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The time-scale associated with flux expulsion

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A simple model problem is solved in order to show that the time-scale associated with the process of flux expulsion is

$$t_{fe} = R_m^{1/3} t_0$$

where t_0 is a time-scale characterising the flow (for example, the eddy turnover time, or inverse shear rate) and R_m is the magnetic Reynolds number. This estimate is in agreement with that of Weiss (1966) based on numerical experiments. By decomposing the vector potential into a product of a rapidly varying part (in space) and a slowly varying part, it is shown how numerical work can be extended to much higher values of R_m than has been achieved hitherto.

1. Introduction

When a steady two-dimensional motion $\mathbf{u}(\mathbf{x})$ with closed streamlines acts upon a magnetic field in the plane of the motion, it is well known that, if $R_m \gg 1$, the field is eventually expelled from regions of closed streamlines, and is ultimately concentrated in layers of thickness $O(R_m^{-1/2})$ at the boundaries of these regions.

The process is described by the equation for the vector potential $A(x, y, t)\mathbf{k}$ of the magnetic field, viz

$$\partial A / \partial t + \mathbf{u} \cdot \nabla A = \eta \nabla^2 A. \quad (1.1)$$

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During an initial phase, diffusion is negligible, and

$$A(\mathbf{x}, t) = A(\mathbf{a}, 0), \quad (1.2)$$

where $\mathbf{x}(\mathbf{a}, t)$ is the position at time t of the fluid particle initially at \mathbf{a} . During this phase, the magnetic field is distorted into a tight double spiral within each eddy, and the magnetic energy increases essentially like t^2 . Obviously the field gradient increases during this process, and so after a time, say t_{fe} , diffusion must become important. This is the stage at which closed field loops form (Parker, 1966), and the process of flux expulsion commences. The magnetic energy within any eddy reaches a maximum at $t \approx t_{fe}$ and then falls off, ultimately to a value of order R_m^{-1} .

The computational study of Weiss (1966) suggested that $t_{fe} \sim R_m^{1/3} t_0$, and that in consequence, $B_{max}^2 \sim R_m^{2/3}$. The purpose of this note is to provide a simple theoretical explanation for this scaling, and to explain why the alternative scaling $t_{fe} \sim R_m^{1/2} t_0$, $B_{max}^2 \sim R_m$ suggested by Moffatt (1978, §3.8) is in fact incorrect.

2. The action of uniform shear on a space-periodic magnetic field

Flux expulsion from an eddy occurs essentially because, at $t = 0$, $\mathbf{u} \cdot \mathbf{B}$ varies (and indeed changes sign) on each closed streamline within the eddy. A much simpler flow and field configuration, with a similar property, is sketched in Figure 1. We suppose that $\mathbf{u} = (\alpha y, 0, 0)$, and that at $t = 0$,

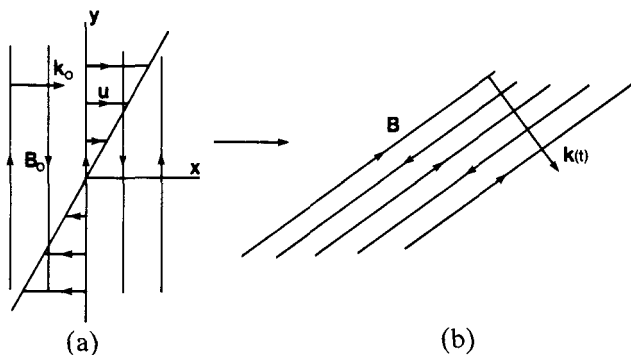


FIGURE 1 Effect of uniform shear on a unidirectional space-periodic magnetic field.

