Annu. Rev. Fluid Mech. 1992. 24 : 281–312

HELICITY IN LAMINAR AND TURBULENT FLOW

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KEY WORDS: Euler equations, turbulence, relaxation, topological constraints

1. INTRODUCTION

The helicity of a fluid flow confined to a domain $\mathscr{D}$ (bounded or unbounded) of three-dimensional Euclidean space $\mathbf{R}^3$ is the integrated scalar product of the velocity field $\mathbf{u}(\mathbf{x}, t)$ and the vorticity field $\boldsymbol{\omega}(\mathbf{x}, t) = \text{curl}\,\mathbf{u}$:

$$\mathscr{H}(t) = \int_{\mathscr{D}} \mathbf{u} \cdot \boldsymbol{\omega}\, dV. \tag{1.1}$$

The quantity $h(\mathbf{x},$
$\mathscr{H}$ are pseudoscalar quantities, i.e. they change sign under change from a right-handed to a left-handed frame of reference (parity transformation). It is important therefore to specify the frame that is used; we shall always use a right-handed Cartesian (or orthogonal curvilinear) frame unless otherwise stated. The simplest (prototype) helical flow is

$$\mathbf{u} = \mathbf{U} + \tfrac{1}{2}\boldsymbol{\Omega} \wedge \mathbf{x}$$

where $\mathbf{U}$ and $\boldsymbol{\Omega}$ are constants. Then

$$\boldsymbol{\omega} = \text{curl}\,\mathbf{u} = \boldsymbol{\Omega}$$

0066-4189/92/0115-0281$02.00

and the helicity density $h(\mathbf{x}) = \mathbf{U}\cdot\mathbf{\Omega}$ is uniform. If $\mathbf{\Omega}$ is parallel to $\mathbf{U}$ (so that $h > 0$), the streamlines are right-handed helices about the z-axis with pitch $p = 4\pi U/\Omega = 4\pi h/\Omega^2$. Hence the origin of the term "helicity," which was introduced into the fluid mechanics literature by Moffatt (1969).

Helicity is important at a fundamental level in relation to flow kinematics because it admits topological interpretation in relation to the linkage or linkages of vortex lines of the flow. Suppose for example that $\boldsymbol{\omega}$ is identically zero except in two closed tubes $\mathscr{T}_1$ and $\mathscr{T}_2$ (Figurc 1), which may be linked. We suppose that $\mathscr{T}_1$ and $\mathscr{T}_2$ are unknotted, and that the vortex lines are untwisted within each tube, i.e. each vortex line is a closed curve passing once round the tube, and unlinked with its neighbors in the same tubc. If the tubes have vanishingly small cross-section, then the integral (1.1) degenerates to the sum of two line integrals round the axes $\mathscr{C}_1$, $\mathscr{C}_2$ of $\mathscr{T}_1$, $\mathscr{T}_2$ respectively:

$$\mathscr{H} = \kappa_1 \oint_{\mathscr{C}_1} \mathbf{u}\cdot d\mathbf{x} + \kappa_2 \oint_{\mathscr{C}_2} \mathbf{u}\cdot d\mathbf{x}, \tag{1.2}$$

where κ_1, κ_2 are the circulations associated with each tube. Now, by Stokes's theorem, the first integral is equal to the flux of vorticity across a surface spanning $\mathscr{C}_1$. This flux is $\pm n\kappa_2$, where n is the linking (or winding) number of $\{\mathscr{C}_1, \mathscr{C}_2\}$, and the $+$ or $-$ is chosen depending on whether the linkage is right- or left-handed. A similar result is found for the second integral. Hence

$$\mathscr{H} = \pm 2n\kappa_1\kappa_2 \tag{1.3}$$

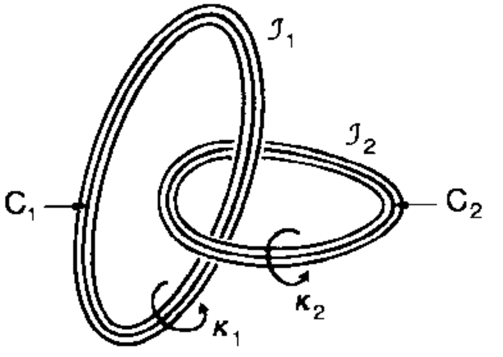

Figure 1 Prototype configuration of linked vortex tubes that gives nonzero helicity.

and thus $\mathscr{H}$ is directly related to the most basic topological invariant of two linked curves.

