q\epsilon^\rho, \qquad S = s\epsilon^\mu \tag{56}$$

where $0 < \epsilon \ll 1$, $0 < \rho < 2$ and $0 < \mu < 2$, and we consider the limiting process $\epsilon \to 0$ with $q = 0(1)$, $s = 0(1)$. It is useful to retain some flexibility at this stage in the choice of ρ and μ; the estimates of Table I would certainly suggest that it would be appropriate to assume that $\mu = \rho$; it will emerge below that the choice $\mu = \rho = \frac{4}{3}$ then has particular significance.

In evaluating eqs. 53 and 55, the presence of the exponential factor $\exp(-2K)$ means that we need to evaluate

$\tau(SK; Q, A)$ only for $K = 0(1)$. To do this, we must first return to the cubic (36) which with the substitutions (56) and $\kappa = SK$ becomes:

$$(1 + \gamma^2) + 4\gamma^2 [q\epsilon^{\rho}(1 + \gamma^2) + i\epsilon^{2-\mu}/sK]^2 = 0 \tag{57}$$

The leading terms of the asymptotic expansions of the three roots γ_n^2 for small ϵ are:

$$\gamma_1^2 \sim -1, \qquad \gamma_2^2 \sim \tfrac{1}{2} iq^{-1}\epsilon^{-\rho}, \qquad \gamma_3^2 \sim -\tfrac{1}{2} iq^{-1}\epsilon^{-\rho} \tag{58}$$

and so:

$$\gamma_1 \sim -i, \qquad \gamma_2 \sim -\tfrac{1}{2}(1 + i)q^{-1/2}\epsilon^{-\rho/2}, \qquad \gamma_3 \sim \tfrac{1}{2}(1 - i)q^{-1/2}\epsilon^{-\rho/2} \tag{59}$$

Correspondingly, from eq. 31:

$$\sigma_1 \sim -isK\epsilon^{\mu-2}, \qquad \sigma_2 \sim -2i, \qquad \sigma_3 \sim 2i \tag{60}$$

and, from eqs. 42, 43 and 46, we then have:

$$\Delta_1 \sim -4, \qquad \Delta_2 \sim 2sK\epsilon^{\mu-2}, \qquad \Delta_3 \sim -2sK\epsilon^{\mu-2} \tag{61}$$

and:

$$\Delta_0 \sim 4isK\epsilon^{\mu-2}, \qquad \Delta \sim -4Kq^{1/2}\epsilon^{\rho/2+\mu-2} - 4i \tag{62}$$

Hence, in particular, from eq. 45:

$$\tau(SK; Q, A) \sim (i + K/K_0)^{-1} \tag{63}$$

where:

$$K_0 = s^{-1}q^{-1/2}\epsilon^{2-\mu-\rho/2} = AS^{-1}Q^{-1/2} = \Omega^{1/2}U_0 d/\lambda^{1/2}H_0 \tag{64}$$

It is now evident that the situation $\mu + \frac{1}{2}\rho = 2$ is critical in the sense that only then is $K_0 = 0(1)$. If $\mu + \frac{1}{2}\rho < 2$, then $K_0 \to 0$ as $\epsilon \to 0$ and $\tau \sim K_0/K$, while if $\mu + \frac{1}{2}\rho > 2$, then $K_0 \to \infty$ as $\epsilon \to 0$ and $\tau \sim -i$. The estimates of Table I give $K_0 = 4$ and clearly therefore the condition $\mu + \frac{1}{2}\rho = 2$ is appropriate. If moreover $\mu = \rho$ (and any other relation would be incompatible with the geophysical estimates) then the choice:

$$\mu = \rho = \frac{4}{3} \tag{65}$$

would appear to provide the best description in the geophysical context. Note that this choice leads to the following estimates of magnetic Reynolds number R_m and Rossby number R_0:

$$R_m = U_0 d/\lambda = AQ^{-1}S^{-1} = q^{-1}s^{-1}\epsilon^{-2/3} \tag{66}$$

$$R_0 = U_0/\Omega L = \beta^{-1}AS = s\beta^{-1}\epsilon^{10/3} \tag{67}$$

It is this small estimate of Rossby number which ensures that inertia effects are negligible, as anticipated in Section 4.

Substitution of eq. 63 in eqs. 53 and 55 now gives:

$$I(X) \sim \int_0^\infty K^{-1}[1 + (K/K_0)^2]^{-1}[(K/K_0)\cos KX + \sin KX] \exp(-2K)F(\beta K)dK \tag{68}$$

and:

$$I_M \sim \int_0^\infty (1 + (K/K_0)^2)^{-1} \exp(-2K)F(\beta K)dK = K_0 I(0) \tag{69}$$

Note that the dependence of these integrals on Q, A and S now appears only via the particular combination $K_0 = AS^{-1}Q^{-1/2}$.