$$\mathbf{B}(\mathbf{x}, 0) = (0, B_0 \cos k_0 x, 0). \quad (2.1)$$

Correspondingly,

$$A(x, y, 0) = -k_0^{-1} B_0 \operatorname{Im} \{e^{ik_0 x}\}, \quad (2.2)$$

and the solution of (1.1) has the form

$$A(x, y, t) = -k_0^{-1} B_0 \operatorname{Im} \{a(t) e^{i\mathbf{k}(t) \cdot \mathbf{x}}\}, \quad (2.3)$$

where

$$a(0) = 1, \quad \mathbf{k}(0) = (k_0, 0, 0). \quad (2.4)$$

It is easily shown that

$$\mathbf{k}(t) = (k_0, -\alpha t k_0, 0), \quad (2.5)$$

so that the wave-fronts of \mathbf{B} are progressively tilted as indicated in Figure 1b, and that

$$da/dt = -\eta \mathbf{k}^2 a,$$

so that

$$a(t) = \exp \left\{ -\int \eta \mathbf{k}^2 dt \right\} = \exp \left\{ -\eta k_0^2 \left(t + \frac{1}{3} \alpha^2 t^3 \right) \right\}. \quad (2.6)$$

The effect of the shear is represented in the term $\frac{1}{3} \alpha^2 t^3$. If $\alpha = 0$, the time-scale of decay of \mathbf{B} is the usual diffusion time-scale $t_\eta = (\eta k_0^2)^{-1}$. If $\alpha \neq 0$, and more particularly if $\alpha \gg \eta k_0^2$, then the time-scale of decay is

$$t_{fe} = (\alpha^2 \eta k_0^2)^{-1/3} = \alpha^{-1} R_m^{1/3}, \quad (2.7)$$

where $R_m = \alpha / \eta k_0^2$.

3. The action of non-uniform shear on a space-periodic magnetic field

Suppose now that $\mathbf{u} = (u(y), 0, 0)$, so that

$$\partial A / \partial t + u(y) \partial A / \partial x = \eta \nabla^2 A. \quad (3.1)$$

Figure 2a, b shows the effect of such a velocity field on a magnetic field given initially by (2.1). Flux expulsion occurs from the region in which $|du/dy| \gg \eta k_0^2$, the field topology changing through the diffusion process. The solution of (3.1) now has the form

$$A(x, y, t) = -k_0^{-1} B_0 \operatorname{Im} \{ a(y, t) e^{i \mathbf{k}(y, t) \cdot \mathbf{x}} \}, \quad (3.2)$$

where

$$\mathbf{k}(y, t) \sim (k_0, -k_0 (du/dy) t, 0) \quad (3.3)$$

and

$$a(y, t) \sim \exp \{ -\eta k_0^2 (t + \frac{1}{3} (du/dy)^2 t^3) \}. \quad (3.4)$$

This solution describes flux expulsion on the time-scale (2.7) where now $\alpha = |du/dy|_{\max}$.

4. Flux expulsion from a single eddy with circular streamlines

Suppose now that

$$\mathbf{u} = (0, s\omega(s), 0) \quad (4.1)$$

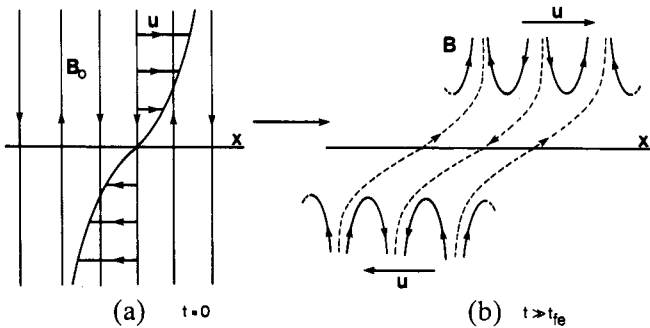


FIGURE 2. Effect of non-uniform shear on the same initial field as in Figure 1. Accelerated ohmic diffusion in the region of maximum shear leads to apparent flux expulsion in a time t_{fe} of order $R_m^{1/3} |du/dy|_{\max}^{-1}$.

in cylindrical polar coordinates (s, θ, z) , and that

$$A(s, \theta, 0) = B_0 s \sin \theta, \quad (4.2)$$

corresponding to a uniform field \mathbf{B}_0 parallel to $\theta = 0$. Then

$$A(s, \theta, t) = B_0 \operatorname{Im} \{f(s, t) e^{i\theta}\}, \quad (4.3)$$

where

$$\partial f / \partial t + i\omega(s)f = (\partial^2 / \partial s^2 + s^{-1} \partial / \partial s - s^{-2})f, \quad (4.4)$$

with $f(s, 0) = s$.

