A model for magnetic reconnection

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If you're not too dreadfully weary,
You may ask: what's the point of this theory?
Just please recollect
That **B***-lines reconnect;*
I hope that will answer your query!

1. Introduction

Magnetic reconnection is a diffusive process whereby the topology of a magnetic field imbedded in a highly conducting fluid may change with time. The process is frequently represented by the simple diagram shown in figure 1 in which the oppositely directed lines of force AB, CD reconnect to the configuration AC, BD. A huge literature emanating from the early papers of Sweet (1958), Parker (1957) and Petschek (1964) has evolved; for a recent thorough exposition including discussion of the important application of the theory in solar, magnetospheric and other contexts, see Priest & Forbes (2000).

Despite the prolonged research on this topic, there seems to be as yet no simple model which demonstrates unambiguously the necessarily time-dependent process indicated in figure 1 which takes account of the curvature of the field lines before and after reconnection. It is the purpose of this short paper to provide such a model. We restrict attention here to two-dimensional evolution in an incompressible fluid; generalisation to the more challenging problem of three-dimensional skewed flux tubes may however also be possible.

2. The initial configuration

It is natural to represent the curves AB, CD in figure 1 by the hyperbola

$$y^2 - k^2x^2 = Y^2 \tag{1}$$

where k and Y are constants. We first construct a magnetic field

K. Bajer and H.K. Moffatt (eds.), Tubes, Sheets and Singularities in Fluid Dynamics, 125–138.

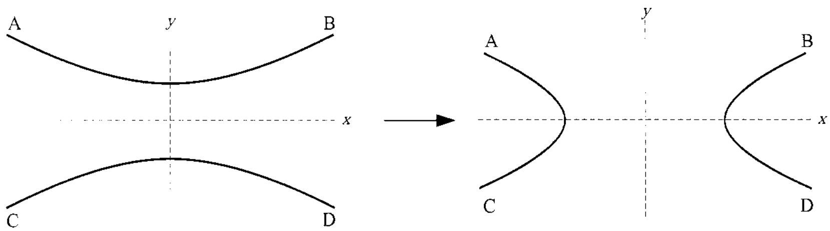

Figure 1. Change of topology associated with magnetic reconnection.

$\mathbf{B} = \nabla \wedge \mathbf{A}$ which is confined to two oppositely directed flux sheets centred on this hyperbola. With $\mathbf{A} = (0, 0, A_0(x, y))$, this is achieved by the choice

$$A_0(x, y) = \tfrac{1}{2}\Phi \tanh\{(y^2 - Y^2 - k^2x^2)/\delta^2\}, \tag{2}$$

where δ is the transverse scale of the flux sheet. We shall adopt non-dimensional variables using δ as the unit of length; so from here on we may take $\delta = 1$. We shall assume further that $Y > 1$ (so that the two flux sheets are 'separated' at $t = 0$, well separated if $Y \gg 1$); and we shall assume that the curvature parameter k is < 1.

The field components associated with (2) are

$$B_x = \frac{\partial A_0}{\partial y} = \Phi y \operatorname{sech}^2(y^2 - Y^2 - k^2x^2) \tag{3}$$

$$B_y = -\frac{\partial A_0}{\partial x} = \Phi k^2 x \operatorname{sech}^2(y^2 - Y^2 - k^2x^2)\,. \tag{4}$$

With $Y \gg 1$, the flux (per unit distance in the z-direction) in the upper sheet is

$$\int_0^\infty B_x dy = A_0(x, \infty) - A_0(x, 0) = \Phi \tag{5}$$

and the flux in the lower sheet is similarly $-\Phi$.

In order to induce rapid reconnection, we subject this field to the irrotational straining flow

$$\boldsymbol{u} = (\alpha x, -\alpha y, 0) \tag{6}$$

with $\alpha > 0$ (Figure 2). We may adopt α^{-1} as the unit of time, so that henceforth $\alpha = 1$. The Lorentz force $\mathbf{j} \wedge \mathbf{B}$ with $\mathbf{j} = \mu_0^{-1}\nabla \wedge \mathbf{B}$ will obviously perturb this flow; but we may adopt (6) as an initial condition for the velocity field, and as an outer boundary condition as $y \to \pm\infty$ for all $t > 0$. The initial streamfunction associated with (6) (with now $\alpha = 1$) is

$$\psi_0(x, y) = xy \tag{7}$$

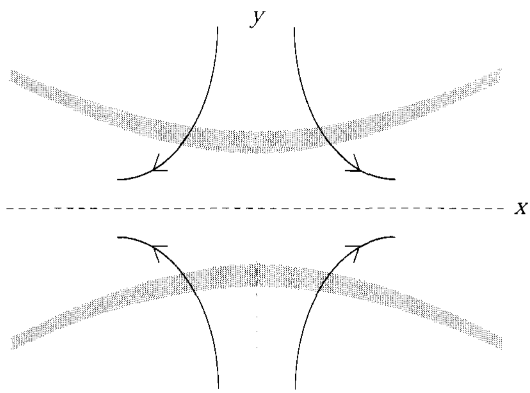

Figure 2. The initial field configuration subjected to imposed irrotational strain.

