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Celt reversals: a prototype of chiral dynamics

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A physically transparent and mathematically streamlined derivation is presented for a third-order nonlinear dynamical system that describes the curious chiral reversals of a celt (rattleback). The system is integrable, and its solutions are periodic, showing an infinite succession of spin reversals. Inclusion of linear dissipation allows any given number of reversals, and a typical celt's observed behaviour is well captured by tuning the dissipation parameters.

1. Introduction

A *celt* (alias rattleback or wobblestone) is a canoe-shaped rigid body with the curious property of spin asymmetry: it tends to spin smoothly in one sense, but when spun in the opposite sense a pitching instability develops which extracts so much energy from the spin that the spin actually reverses in sign. This *chiral* behaviour was first discussed by the meteorologist G. T. Walker [9], who recognized that it results from misalignment of the celt's principal axes of inertia and its axes of curvature at the point of contact with the table. Many subsequent authors have revisited this phenomenon, analysing aspects of the stability of steady spin [5, 7] (these Russian works seem to have been largely overlooked in the West), [1], or conducting a numerical simulation of the full nonlinear equations of motion [3, 4], some including dissipation from slip [2].

Our objective here is to illuminate, streamline and extend previous analyses, in order to provide a transparent derivation of a simple system of equations ((5.5) and (7.1) below) that captures the celt's reversal mechanism. The scalar form of (5.5) was derived earlier by Markeev and Paskal [5, 7]. Our derivation differs from theirs in that

(i) at each step the physical meaning of the approximation is made explicit,

(ii) we pinpoint the *chirality* and *Coriolis terms* that are the direct physical causes of the phenomenon.

The key physical observation is that, even from rest, if we tap a celt so as to make it *pitch*, it spontaneously spins in one sense, whereas if we make it *roll*, it spins in the opposite sense (acting like a reversed celt); whatever we do, theory must account for this. The key mathematical idea is *separation of time-scales*, slow and fast. The mean torque from pitching and rolling was calculated also in [2] by averaging over these fast oscillations.

Chiral dynamics is common in nature, yet very little studied. We like to think of the celt as a model, and of its equations (5.5), (7.1) as the *simplest prototype of chiral dynamics*. For example, the parallel between the celt equations and those of the $\alpha\omega$-dynamo is suggestive.

1.1. On the word 'celt'

Though always confused with 'Celt' (pron. /`kelt`/) as in 'Celtic people', it is a separate word pronounced /`selt`/, asserts the *Oxford English Dictionary* (2nd edn, 1989): 'An implement with chisel-shaped edge, of bronze or stone (but sometimes of iron), found among the remains of prehistoric man...'. Etymologically, it has come into existence by mistake:

> [t]he received or Clementine text of the Vulgate has in *Job* xix.24 *Stylo ferreo, et plumbi lamina, vel celte sculpantur in silice*; but, though this is the reading of some MSS., the Codex Amiatinus and others read *certe* 'surely'[...] the independent evidence for a word *celtes* or *celte* is slender[...] *celtes*, whatever its origin and character, was assumed, on the authority of the Vulgate, to be a genuine word; and as such, the term was admitted into the technical vocabulary of Archæology, about 1700[...] the general adoption of the word by antiquaries was influenced by a fancied etymological connexion with [the other word 'Celt'].

The Authorized Version translates Job 19:23, 24 as

> Oh that my words were now written! Oh that they were printed in a book!
>
> That they were graven with an iron pen and lead in the rock for ever!

Indeed, the word in question לָעַד (*laad*) in the original Hebrew text means 'for certain', 'for ever'.

2. Pitching and rolling

We initially neglect dissipative effects which, though present in practice, are incidental to the reversal mechanism and best incorporated at a later stage (see § 7). Moreover, we focus on a body of specific geometry; the analysis is readily generalized.

Consider a uniform solid ellipsoid of mass M with surface $x^2/a^2+y^2/b^2+z^2/c^2 = 1$, where $a > b > c$. It can rest in stable equilibrium on a table $z = -c$ with z-axis vertically up and gravity $(0, 0, -g)$. We work in dimensionless variables adopting M, c, $\sqrt{c/g}$ as units of mass, length, time (so that $M = c = g = 1$). The point of contact P then has coordinates $(0, 0, -1)$, and the principal moments of inertia at P are $\alpha = (b^2 + 6)/5$, $\beta = (a^2 + 6)/5$, $\gamma = (a^2 + b^2)/5$. Near P the contact surface is locally $z = -1 + x^2/2a^2 + y^2/2b^2$.