If β is small (and again this is suggested by the estimates of Table I), then it is reasonable to evaluate the expressions 54, 68 and 69 by replacing $F(\beta K)$ by the leading term of its expansion for small βK. If $F(\beta K) = 0[(\beta K)^{2n}]$ as $\beta K \to 0$, then eq. 52 reduces to:

$$R_n(X,K_0) = \frac{2^{1/2}\int_0^\infty K^{2n-1}\,[1+(K/K_0)^2]^{-1}\,[(K/K_0)\cos Kx + \sin KX]\,\exp(-2K)\,dK}{[\int_0^\infty K^{2n}\,\{1+(K/K_0)^2\}^{-1}\,\exp(-2K)\,dK]^{1/2}} \tag{70}$$

with limiting forms, corresponding to the possibilities $n = 1, 2$:

$$R_1(X,0) = \frac{1}{1+\frac{1}{4}X^2}, \qquad R_1(X,\infty) = \frac{X}{2^{1/2}(1+\frac{1}{4}X^2)^2} \tag{71}$$

and:

$$R_2(X,0) = \frac{1-\frac{3}{4}X^2}{(1+\frac{1}{4}X^2)^3}, \qquad R_2(X,\infty) = -\frac{3^{1/2}X(1-\frac{1}{4}X^2)}{(1+\frac{1}{4}X^2)^4} \tag{72}$$

The functions $R_2(X,K_0)$ are sketched in Fig. 2 for four values of K_0 (0, 0.425, 4.25 and ∞). Note that as K_0 increases, the maximum value of $R_2(X,K_0)$ decreases from 1 to 0.67, and the value X_m of X at which this maximum occurs decreases from 0 to -0.65. In the geophysical context this would correspond to a longitude shift of approximately 20°. Note that, since $X_m < 0$, the apparent displacement of the magnetic pattern relative to the gravitational pattern is in the direction of the mean flow U_0 (so that a negative displacement of Ψ is needed to achieve maximum correlation).

Also included in Fig. 2 is the correlation:

$$R(\varphi_0) = \langle \Phi(\theta,\varphi)\Psi(\theta,\varphi-\varphi_0)\rangle / \langle \Phi^2\rangle^{1/2}\langle \Psi^2\rangle^{1/2} \tag{73}$$

where the angular brackets represent an average over latitude θ and longitude φ, as obtained by Hide and Malin (1970) from observational data for the year 1965. The qualitative similarity between the theoretical curves and the observational curve, and in particular the order of magnitude of the maximum correlation (0.72 for the case $K_0 = 4.25$ which most nearly represents the geophysical situation) is striking. However, as mentioned in the introduction, it must be emphasised that the westward drift of the geomagnetic field cannot emerge from a steady model; this could only appear if transient or other time-dependent effects were built into the model at the outset.

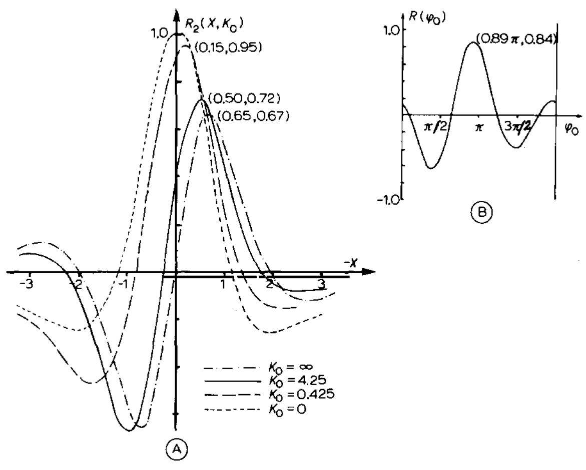

Fig. 2. A. The curves $R_2(X, K_0)$ for $K_0 = 0, 0.425, 4.25, \infty$, as given by eq. 70; the maximum correlations and the corresponding values of $-X$ are as indicated.
B. The correlation function $R(\varphi_0)$ (eq. 73) as determined by Hide and Malin (1970) from observational data for the year 1965.