In general, of course, a vorticity distribution $\boldsymbol{\omega}(\mathbf{x})$ cannot be simply decomposed into a set of nonoverlapping vortex tubes; indeed the vortex lines of a general three-dimensional flow may be expected to diverge exponentially in the manner characteristic of third-order dynamical systems with chaotic trajectories. There is however always a simple decomposition of $\boldsymbol{\omega}$ into a sum of *three* overlapping fields for which the above interpretation of helicity remains meaningful (Figure 2). If $\mathbf{u} = [u_1(x,y,z), u_2(x,y,z), u_3(x,y,z)]$, then wc definc

$$\boldsymbol{\omega}_1 = (0, \partial u_1/\partial z, -\partial u_1/\partial y) \tag{1.4}$$

and similarly (by cyclic permutation) for $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$, so that $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3$. The vortex lines of the field $\boldsymbol{\omega}_1(\mathbf{x})$ are the closed curves $u_1 =$ constant, $x =$ constant and the associated "self-helicity" is zero; similarly for $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$. The helicity is

$$\mathscr{H} = \sum_{n \neq m} \mathscr{H}_{nm}, \mathscr{H}_{nm} = \int \mathbf{u}_n \cdot \boldsymbol{\omega}_m \, dV = \int \mathbf{u}_m \cdot \boldsymbol{\omega}_n \, dV \tag{1.5}$$

and $\mathscr{H}_{nm}$ is the "degree of linkage" of the two vorticity fields $\boldsymbol{\omega}_n$ and $\boldsymbol{\omega}_m$.

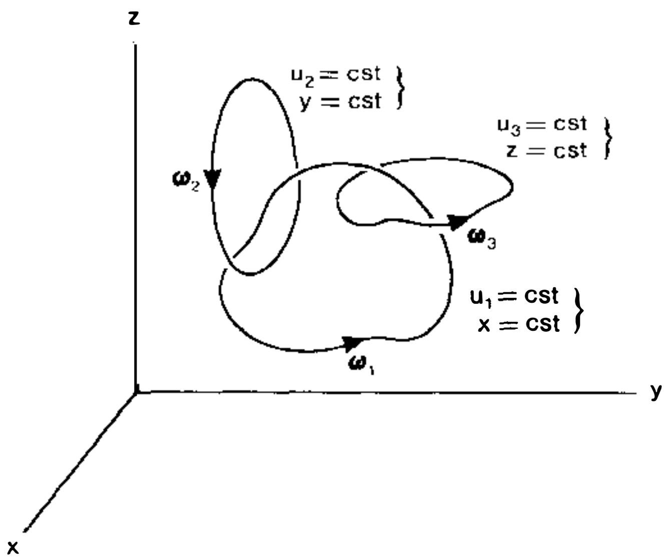

Figure 2 Decomposition of arbitrary vorticity field into three linked fields, each of which has trivial topology.

Although the concept of helicity is relatively recent in the fluid mechanical context, its roots go back to the seminal contributions of Helmholtz (1858) and Kelvin (1869). Kelvin recognized that, under evolution governed by the Euler equations for an ideal fluid, vortex lines behave like material lines, or, in modern parlance, they are "frozen in the fluid." This result is a consequence of the fact that the flux of vorticity through any open surface bounded by a curve moving with the fluid is conserved (again under Euler evolution). If the moving curve is chosen to be itself a closed vortex line, then it follows that the vorticity linked by this vortex line is conserved. It is this double application of the Helmholtz-Kelvin theorems that implies the "inviscid invariance" of helicity, a result that immediately gives helicity a status comparable to energy in the dynamics of ideal fluids.

The helicity conservation theorem may be stated as follows. Consider the velocity field $\mathbf{u}(\mathbf{x}, t)$ of an inviscid barotropic $[p = p(\rho)]$ fluid flowing under conservative body forces (note that these are precisely the conditions under which Kelvin's circulation theorem holds). Let $S(t)$ be any closed surface, moving with the fluid, on which $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ (a condition that persists for all t if it holds at $t = 0$, by virtue of the fact that the vortex lines are frozen in the fluid). Let $\mathscr{H}_S(t) = \int_V \mathbf{u} \cdot \boldsymbol{\omega} \, dV$ be the helicity in the volume V inside S. Then $d\mathscr{H}_S(t)/dt = 0$, so that $\mathscr{H}_S(t) = \text{constant}$.