3. The evolution equations

Consider first the magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \mathbf{B}) + \eta \nabla^2 \mathbf{B}\,, \tag{8}$$

where η is the (dimensionless) magnetic diffusivity parameter. For the two-dimensional situation, we take $\mathbf{B} = \nabla \wedge \mathbf{A}$ with

$$\mathbf{A} = (0, 0, A(x, y, t))\,, \qquad \nabla \cdot \mathbf{A} = 0\,. \tag{9}$$

Note that $\boldsymbol{u} \wedge \mathbf{B}$ is in the z-direction and $\nabla \cdot (\boldsymbol{u} \wedge \mathbf{B}) = 0$. There is no applied electric field in the z-direction. In these circumstances, (8) may be 'uncurled' to give

$$\frac{\partial \mathbf{A}}{\partial t} = \boldsymbol{u} \wedge \mathbf{B} + \eta \nabla^2 \mathbf{A}\,. \tag{10}$$

Equivalently, with $\boldsymbol{u} = (\partial\psi/\partial y\,, -\partial\psi/\partial x, 0)$ and $\psi = \psi(x, y, t)$,

$$\frac{\partial A}{\partial t} + \frac{\partial(A, \psi)}{\partial(x, y)} = \eta \nabla^2 A\,. \tag{11}$$

The Navier-Stokes equation including the Lorentz force is, in standard notation,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{j} \wedge \mathbf{B} + \nu \nabla^2 \boldsymbol{u}\,. \tag{12}$$

With vorticity $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u} = (0, 0, -\nabla^2\psi)$, the vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{\omega}) + \frac{1}{\rho}\nabla \wedge (\mathbf{j} \wedge \mathbf{B}) + \nu \nabla^2 \boldsymbol{\omega} \tag{13}$$

or equivalently

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial(\nabla^2\psi, \psi)}{\partial(x,y)} = \frac{1}{\mu_0 \rho} \frac{\partial(\nabla^2 A, A)}{\partial(x,y)} + \nu \nabla^4 \psi \,. \tag{14}$$

Equations (11) and (14), coupled with the initial conditions $A = A_0(x,y)$, $\psi = xy$ at $t = 0$ and the outer boundary conditions

$$A \to 0\,,\ \psi \sim xy \qquad \text{as} \qquad y \to \pm\infty \tag{15}$$

provide a complete specification of the problem. Note the symmetries, A even in x and y, ψ odd in x and y which are respected by the evolution equations.

4. Boundary-layer approximation

As the straining flow sweeps the two sheets towards each other, it is obvious that the field gradient in the y-direction will increase, while that in the x-direction will diminish. In these circumstances, a boundary-layer approach is legitimate. We expand A and ψ in the form

$$\psi = xf(y,t) + \tfrac{1}{6}x^3 s(y,t) + O(x^5) \tag{16}$$

$$A = \tfrac{1}{2}\Phi a(x,y,t) = \tfrac{1}{2}\Phi[g(y,t) + \tfrac{1}{2}x^2 h(y,t) + O(x^4)] \tag{17}$$

where f is odd in y, and g and h are even in y. Note that then,

$$\nabla^2\psi = x(f_{yy} + s) + O(x^3)\,, \tag{18}$$

$$\nabla^2 a = (g_{yy} + h) + O(x^2)\,. \tag{19}$$

In each case, the leading-order term will be dominated by the part involving the y-derivatives, i.e.

$$\nabla^2\psi \sim x f_{yy}\,,\ \nabla^2 a \sim g_{yy}\,. \tag{20}$$

With this boundary-layer approximation, we have

$$\frac{\partial(a,\psi)}{\partial(x,y)} \approx -f g_y + x^2(h f_y - \tfrac{1}{2} f h_y) + O(x^4)\,, \tag{21}$$

$$\frac{\partial(\nabla^2\psi,\psi)}{\partial(x,y)} \approx x(f_y f_{yy} - f f_{yyy}) + O(x^3)\,, \tag{22}$$

$$\frac{\partial(\nabla^2 a, a)}{\partial(x,y)} \approx x(g_y h_{yy} - h g_{yyy}) + O(x^3)\,, \tag{23}$$