The results of §§2 and 3 now suggest the best way to proceed. If $\eta = 0$, the solution of (4.4) is

$$f(s, t) = f(s, 0) e^{-i\omega t} = s e^{-i\omega t}. \quad (4.5)$$

The function $e^{-i\omega t}$ is a rapidly varying function of s , when t is large. When $\eta \neq 0$, let

$$f(s, t) = e^{-i\omega t} g(s, t), \quad g(s, 0) = s, \quad (4.6)$$

so that

$$\partial f / \partial s = (-it\omega'g + \partial g / \partial s) e^{-i\omega t}, \quad (4.7)$$

$$\partial^2 f / \partial s^2 = (-t^2\omega'^2g - it\omega''g - 2it\omega'\partial g / \partial s + \partial^2 g / \partial s^2) e^{-i\omega t}. \quad (4.8)$$

Substitution in (4.4), and retaining only the term on the right which increases like t^2 , we have

$$\partial g / \partial t = -\eta(\omega'^2 t^2 g + \dots), \quad (4.9)$$

giving

$$g(s, t) \sim s e^{-1/2\eta\omega'^2 t^3}, \quad (4.10)$$

which again describes flux expulsion on the time-scale

$$t_{fe} = R_m^{1/3} (s\omega')^{-1}, \quad (4.11)$$

in agreement with the results of Weiss (1966).

The asymptotic symbol \sim in (4.10) needs interpretation in terms of a double limiting process

$$|s\omega't| \gg 1 \quad \text{and} \quad |\eta t^2 \omega''| \ll 1, \quad (4.12)$$

the latter arising from back-substitution of the solution (4.10) in (4.8) to see at what stage the term $\partial^2 g / \partial s^2$ becomes comparable with $t^2 \omega'^2 g$. The solution (4.10) is thus valid for

$$1 \ll \omega_0 t \ll R_m^{1/2}, \quad (4.13)$$

where ω_0 is a typical value of $|s\omega'|$, and it is supposed that $|\omega'/s\omega''| = O(1)$. The flux expulsion time $t_{fe} = \omega_0^{-1} R_m^{1/3}$ is within the range (4.13), so that the description given by (4.10) is self-consistent.

The important point to note is that, for t in the range (4.13), the function $g(s, t)$ defined by (4.6) is slowly varying as a function of s , whereas $f(s, t)$ is rapidly varying. Computer experiments based on the exact equation for $g(s, t)$ have in fact been carried out for values of R_m up to 10^9 (Kamkar, 1981), and the $R_m^{1/3}$ behaviour for t_{fe} (defined as the value of t for which the field perturbation energy is maximal) persists, as expected, to these high values. Computer experiments based on the equation for f fail for $R_m \gtrsim 10^3$ due to inadequate radial resolution of the developing field structure.

5. Discussion

It remains to explain why the argument given by Moffatt (1978, §3.8), though appealing in its simplicity, is in fact incorrect. This argument involved simple evaluation of the diffusion term $\eta \nabla^2 A$ in (1.1) on the basis of the Lagrangian solution (1.2), and the assertion that, when $\eta \nabla^2 A$ becomes of the same order as either term on the left of (1.1), neglect of diffusion is no longer valid. It is this argument that leads to the estimate $t_{fe} \sim R_m^{1/2} t_0$ referred to in the introduction. The reason that diffusion becomes significant at an *earlier stage* ($\sim R_m^{1/3} t_0$) is that, whereas the $\mathbf{u} \cdot \nabla A$ term in (1.1) leads to periodic variation of A at any fixed point (with period of order t_0) the diffusion term $\eta \nabla^2 A$ is cumulative in its effect, which must therefore be estimated *by an integration from zero to t* , rather than simply an evaluation at time t ; it is this integration which leads to the crucial t^3 term in (2.6). It is rather interesting that the normal procedure for neglecting a 'small' term in an equation, viz "neglect it, solve the equation,

then evaluate the neglected term to see whether it was indeed negligible" is here unreliable and gives a misleading result!

Acknowledgements

We are grateful to Dr Nigel Weiss, whose disbelief in the $R_m^{1/2}$ estimate led to a continuing investigation of the problem, and to Dr John Chapman, who contributed to some of our earlier discussions.

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Reprinted from:

Stellar & Planetary Magnetism
(ed. A.M. Soward)
Gordon & Breach, London, 1983