If this ellipsoid is disturbed from rest, then, assuming no slip, the linearized equations for the coordinates $x(t), y(t)$ of the now moving point of contact P are $\ddot{x} + \omega_1^2 x = 0$, $\ddot{y} + \omega_2^2 y = 0$, where the dot denotes time-derivative and the frequencies $\omega_{1,2}$ are given by $\omega_1^2 = 5(a^2-1)/(a^2+6)$, $\omega_2^2 = 5(b^2-1)/(b^2+6)$. There are thus two uncoupled modes of oscillation: *pitching* $(x, y) = (1, 0)\exp \mathrm{i}\omega_1 t$ and *rolling* $(x, y) = (0, 1)\exp \mathrm{i}\omega_2 t$. We suppose that $b - 1$ is of order 1, so that both $\omega_{1,2}$ are also of order 1. In fact, $\omega_1^2 > \omega_2^2$ and $\omega_1^2 \sim 5$ if $a \gg 1$. If the ellipsoid spins about the z-axis, then these two modes get coupled, but they remain stable as long as the no-slip condition persists.

3. Chiral distortion

Now redistribute the mass inside the ellipsoid, causing its principal axes to be rotated by a small angle about the vertical relative to the principal axes of curvature at P. The resulting body does not in itself exhibit chirality, since a rigid-body motion brings it into coincidence with its mirror image; however, with the orientation of gravity toward the table, the system 'body plus table' does exhibit chirality. This is our celt. If we still refer to xyz aligned with the axes of inertia, then it is the axes of curvature that are rotated by a small angle, and the surface near P is locally

$$z = -1 + \frac{x^2}{2a^2} + \chi\frac{xy}{a^2} + \frac{y^2}{2b^2},$$

where the *chirality parameter* χ (as in $\chi\varepsilon\acute{\iota}\rho$ 'hand'), positive or negative, is small.

4. Chiral instability

This body can spin with angular velocity $(0, 0, n)$, but the effect of chirality $\chi \neq 0$ is to destabilize pitching or rolling, or both. To simplify matters, we suppose $a \gg b$, i.e. the celt (like most toy models) is long in the x-direction compared with its transverse length-scale of order 1. We suppose further that $|n| \ll 1$, $|\chi| \ll 1$, so that the pitching and rolling modes are weakly perturbed by spin and chirality (as in most toy experiments). In these circumstances, the angular momentum equation and the no-slip condition lead, as in [1], to linearized equations for the coupled modes of oscillation which we arrange, at leading order in $a^2 \gg 1$, in the form

$$\begin{aligned}
\ddot{x} + \omega_1^2 x + \chi\ddot{y} &= -6n^2 x - 12a^{-2}n\chi\dot{x} - (n^2 - 5a^{-2})\chi y - (12b^{-2} - 5)n\dot{y},\\
\ddot{y} + \omega_2^2 y + N\dot{x} &= -b^2 a^{-2}\chi\ddot{x} - (1 + 6b^{-2})\\
&\qquad \times (6(1 - b^{-2})n^2 y - 12a^{-2}n\chi\dot{y} - (11 + b^2)a^{-2}\chi x),
\end{aligned}$$

where $N = 5(1 + 6b^{-2})n$. The reason for arranging the equations in this form is that, as checked by standard analysis, all terms collected on the right merely perturb the (real) frequencies $\omega_{1,2}$ of the pitching and rolling modes, and do not induce any instability; by contrast, $\chi\ddot{y}$ and $N\dot{x}$ placed on the left act in conjunction to shift these frequencies off the real axis, so these terms alone are responsible for instabilities. A small shift of frequency along the real axis is immaterial: the essential dynamics is thus adequately described if we ignore the terms on the right, giving the strikingly simple system

$$\ddot{x} + \omega_1^2 x + \chi\ddot{y} = 0, \qquad \ddot{y} + \omega_2^2 x + N\dot{x} = 0. \tag{4.1}$$

To recap, the two crucial perturbative effects we have retained are

(i) a small change in the pitching mode due to chirality χ, and

(ii) a small change in the rolling mode due to the Coriolis effect of the spin N.

In retrospect the interpretation of (4.1) is plausible from intuitive considerations. We note that (4.1) is reminiscent of the equations of the $\alpha\omega$-dynamo (see, for example, [6]):

$$\frac{\partial A}{\partial t} = \alpha B + \lambda \nabla^2 A, \qquad \frac{\partial B}{\partial t} = (\nabla \wedge A\boldsymbol{i}_y) \cdot \nabla U + \lambda \nabla^2 B,$$

χ and N being replaced by α (related to helicity) and shear ∇U.