This result was discovered by Moreau (1961), although it was foreshadowed by an earlier, and closely related, result of Woltjer (1958) in magnetohydrodynamics (see below). The result was rediscovered, together with its topological interpretation, by Moffatt (1969). The conditions under which the theorem holds are important, and have been frequently misunderstood. It may therefore be useful to repeat the proof, in its simplest form, here. The governing equations are

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F} \tag{1.6}$$

and, in consequence (taking due account of mass conservation)

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\right) \cdot \nabla \mathbf{u}, \tag{1.7}$$

where ρ is the fluid density, $p = p(\rho)$ the pressure, $\mathbf{F} = -\nabla\varphi$ the body force distribution, and $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$ the material (or Lagrangian) derivative. Writing $\mathscr{H}_S$ in the form

$$\mathscr{H}_S = \int_V \left(\frac{h}{\rho}\right) \rho \, dV, \tag{1.8}$$

we then have

$$\frac{d\mathscr{H}_S}{dt} = \int_V \frac{D}{Dt}\left(\frac{h}{\rho}\right)\cdot \rho\, dV. \tag{1.9}$$

Now

$$\frac{D}{Dt}\left(\frac{h}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)Q = \frac{1}{\rho}\nabla\cdot(\omega Q), \tag{1.10}$$

where

$$Q = \tfrac{1}{2}\mathbf{u}^2 - e - \varphi \tag{1.11}$$

and $e = \int dp/\rho$, the enthalpy per unit mass. Hence from (1.7),

$$\frac{d\mathscr{H}_S}{dt} = \int_V \nabla\cdot(\omega Q)\, dV = \int_S (\mathbf{n}\cdot\boldsymbol{\omega})Q\, dS = 0, \tag{1.12}$$

and the result follows.

In the special case in which the fluid is incompressible and ρ = constant, the above argument remains valid. Equation (1.8) then shows that

$$\frac{\partial h}{\partial t} + \nabla\cdot(\mathbf{u}h - \omega Q) = 0 \tag{1.13}$$

so that the *flux* of helicity is given by

$$\mathbf{F}_h = \mathbf{u}h - \omega Q. \tag{1.14}$$

It is important to note that, to every vorticity surface S (on which $\boldsymbol{\omega}\cdot\mathbf{n} = 0$) there corresponds a helicity invariant. If the vortex lines of a flow lie on a family of (nested) surfaces, then there is a corresponding family of helicity invariants. In general, however, as observed above, the vortex lines of a fully three-dimensional flow do not lie on surfaces, but wander in chaotic manner throughout the flow domain (see for example Dombre et al 1986, Bajer & Moffatt 1990); in this case there is only one helicity invariant for each subdomain within which the vortex lines are chaotic. Note however that, if $\boldsymbol{\omega}\cdot\mathbf{n} \neq 0$ on the boundary $\partial\mathscr{D}$ of the domain, then the total helicity of the flow is not invariant, since the essential surface condition is not satisfied. Helicity is then either created or destroyed in a boundary layer on $\partial\mathscr{D}$ (see the discussion of Moffatt 1969, Section 3). Viscosity is of course responsible for the reconnection of vortex lines and correspondingly for the evolution of helicity—effects explicitly revealed in the computations of Kida & Takaoka (1988) and recently by Aref & Zawadski (1991).

If the fluid fills all space (i.e. $\mathscr{D} = \mathbf{R}^3$), and if $|\boldsymbol{\omega}|$ is exponentially small as $|\mathbf{x}| \to \infty$, then the helicity integral is clearly convergent and constant since the surface integral in (1.10) (over the surface at infinity) vanishes. If $\boldsymbol{\omega}$ is a stationary random function of $\mathbf{x}$, as appropriate in the case of a field of homogeneous turbulence, then we may define the mean helicity

$$\mathscr{H} = \langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle, \tag{1.15}$$

where the angular brackets indicate either an ensemble average or (equivalently) a space average. We then have the result that, again under the three Kelvin conditions (inviscid fluid, barotropic flow, and conservative body forces), $\mathscr{H}$ = constant. If $\mathscr{H} \neq 0$, then the turbulence "lacks reflectional symmetry." Conversely, for any field of turbulence that is reflectionally symmetric (i.e. invariant under parity transformations) then necessarily $\mathscr{H} = 0$.

