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CORNER SINGULARITIES IN THREE-DIMENSIONAL STOKES FLOW

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Stokes flow in a corner region driven by a weakly three-dimensional stirring mechanism at some distance from the corner is considered. It is shown that, when the stirring is antisymmetric about the bisecting plane $\theta = 0$, the flow near the corner exhibits the same type of eddy structure as is familiar from the two-dimensional theory. When the stirring is symmetric about $\theta = 0$ however, a non-oscillatory component is in general present (driven by conditions far from the corner), and this component dominates over the oscillatory component near $r = 0$.

1. Introduction

It is well known (Moffatt 1964a) that if an incompressible viscous fluid, contained between two plane rigid boundaries $\theta = \pm\alpha$ with $\alpha < \alpha_c \approx 73.5°$, is subjected to an arbitrary two-dimensional stirring at some distance from the corner, then under the Stokes approximation the flow near the corner exhibits an infinite sequence of eddies of alternating sense of rotation. These eddies are geometrically and dynamically self-similar and get rapidly weaker as the corner is approached.

It is natural to enquire whether these eddies survive when the stirring mechanism is no longer two-dimensional. For simplicity, we shall suppose that this stirring mechanism is only weakly dependent on the coordinate z parallel to the intersection of the two planes. In the following sections, we consider two specific stirring mechanisms, the first generating a flow that is antisymmetric about the bisecting plane $\theta = 0$, the second a flow that is symmetric. The conclusions concerning the formation of eddies are very different in these two cases.

2. General formulation

We shall use cylindrical polar coordinates (r, θ, z) where r represents the distance from the intersection of the planes $\theta = \pm\alpha$. The velocity field $\boldsymbol{u} = (u, v, w)$ admits the general 'toroidal/poloidal' decomposition

$$\boldsymbol{u} = \nabla\wedge(\boldsymbol{e}_z\psi) + \nabla\wedge\nabla\wedge(\boldsymbol{e}_z\chi)\,, \tag{2.1}$$

where $\psi(r,\theta,z)\,,\chi(r,\theta,z)$ are scalar fields; thus

$$u = r^{-1}\psi_\theta + \chi_{rz}\,,\ v = -\psi_r + r^{-1}\chi_{\theta z}\,,\ w = -\nabla_2^2\chi\,, \tag{2.2}$$

where suffices indicate differentation, and

$$\nabla_2^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \nabla^2 - \frac{\partial^2}{\partial z^2} \tag{2.3}$$

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is the two-dimensional Laplacian operator. Note that in the two-dimensional limit ($\partial/\partial z \to 0$), we have $u = r^{-1}\psi_\theta$, $v = -\psi_r$, i.e. $\psi(r,\theta)$ is then the stream-function of the flow.

The vorticity field $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$ is given from (2.1) by

$$\boldsymbol{\omega} = \nabla \wedge \nabla \wedge (\boldsymbol{e}_z \psi) - \nabla \wedge \left(\boldsymbol{e}_z \nabla^2 \chi\right), \tag{2.4}$$

and it follows that

$$\begin{aligned}\nabla^2 \boldsymbol{\omega} &= \nabla \wedge \nabla \wedge \left(\boldsymbol{e}_z \nabla^2 \psi\right) - \nabla \wedge \left(\boldsymbol{e}_z \nabla^4 \chi\right) \\ &= -\nabla_2^2 \nabla^2 \psi \boldsymbol{e}_z + \nabla_2 \left(\nabla^2 \psi\right)_z + \left(\boldsymbol{e}_z \wedge \nabla_2\right) \nabla^4 \chi .\end{aligned} \tag{2.5}$$

The Stokes equation in the form $\nabla^2 \boldsymbol{\omega} = 0$ is thus satisfied if and only if

$$\nabla_2^2 \nabla^2 \psi = 0, \tag{2.6}$$

and

$$\nabla_2 \frac{\partial}{\partial z}\left(\nabla^2 \psi\right) + \left(\boldsymbol{e}_z \wedge \nabla_2\right) \nabla^4 \chi = 0 . \tag{2.7}$$

Thus $\nabla^2 \psi$ satisfies the two-dimensional Laplace equation; and (2.7) (the Cauchy-Riemann equations) expresses the fact that

$$\frac{\partial}{\partial z}\left(\nabla^2 \psi\right) + i \nabla^4 \chi \tag{2.8}$$

is an analytic function of $x + iy = re^{i\theta}$.