Stability and instability are governed not by χ and N separately, but by their product $N\chi$. With $(x, y) \sim \exp \mathrm{i}\omega t$ (assuming N is constant), the characteristic equation for ω is $(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) + \mathrm{i}N\chi\omega^3 = 0$. For small $|N\chi|$, the perturbed roots are $\omega_{1,2} - \mathrm{i}\sigma_{1,2}$, where $\sigma_{1,2}$ are given by

$$2(\omega_1^2 - \omega_2^2)\sigma_1 = N\chi\omega_1^2, \qquad 2(\omega_1^2 - \omega_2^2)\sigma_2 = -N\chi\omega_2^2,$$

and the corresponding eigenvectors are

$$(x_1, y_1) = \left(1, \frac{-\mathrm{i}N}{\omega_1^2 - \omega_2^2}\right), \qquad (x_2, y_2) = \left(\frac{\chi\omega_2^2}{\omega_1^2 - \omega_2^2}, 1\right).$$

If $N\chi > 0$, then pitching $\sim \exp(\mathrm{i}\omega_1 + \sigma_1)t$ is unstable with growth rate σ_1 while rolling $\sim \exp(\mathrm{i}\omega_2 + \sigma_2)t$ is stable with decay rate $|\sigma_2|$. If $N\chi < 0$, then the situation is reversed: pitching becomes stable and rolling unstable. This agrees well with the observed behaviour of long thin celts.

5. Two time-scales and conservation of energy

The growth rates $\sigma_{1,2}$ of the above instabilities are both of order $|N\chi|$, small compared with the frequencies $|\omega_{1,2}|$ of the pitching and rolling modes. Hence, N must vary on this *slow time-scale*. We use quasi-steady analysis to track the evolution of the amplitudes $A_0(t)$, $B_0(t)$ of pitching and rolling, which evidently satisfy $\mathrm{d}A_0/\mathrm{d}t = \sigma_1 A_0$, $\mathrm{d}B_0/\mathrm{d}t = \sigma_2 B_0$. In terms of a slow-time variable

$$\tau = \frac{|\chi|}{2(\lambda - 1)} t, \quad \text{where } \lambda = \left(\frac{\omega_1}{\omega_2}\right)^2,$$

these become (taking for the moment $\chi > 0$)

$$\frac{\mathrm{d}A_0}{\mathrm{d}\tau} = \lambda N A_0, \qquad \frac{\mathrm{d}B_0}{\mathrm{d}\tau} = -N B_0, \tag{5.1}$$

where we now regard $N = N(\tau)$ as varying on the slow time-scale. The phases of the oscillations play no part in this analysis.

To describe the slow-time variation of N, we need only appeal to conservation of energy. The energies E_1 in pitching and E_2 in rolling are given at leading order by $2E_1 = \beta(\dot{x}^2 + \omega_1^2 x^2) = \beta\omega_1^2 A_0^2$ and $2E_2 = \alpha\omega_2^2 B_0^2$, while the energy E_3 in the spin n about the z-axis is given by $2E_3 = \gamma n^2$. If dissipation is neglected, the total energy $E = E_1 + E_2 + E_3$ is conserved: $\beta\omega_1^2 A_0^2 + \alpha\omega_2^2 B_0^2 + \gamma n^2 = 2E$, an ellipsoid in the (A_0, B_0, n) phase space. With rescaled amplitudes

$$A = 5(1 + 6b^{-2})\sqrt{\beta/\gamma}A_0, \qquad B = 5(1 + 6b^{-2})\sqrt{\alpha/\gamma}B_0,$$

the ellipsoid is transformed to a sphere

$$A^2 + B^2 + N^2 = \text{const.} \tag{5.2}$$

in the (A, B, N) phase space, while (5.1) reads

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \lambda NA, \qquad \frac{\mathrm{d}B}{\mathrm{d}\tau} = -NB. \tag{5.3}$$

From (5.2) and (5.3), the equation for N is

$$\frac{\mathrm{d}N}{\mathrm{d}\tau} = -\lambda A^2 + B^2. \tag{5.4}$$

In the derivation so far the chirality parameter χ, which is hidden in the slow time τ, has been taken to be positive. When χ is negative, $\mathrm{d}/\mathrm{d}\tau$ acquires the sign of χ, and the three scalar equations (5.3) and (5.4) combine as a clean vector equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} A \\ B \\ N \end{pmatrix} = \operatorname{sgn}\chi \begin{pmatrix} B \\ \lambda A \\ 0 \end{pmatrix} \wedge \begin{pmatrix} A \\ B \\ N \end{pmatrix}. \tag{5.5}$$