In this latter case, it may nevertheless happen that higher moments of the helicity distribution are constant (Levich & Tsinober 1983) and exert an influence on the statistics of the flow. Suppose for example that the space is divided into cells V_i bounded by surfaces S_i (which move with the fluid) on each of which the condition $\mathbf{n} \cdot \boldsymbol{\omega} = 0$ is satisfied (for all t), and let $h^{(i)} = \int_{V_i} \mathbf{u} \cdot \boldsymbol{\omega} \, dV$ be the helicity of the flow in the cell V_i. Then each $h^{(i)}$ is an inviscid invariant of the flow. Consider now a large volume V containing many such cells. We may define the moments

$$\mathscr{H}_n = \lim_{V \to \infty} \frac{1}{V} \sum_i h^{(i)n}$$

and these are all inviscid invariants. $\mathscr{H}_1$ is the mean helicity of the flow as previously defined. If the $h^{(i)}$ are randomly distributed with equal probability of positive and negative values, then

$$\sum_i h^{(i)} \sim V^{1/2}$$

and $\mathscr{H}_1 = 0$. However, all even moments (and in particular $\mathscr{H}_2$) are finite and nonzero; and although the mean helicity is zero, the fluctuations about the mean have constant variance.

The above argument depends critically upon the cell decomposition of the vorticity field. It cannot easily be extended to moments of the form $\langle (\mathbf{u} \cdot \boldsymbol{\omega})^n \rangle$, which do not in general appear to be invariant (except when $n = 1$).

2. THE TOPOLOGICAL NATURE OF THE HELICITY INVARIANT

We have already referred above to the interpretation of the helicity invariant in terms of the linkage of vortex tubes in the flow. Invariance of the helicity is then directly associated with invariance of the topology of the vorticity field.

Similarly, any solenoidal vector field that is convected without diffusion by a flow will have conserved topology and an associated helicity invariant. The prototype is the magnetic field **B** in a perfectly conducting fluid, which satisfies the frozen-field equation

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{u} \wedge \mathbf{B}), \qquad \nabla \cdot \mathbf{B} = 0, \tag{2.1}$$

where **u** is the convecting velocity field. The formal analogy between this equation and the vorticity equation of inviscid fluid dynamics is well-known. The difference however is that in the context of Equation (2.1), **B** is no longer constrained to be equal to curl **u**. In other words, Equation (2.1) admits consideration of far wider class of initial conditions.

If $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$, and if $\mathbf{B} \cdot \mathbf{n} = 0$ on a moving surface S containing volume V, then the magnetic helicity invariant is

$$\mathscr{H}_M = \int_V \mathbf{A} \cdot \mathbf{B}\, dV. \tag{2.2}$$

Note that this quantity is gauge invariant—i.e. invariant under replacement of **A** by $\mathbf{A} + \nabla\Psi$, where Ψ is any single valued scalar field. This type of invariance should not be confused with the time-invariance of $\mathscr{H}_M$, which follows from Equation (2.1). This latter invariance was discovered by Woltjer (1958). Integrals having this structure were encountered previously by Whitehead (1947) in a study of the famous Hopf invariant of topological mappings from S^3 to S^2. The integral (2.2) was described as the "asymptotic Hopf invariant" by Arnold (1974), a description that is appropriate when the **B**-lines are not simple closed curves, but may for example wander chaotically (see Arnold & Khesin 1992, this volume).

A field **B** has "trivial" topology if each **B**-line is an unknotted closed curve that may be shrunk to a point without having to cut through any other **B**-line in the process. By virtue of the arguments advanced in the previous section, the helicity of such a field is clearly zero. Starting with such a field, we may however construct a topologically nontrivial field by "surgical operation" as follows.

Let us start with a circular flux tube of small cross-section, in which each **B**-line is a circle running parallel to the axis of the tube. The helicity in this initial state is zero. Suppose that the magnetic flux (i.e. the integral of **B** across the cross-section of the tube) is Φ. Imagine that we now cut the tube at any section, twist it through an angle $2\pi\gamma$, and reconnect. If γ is a rational number, p/q, where p and q are coprime, then each **B**-line in the tube (with the exception of the central axis) is a torus knot, denoted $K_{p,q}$. Moreover, each pair of **B**-lines is now linked; i.e. the topology is definitely nontrivial.

If $\gamma = 1$, then each pair of **B**-lines is simply linked, and the helicity may be calculated by simply summing the contributions from each pair of elemental tubes:

$$\mathscr{H} = 2\int_0^{\Phi} \phi \, d\phi = \Phi^2. \tag{2.3}$$

More generally, for arbitrary values of γ, $\mathscr{H} = \gamma\Phi^2$. Obviously, by this type of surgery (i.e. by the introduction of twist in a tube), we may alter the helicity by any desired amount.