Under the assumption that all fields are weakly varying in the z-direction, we have

$$\frac{\partial^2}{\partial z^2} \ll \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2.9}$$

provided $r = O(1)$; hence it is legitimate to replace ∇^2 by ∇_2^2 in the above equations, an approximation that clearly improves as $r \to 0$. In particular, asymptotically,

$$\nabla_2^4 \psi = 0 \tag{2.10}$$

and (using (2.2))

$$\frac{\partial}{\partial z}\left(\nabla_2^2 \psi\right) - i \nabla_2^2 w \tag{2.11}$$

is an analytic function of $re^{i\theta}$.

3. Flow antisymmetric about $\theta = 0$

To be specific, let us suppose that the fluid is confined to the domain

$$|\theta| < \alpha, \; r < R, \tag{3.1}$$

and that the stirring is provided at the boundaries $\theta = \pm\alpha$ in such a way that

$$\boldsymbol{u} = (\pm U(r,z), 0, 0) \qquad \text{on} \quad \theta = \pm\alpha \tag{3.2}$$

and $\boldsymbol{u} = 0$ on $r = R$. In order to generate weakly three-dimensional flow, we suppose that the imposed boundary velocity U satisfies

$$|\partial U/\partial z| \ll |\partial U/\partial r| . \tag{3.3}$$

We suppose moreover that U is nonzero only in a bounded region

$$r_0 < r < r_1 \quad , \quad |z| < z_0 \tag{3.4}$$

where $r_0 > 0$ and $z_0 \gg R$ (consistent with (3.3)).

In terms of ψ and χ, the boundary conditions (3.2) on $\theta = \pm\alpha$ become

$$\begin{aligned} r^{-1}\psi_\theta + \chi_{rz} &= \pm U(r,z)\,, \\ -\psi_r + r^{-1}\chi_{\theta z} &= 0\,, \\ -\nabla_2^2\chi &= 0\,. \end{aligned} \tag{3.5}$$

Under the assumption of weak z-dependence, the z-derivative terms in (3.5a,b) are small, and at leading order, these boundary conditions are simply

$$r^{-1}\psi_\theta = \pm U(r,z)\,,\ \psi_r = 0 \text{ on } \theta = \pm\alpha\,. \tag{3.6}$$

We have also

$$r^{-1}\psi_\theta = \psi_r = 0 \text{ on } r = R\,. \tag{3.7}$$

We must first solve the problem (from (2.9))

$$\nabla_2^4\psi = 0\,, \tag{3.8}$$

subject to the boundary conditions (3.6), (3.7). There is a unique solution (up to an arbitrary additive function of z which does not contribute to the velocity field), $\Psi(r,\theta,z)$ say. We are particularly interested in the asymptotic behaviour of Ψ near $r = 0$. From earlier two-dimensional studies (Moffatt 1964a,b) it is known that (under the assumed conditions on $U(r,z)$) this is given by

$$\Psi(r,\theta,z) \sim \mathrm{Re}A(z)r^\lambda \left[\frac{\cos\lambda\theta}{\cos\lambda\alpha} - \frac{\cos(\lambda-2)\theta}{\cos(\lambda-2)\alpha}\right]\,, \tag{3.9}$$

where λ is the (complex) root of the transcendental equation

$$\sin 2(\lambda-1)\alpha + 2(\lambda-1)\sin 2\alpha = 0 \tag{3.10}$$

having smallest real part satisfying $\mathrm{Re}(\lambda) > 1$. The (complex)coefficient $A(z)$ is determined in principle (at each z) by the boundary function $U(r,z)$.