and so, to leading order, (11) yields

$$g_t = f g_y + \eta g_{yy} , \tag{24}$$

$$h_t = -2h f_y + f h_y + \eta h_{yy} , \tag{25}$$

and (14) yields

$$f_{yyt} = f f_{yyy} - f_y f_{yy} - M^2 \left(h g_{yyy} - h_{yy} g_y \right) + \eta f_{yyyy} , \tag{26}$$

where $M = \frac{1}{2}(\mu_0 \rho)^{-1/2}\Phi$, a measure of the strength of the initial field. This equation integrates with respect to y (as expected within the boundary-layer approximation) to give

$$f_{yt} = 1 + f f_{yy} - f_y^2 - M^2 (h g_{yy} - h_y g_y) + \nu f_{yyy} , \tag{27}$$

where the constant of integration (+1) is fixed by the boundary conditions

$$f \sim y \, , g \to 1 \, , \ h \to 0 \qquad \text{as} \qquad y \to \pm\infty \, . \tag{28}$$

We have thus reduced the problem to three nonlinear coupled evolution equations (24), (25), (27). Clearly the initial condition on f is just

$$f(y, 0) = y \, . \tag{29}$$

To obtain initial conditions on g and h, we expand (2) in the form

$$A_0(x, y) = \tfrac{1}{2}\Phi \left[\tanh(y^2 - Y^2) - k^2 x^2 \mathrm{sech}^2(y^2 - Y^2) + O(x^4) \right] , \tag{30}$$

so that, from (4.2), these initial conditions are

$$g(y, 0) = \tanh(y^2 - Y^2) \tag{31}$$

$$h(y, 0) = -2k^2 \mathrm{sech}^2(y^2 - Y^2) \, . \tag{32}$$

5. Weak field limit

If $M \ll 1$, then the field behaves, at least in the initial stages, as a passive vector field. We then have $f = y$, and (24), (25) become

$$g_t = y g_y + \eta g_{yy} \tag{33}$$

$$h_t = -2h + y h_y + \eta h_{yy} \, . \tag{34}$$

If diffusion is negligible ($\eta = 0$) then the field is 'frozen in',

$$A(x, y, t) = A_0(x e^{-t}, y e^{t}) \, , \tag{35}$$

and so

$$g(y,t) = \tanh(y^2 e^{2t} - Y^2) \tag{36}$$

$$h(y,t) = -2k^2 e^{-2t} \mathrm{sech}^2 (y^2 e^{2t} - Y^2)\,. \tag{37}$$

We then have

$$h g_{yy} - h_y g_y = -4k^2 \mathrm{sech}^4 (y^2 e^{2t} - Y^2)\,. \tag{38}$$

Thus, although the magnetic field strength increases exponentially (like e^t), the relevant part of the Lorentz force in (27) (that part not compensated by the pressure field) remains $O(M^2)$; this is because the curvature of the field lines near $x = 0$ decreases in proportion to e^{-t}. There may be a weak cumulative effect of the Lorentz force over a long time; but the perturbation velocity field driven by this Lorentz force certainly does not increase exponentially, and remains $O(M^2)$ relative to the imposed strain field.

When $\eta \ll 1$, this frozen field development lasts only for so long as the diffusion terms in (33), (34) remain negligible. From (36), ηg_{yy} becomes comparable with g_t when

$$2t = O(\ln(1/\eta)) \tag{39}$$

at which stage the (dimensionless) field strength is of order $\eta^{-1/2}\Phi$. From this point on, diffusive reconnection proceeds in a layer whose thickness is $O(\eta^{-1/2})$ and presumably does not decrease much below this.

6. Saddle point reconnection

Being even in y, the functions $g(y,t)$, $h(y,t)$ admit expansions

$$g(y,t) = g_0(t) + \tfrac{1}{2} y^2 g_2(t) + O(y^4)\,, \tag{40}$$

$$h(y,t) = h_0(t) + \tfrac{1}{2} y^2 h_2(t) + O(y^4)\,, \tag{41}$$

where, from (31), (32),

$$g_0(0) = -\tanh Y^2\,,\ g_2(0) = 2\mathrm{sech}^2 Y^2\,, \tag{42}$$

$$h_0(0) = -2k^2 \mathrm{sech}^2 Y^2\,,\ h_2(0) = -8k^2 \mathrm{sech}^2 Y^2 \tanh Y^2\,. \tag{43}$$

Of course, for $Y^2 \gg 1$, we have

$$g_0(0) \approx -1\,,\ g_2(0) = h_0(0) = h_2(0) \approx 0\,. \tag{44}$$

Near the origin, the **B**-lines $A =$ cst. are hyperbolic and given by

$$h_0(t) x^2 + g_2(t) y^2 = \text{ cst.} \tag{45}$$

The separatrices through O are the lines

$$y/x = \pm(-h_0(t)/g_2(t))^{1/2}. \tag{46}$$

Note that at $t = 0$, although the field is exponentially weak near O, these separatrices are well-defined as $y/x = \pm k$. Under the frozen-field evolution (36), (37), they move with the fluid to position

$$y/x = \pm k e^{-2t} \tag{47}$$

i.e. the angle between the separatrices decreases exponentially.