Suppose that $\chi > 0$. It is obvious now that, if the pitching mode A is excited when $N = 0$, then $\mathrm{d}N/\mathrm{d}\tau < 0$, i.e. the celt begins to spin in the negative sense, whereas if the rolling mode B is excited, then $\mathrm{d}N/\mathrm{d}\tau > 0$, i.e. it begins to spin in the positive sense but with weaker angular acceleration because $\lambda > 1$. The senses are reversed when $\chi < 0$. The behaviour of the system on the slow time-scale is completely described by the three-dimensional dynamical system (5.5), with quadratic nonlinearity, involving the sign ($\pm$) of chirality χ and a single parameter λ.

In terms of the constant energy E, N is of order $\sqrt{E}/a$; the slow time-scale is therefore of order $a/\sqrt{E}|\chi|$, and the condition for validity of the two-time-scale approach adopted here is $E\chi^2 \ll a^2$.

6. Another conservation law, phase-space trajectories and multiple reversals

From (5.5) we see that $(B, \lambda A, 0)$ is perpendicular to $\mathrm{d}/\mathrm{d}\tau(A, B, N)$, yielding another conservation law $AB^\lambda = \text{const.}$, a family of quasi-hyperbolic cylinders in the (A, B, N) phase space. Trajectories of the system are closed curves of intersection of these surfaces with the spheres (5.2). Therefore, in the absence of dissipation the behaviour of the celt is *time-periodic*, and its spin will *reverse an infinite number of times*, reversal being induced in turn by the pitching instability when $N\chi > 0$ and by the rolling instability when $N\chi < 0$. Figure 1 shows a sample solution exhibiting this periodicity.

Figures 2 and 3 show another interesting solution. If we tap the celt at a suitable spot between the x- and y-axes to make $B^2 = \lambda A^2$, then it pitches and rolls in

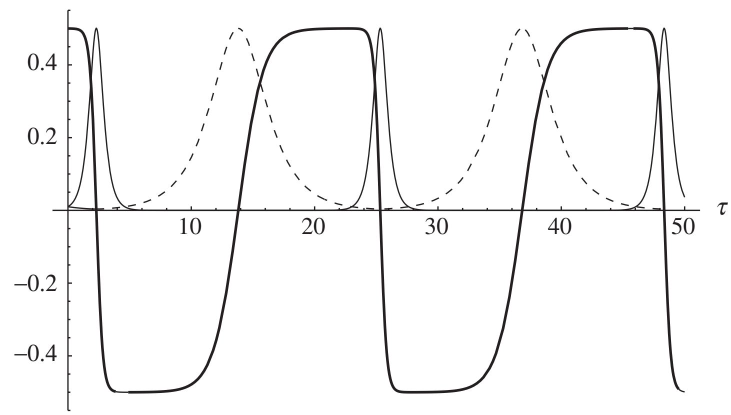

Figure 1. Solutions of the conservative equation (5.5) with $\lambda = 4$ and $A(0) = B(0) = 0.01$, $N(0) = 0.5$. The thin curve is pitching A, the dashed curve is rolling B, the thick curve is spin N. Rapid reversals from N positive to negative are induced by pitching instability, slow reversals from N negative to positive by rolling instability. Both reversals are approximated by tanh, the solution for N when A or B is 0.

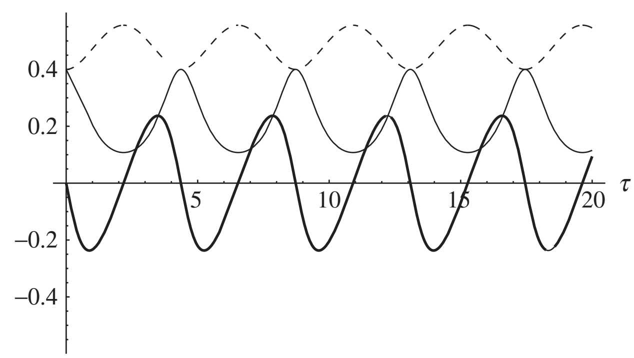

Figure 2. The same graph as in figure 1, this time from $A(0) = B(0) = 0.4$, $N(0) = 0$.

synchrony with no spin. The perturbed motion near this steady state is stable, N wobbling to and fro but not inducing a net spin in either sense; A, B, N all oscillate sinusoidally, with a period that depends on the amplitude of the initial perturbation.