Now we have to find $w(r,\theta,z)$ at leading order. To do this, note that $\nabla_2^2\Psi$ is a harmonic function, and so can be expressed in the form

$$\nabla_2^2\Psi = \mathrm{Re}\mathcal{F}\left(re^{i\theta},z\right) \tag{3.11}$$

for some analytic function $\mathcal{F}$ of the complex variable $re^{i\theta}$. We may note from (3.9) that

$$\mathcal{F}\left(re^{i\theta},z\right) \sim -2(\lambda-2)A(z)\left(re^{i\theta}\right)^{\lambda-2}\sec(\lambda-2)\alpha \tag{3.12}$$

as $r \to 0$. From (2.11), it follows that

$$\nabla_2^2 w = -\mathrm{Im}\ \partial\mathcal{F}\left(re^{i\theta},z\right)/\partial z = H(r,\theta,z)\,, \mathrm{say}\,, \tag{3.13}$$

and that, taking account of the antisymmetry in θ,

$$\nabla_2^2 w \sim 2\mathrm{Im}\left[(\lambda-2)A'(z)r^{\lambda-2}\sin(\lambda-2)\theta\sec(\lambda-2)\alpha\right] \tag{3.14}$$

as $r \to 0$.

The equation

$$\nabla_2^2 w = H(r,\theta,z) \tag{3.15}$$

must now be solved, subject to the boundary condition

$$w = 0 \text{ on } \theta = \pm\alpha \text{ and on } r = R. \tag{3.16}$$

There is obviously a particular integral $w^{(P)}$ which, from (3.14) and symmetry considerations, has the behaviour

$$w^{(P)} \sim \operatorname{Im} \frac{A'(z)r^{\lambda}}{\cos(\lambda-2)\alpha}\left[\frac{\sin\lambda\theta}{\sin\lambda\alpha} - \frac{\sin(\lambda-2)\theta}{\sin(\lambda-2)\alpha}\right] \tag{3.17}$$

as $r \to 0$. There is also however a complementary function $w^{(C)}$ satisfying $\nabla_2^2 w^{(C)} = 0$ which is inevitably present in order that the combined solution

$$w = w^{(C)} + w^{(P)} \tag{3.18}$$

satisfy the condition $w = 0$ on $r = R$ (cf. Moffatt & Duffy 1980, where this behaviour is analysed in detail). The asymptotic form of $w^{(C)}$ near $r = 0$ is

$$w^{(C)} \sim B(z) r^{\nu} \sin\nu\theta\,, \tag{3.19}$$

where

$$\nu\alpha = \pi\,, \tag{3.20}$$

and $B(z)$ is a (real, slowly varying) function of z, again determined in principle by $U(r,z)$.

Since $\partial w^{(C)}/\partial z \neq 0$, there is an associated flow $\left(u^{(C)}, v^{(C)}\right)$ in each plane $z =$ cst driven by the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r u^{(C)}\right) + \frac{1}{r}\frac{\partial v^{(C)}}{\partial\theta} = -\frac{\partial w^{(C)}}{\partial z} = -B'(z) r^{\nu}\sin\nu\theta\,. \tag{3.21}$$

The solution satisfying the no-slip condition on $\theta = \pm\alpha$ is

$$u^{(C)} = -\frac{B'(z)}{\nu+1} r^{\nu+1}\sin\nu\theta\,,\ v^{(C)} = 0\,. \tag{3.22}$$

Thus the flow $\left(u^{(C)}, 0, w^{(C)}\right)$ has streamlines on planes $\theta =$ cst.