At time $t = 0$, the flux crossing the positive y-axis is Φ; at time t, it is

$$\begin{aligned} \Phi_x(t) &= \int_0^\infty B_x|_{x=0} dy = A(0, \infty, t) - A(0, 0, t) \\ &= \tfrac{1}{2}\Phi\left[1 - g_0(t)\right]. \end{aligned} \tag{48}$$

The difference,

$$\Phi - \Phi_x(t) = \tfrac{1}{2}\Phi\left(1 + g_0(t)\right), \tag{49}$$

is the flux that has reconnected and is associated with the field in the region

$$0 < y < k e^{-2t} x, \tag{50}$$

(see figure 3). Clearly, this reconnection process occurs in an ϵ-neighbour-

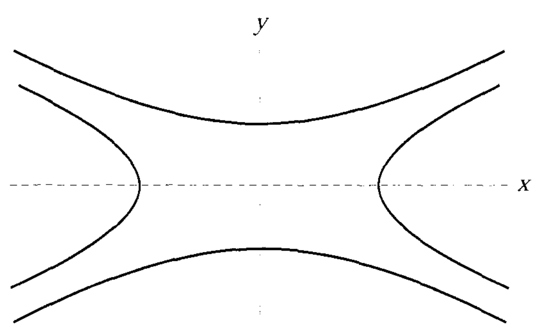

Figure 3. Field configuration during reconnection.

hood of the saddle point of the field $\mathbf{B}$ where $\epsilon = O(\eta^{1/2})$, and cannot occur in the absence of such a saddle point. Figure 4 shows the computed evolution of $g_0(t)$ based on equations (24), (25), (27), for the following parameter values:

$$M = 1, \ \nu = \eta = 0.01, \ Y = 1.5, \ k = 0.5. \tag{51}$$

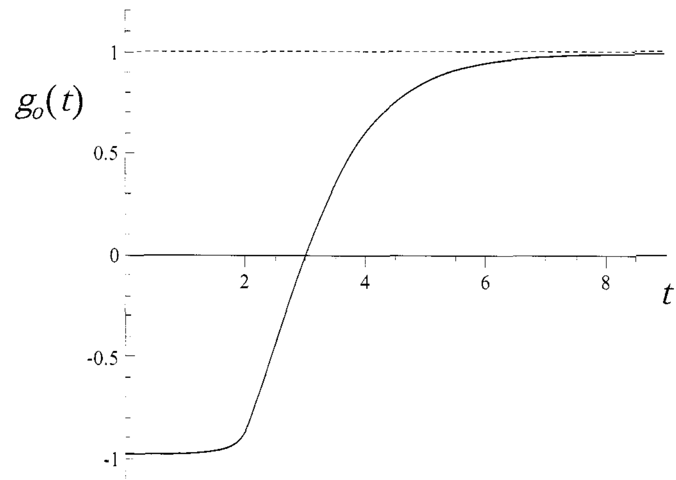

Figure 4. Evolution of the function $g_0(t)$ based on numerical solution of (24), (25) and (27); note (i) the initial 'frozen-flux' period of constancy of $g_0(t)$, and (ii) the subsequent monotonic increase from -1 to 1 (so that, from (48), $\Phi_x(t)$ decreases from Φ to zero).

As expected, $g_0(t)$ remains constant at near -1 for small t, while diffusion is still negligible; it then rises quite rapidly and tends to $+1$, the reconnection process being 99% complete by $t \approx 8$.

Contrary to received opinion, there is no indication here that the Lorentz force has a dominating effect on this process. The time-scale of the process is $O(1)$, i.e. $O(\alpha^{-1})$ when we return to dimensional variables; this is because, however small the value of η, the transverse scale decreases to $O(\eta/\alpha)^{1/2}$ before the onset of diffusion, and on dimensional grounds the reconnection time-scale is necessarily $O(\alpha^{-1})$. Of course this conclusion may need modification if the magnetic field parameter M is greatly increased. Numerical experiments over the space of the parameters M, ν and η are in progress.

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