7. Effects of dissipation

Dissipation is associated with slipping friction at the point of contact, and with air viscosity in the immediate vicinity of this point. These effects, for which no adequate theory is as yet available, may be modelled in a semi-empirical manner by including *simple linear dissipation* in (5.5), giving the modified system

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} A \\ B \\ N \end{pmatrix} = \operatorname{sgn}\chi \begin{pmatrix} B \\ \lambda A \\ 0 \end{pmatrix} \wedge \begin{pmatrix} A \\ B \\ N \end{pmatrix} - \begin{pmatrix} \mu_1 A \\ \mu_2 B \\ \mu_3 N \end{pmatrix}. \tag{7.1}$$

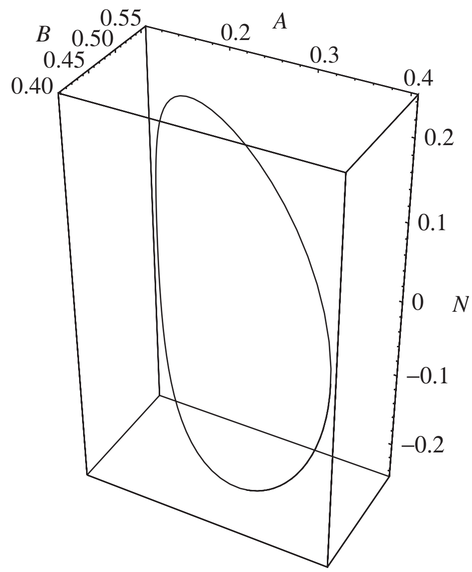

Figure 3. In this phase portrait, N is vertical, A and B are horizontal. Perturbed motions around the non-spinning steady state of synchronous pitching and rolling are stable.

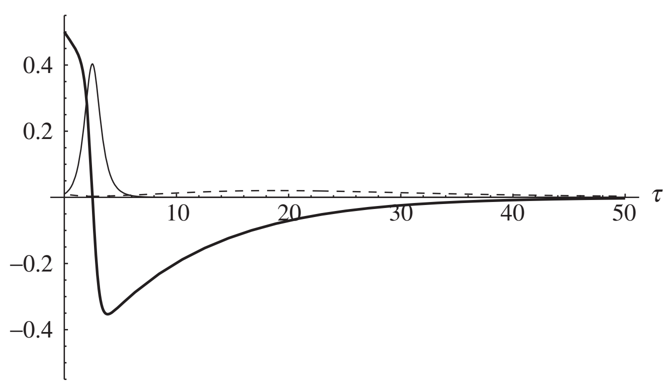

Figure 4. Solution of the dissipative equation (7.1) with $\lambda = 4$, $\mu_1 = 0.04$, $\mu_2 = 0.08$, $\mu_3 = 0.1$; initial conditions are as in figure 1. The pitching instability still induces a spin reversal, but the subsequent rolling instability is not strong enough to induce a second reversal against dissipation.

We can tune the dissipation parameters μ_1, μ_2, μ_3 to realize any number of reversals before the ultimate decay of energy to zero. Walker [9] claimed to have designed a celt that reversed four times, and predicted that celts capable of 'twelve or fifteen reversals' could be constructed. In the Cavendish Laboratory, Pippard [8] fashioned one from a slice of a Rhine wine bottle, which 'may reverse sense four or, rarely, five times'. Figure 4 shows the solution when the dissipation parameters have values that realize a single strong reversal. The contrasting solution when the initial spin is in the opposite sense is shown in figure 5. The curves mimic closely the behaviour of real celts.

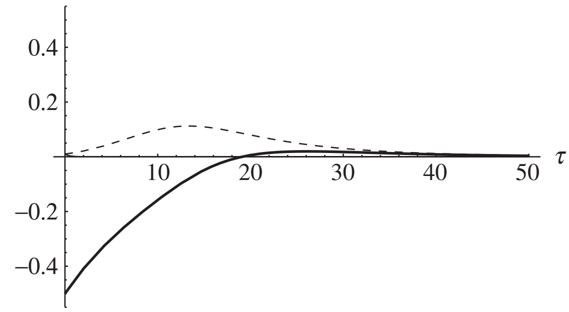

Figure 5. As figure 4, this time with $N(0) = -0.5$. A single weak reversal is induced very late ($\tau \sim 19$) by rolling instability. Pitching instability is not excited.

Acknowledgments

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