We now see that there are in general two contributions to the velocity components (u,v) in the neighbourhood of $r = 0$, the first proportional to $r^{\lambda-1}$ being oscillatory (since λ is complex) and the second, arising from the weak z-dependence, proportional to $r^{\nu+1}$ being non-oscillatory (since ν is real). The oscillatory component dominates provided

$$\operatorname{Re}(\lambda-1) < \nu + 1 = \frac{\pi}{\alpha} + 1\,, \tag{3.23}$$

or equivalently, provided

$$\xi_1 = 2\alpha \operatorname{Re}(\lambda-1) < 2\pi + 2\alpha\,. \tag{3.24}$$

The value of ξ_1 was calculated by Moffatt (1964a) and is less than 4.51 for all values of $\alpha < \alpha_c$ for which λ is complex; hence, for $\alpha < \alpha_c$, the oscillatory term dominates, and the projection (u,v) of the flow on every plane $z =$ cst ($|z| < z_0$) exhibits the familiar corner eddy structure.

4. Flow symmetric about $\theta = 0$

The treatment here is very similar, but the conclusion is dramatically different. We now suppose that the boundary condition (3.2) is replaced by

$$\boldsymbol{u} = (U(r,z), 0, 0) \qquad \text{on} \qquad \theta = \pm\alpha\,, \tag{4.1}$$

all other conditions remaining as before. The streamfunction $\Psi(r,\theta,z)$ is now an odd function of θ, and for $r \to 0$ has the form

$$\Psi(r,\theta,z) \sim \mathrm{Re} C(z) r^{\lambda} \left[\frac{\sin\lambda\theta}{\sin\lambda\alpha} - \frac{\sin(\lambda-2)\theta}{\sin(\lambda-2)\alpha} \right], \tag{4.2}$$

where λ is now the (complex) root of

$$\sin 2(\lambda-1)\alpha - 2(\lambda-1)\sin 2\alpha = 0 \tag{4.3}$$

having smallest real part with $\mathrm{Re}(\lambda) > 1$. Note the change of sign between (3.10) and (4.3).

The complementary function corresponding to (3.19) is now

$$w^{(C)} \sim D(z) r^{\nu} \cos\nu\theta\,, \tag{4.4}$$

where now

$$\nu\alpha = \pi/2\,. \tag{4.5}$$

Again, we have two contributions to the flow components (u,v) near $r=0$, proportional to $r^{\lambda-1}$, $r^{\nu+1}$ respectively. We find now however that

$$\xi_2 = 2\alpha \mathrm{Re}(\lambda - 1) > 2\alpha(\nu+1) = \pi + 2\alpha \tag{4.6}$$

for all $\alpha \lesssim 78°$ (actually $\xi_2 \geqslant 7.50$, for symmetric flow – Moffatt 1964a). Hence, it is now the *non-oscillatory* component proportional to $r^{\nu+1}$ that dominates near $r=0$, and there are therefore at most a finite number of eddies near the corner in the symmetric case. Of course the non-oscillatory component $u^{(C)}$ is proportional to $D'(z)$ (cf 3.22) and becomes weaker if the three-dimensionality is reduced (so that $D'(z) \to 0$).

5. Conclusions

Stokes flow in a neighbourhood of a sharp corner may, by linearity, be represented as the sum of ingredients that are symmetric and antisymmetric about the bisecting plane $\theta = 0$. We have supposed that the stirring mechanism is located at some distance from the corner, and that it is weakly dependent on the coordinate z parallel to the corner. We have shown that the antisymmetric ingredient of the flow then exhibits the eddy structure that is familiar from the two-dimensional situation. The symmetric ingredient however is dominated near $r=0$ by a non-oscillatory contribution $\left(u^{(C)}, 0, w^{(C)}\right)$ in planes $\theta = \mathrm{cst}$ which obliterates the corner eddies.

These conclusions are compatible with earlier results of Sano & Hasimoto (1980) who considered the flow associated with a Stokeslet imbedded in the corner region; they are also compatible with the results of a parallel study (Mak & Moffatt 1997) of the three-dimensional flow between two parallel planes $z = \pm z_0$ under general localised boundary forcing – again in that case, flow antisymmetric about $z=0$ exhibits eddies outside the region of forcing, whereas for flow symmetric about $z=0$ a non-oscillatory contribution dominates the far-field.

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