7. The physical character of the three wave modes

The values of γ_2 and γ_3 given by eq. 59 indicate that the corresponding velocity and magnetic perturbations have a boundary layer structure, the thickness of the layer being $0(\delta)$ where:

$$\delta = q^{1/2}\epsilon^{\rho/2}k^{-1} = Q^{1/2}k^{-1} \tag{74}$$

Since, from eq. 63, $\Delta_2 = -\Delta_3$ in the limit considered, the energy densities of these modes are equal; in fact:

$$|\hat{u}_2|^2 = |\hat{u}_3|^2 = \frac{8S^2U_0^2d^2k^4|\hat{\eta}|^2}{A^2|\Delta|^2}\exp(kz/Q^{1/2}) \tag{75}$$

and:

$$|\hat{h}_2|^2 = |\hat{h}_3^2| = 4S^{-2}K^{-2}|\hat{u}_{2,3}|^2 \tag{76}$$

Note that, for $K = 0(1)$, the magnetic energy density is a factor $0(S^{-2})$ greater than the kinetic energy density. Both modes are highly helical; in fact, from eq. 34, the helicity density (Moffatt, 1970) in either mode (n = 2 or 3) is given by:

$$\mathcal{H}_n = \mathrm{Re}(\hat{u}_n^\star \cdot \mathrm{i}\boldsymbol{k}\wedge\hat{u}_n) = 2k|\hat{u}_n|^2\,\mathrm{Re}(\mathrm{i}\gamma_n/\sigma_n) = \tfrac{1}{2}k|\hat{u}_n|^2q^{-1/2}\epsilon^{-\rho/2} = \tfrac{1}{2}k|\hat{u}_n|^2Q^{-1/2} \tag{77}$$

This positive helicity in both modes is associated with the flux of energy away from the boundary in the direction $-\boldsymbol{\Omega}$. (Moffatt, 1970, §2; Soward, 1975). A region of concentrated helicity near the boundary is of potential importance in the context of dynamo regeneration of the large-scale magnetic field.

The mode corresponding to suffix $n = 1$, with $\gamma_1 = -\mathrm{i}$ has a totally different character. The velocity and magnetic field in this mode are given by:

$$\hat{u}_1(k) = \mathrm{Re}\frac{4U_0\hat{\eta}(k)}{\mathrm{d}\Delta}\left(\mathrm{i}K, \frac{\epsilon^2}{S}, K\right)\exp[k(\mathrm{i}x+z)], \qquad \hat{h}_1(k) = \mathrm{Re}\frac{4U_0\hat{\eta}(k)}{\epsilon^2 d\Delta}\left(\mathrm{i}K, \frac{\epsilon^2}{S}, K\right)\exp[k(\mathrm{i}x+z)] \tag{78}$$

and are irrotational to leading order (i.e. neglecting the small y-components). The mode is therefore almost unaffected by ohmic dissipation. The kinetic energy density $|\hat{u}_1|^2$ is smaller by a factor $0(\epsilon^4S^{-2})$ than that in modes 2 and 3; it is however distributed throughout a layer of thickness $0(k^{-1})$ in the fluid. The magnetic energy density $|\hat{h}_1|^2$ is of the same order of magnitude as $|\hat{h}_2^2|^2$ and $|\hat{h}_3|^2$. If E_n and M_n (n = 1, 2, 3) are the total kinetic and magnetic energies in the three modes (integrated over z) then, in the case $\rho = \mu = \frac{4}{3}$:

$$M_1 = 0(\epsilon^{-2/3})M_{2,3} = 0(\epsilon^{-10/3})E_{2,3} = 0(\epsilon^{-4})E_1 \tag{79}$$

It may readily be verified that the mode $n = 1$ is non-helical in the limit considered.

The term $\boldsymbol{U}_0 \cdot \nabla\boldsymbol{h}$ in eq. 24 introduces a distinction between the positive and negative x-directions, and is responsible for the final asymmetry in the correlation functions sketched in Fig. 2. In the $n = 1$ mode, in which $\nabla^2\boldsymbol{h} = 0$ at lowest order in ϵ (since $\gamma_1^2 = -1$), this convection term is of dominant importance. In the other two modes, the relative importance of $\boldsymbol{U}_0 \cdot \nabla\boldsymbol{h}$ and $\lambda\nabla^2\boldsymbol{h}$ in eq. 24 is given by:

$$\frac{|\boldsymbol{U}_0 \cdot \nabla\boldsymbol{h}|}{|\lambda\nabla^2\boldsymbol{h}|} \sim \frac{U_0k\delta^2}{\lambda} = \frac{\beta A}{S} = \frac{\beta}{s}\epsilon^{2-\mu} \tag{80}$$

so that, since $\mu < 2$, diffusion dominates in these modes (due to the small vertical length scale) and convection is unimportant. If U_0 is gradually increased from small values (so that K_0 increases in proportion) the gradual shift in the correlation pattern from symmetric to antisymmetric form (Fig. 2A) is therefore entirely due to the influence of convection in the $n = 1$ mode; the dominant magnetic energy in this mode (eq. 79) ensures that structural changes in the mode are transmitted via the boundary conditions at the interface into the mantle region. It seems in fact as if the $n = 1$ mode plays the dominant role in determining the correlation pattern; the $n = 2$ and 3 modes play a passive role, driven by the $n = 1$ mode via its interaction with the interface.

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