If the axis of the initial flux tube is a knotted closed curve C, then the field helicity arises partly through the nontrivial topology of the knot, and partly through the twist of the field within the tube. However, since the latter contribution can be varicd by an arbitrary amount with the same type of surgical operation as indicated above, the total helicity of the field may be adjusted to take any value of the form $\gamma\Phi^2$. This accounts for the differences in the evaluation of $\mathscr{H}$ for the particular example of the trcfoil knot in Moffatt (1969) and Berger & Field (1984), who adopted a particular convention for the field twist in the knotted tube. The most natural convention is perhaps to choose the twist that makes $\mathscr{H} = 0$, i.e. to compensate the helicity associated with torsion of the knot by helicity associated with twist within the knot tube. For this choice of twist (described as "zero-framing" in the topological literature) the linking number of any two **B**-lines in the tube is zero—although, paradoxically, they still generally remain linked (like the Whitehead link in which a circle links a figure of eight, once in each sense).

3. HELICITY AND THE TURBULENT DYNAMO

Helicity plays a central role in dynamo theory, i.e. the theory that is concerned with the growth of magnetic fields in electrically-conducting fluids, a subject that has been extensively treated in a number of research monographs (Moffatt 1978, Parker 1979, Krause & Rädler 1980, Zeldovich et al 1983). Modern treatment of the turbulent dynamo may be said to date from the discovery of the α-effect by Steenbeck et al (1966) (see Roberts & Stix 1971), although again, the seeds of this discovery may be traced to earlier work, particularly that of Parker (1955) and Braginskii (1964). This discovery is of fundamental importance, and has totally transformed our understanding of the dynamo mechanism. It provides one of the most remarkable examples of how order (in the form of a large-scale magnetic field) can arise out of chaos (in the form of small-scale turbulence with zero mean). The essential ingredient however is that the statistics of the turbulence should lack reflectional symmetry, the simplest manifestation being a nonzero mean helicity.

We describe here the mechanism of the α-effect in its simplest form. The equation for the magnetic field, allowing for the finite conductivity of the medium, is

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{u} \wedge \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{3.1}$$

where η is the magnetic diffusivity of the medium (inversely proportional to electrical conductivity), and $\mathbf{u}$ is the turbulent velocity field. We decompose $\mathbf{B}$ in the form

$$\mathbf{B} = \mathbf{B}_0(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t), \tag{3.2}$$

where $\mathbf{B}_0$ is the "mean" field varying on a scale L large compared with the scale l of the turbulence, and $\mathbf{b}$ is the fluctuation field generated by the turbulence. The mean and fluctuating parts of Equation (3.1) are then

$$\frac{\partial \mathbf{B}_0}{\partial t} = \operatorname{curl} \mathscr{E} + \eta \nabla^2 \mathbf{B}_0 \tag{3.3}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \operatorname{curl}(\mathbf{u} \wedge \mathbf{B}_0) + \operatorname{curl} \mathbf{G} + \eta \nabla^2 \mathbf{b}, \tag{3.4}$$

where $\mathscr{E} = \langle \mathbf{u} \wedge \mathbf{B} \rangle$ and $\mathbf{G} = \mathbf{u} \wedge \mathbf{B} - \mathscr{E}$.

Equation (3.4) establishes a linear relationship between $\mathbf{b}$ and $\mathbf{B}_0$, and so between $\mathscr{E}$ and $\mathbf{B}_0$. This relationship in general admits expansion in the form

$$\mathscr{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \dots, \tag{3.5}$$

where the (pseudo) tensor coefficients α_{ij}, β_{ijk}, etc. are determined by the statistical properties of the turbulence and the parameter η. Explicit determination of these coefficients requires solution of the fluctuation Equation (3.4).

The simplest situation is that in which the magnetic Reynolds number of the turbulence, $R_m = u_0 l/\eta$,
velocity). In this case, the awkward term curl $\mathbf{G}$ in (3.4) may be neglected, and the resulting linear equation may be solved by standard Fourier techniques. The result for the leading coefficient α_{ij} is

$$\alpha_{ij} = -\eta \int \frac{k_i k_j H(\mathbf{k}, \omega)}{\omega^2 + \eta^2 k^4} \, d\mathbf{k} \, d\omega \tag{3.6}$$

(see, for example, Moffatt & Proctor 1982), where $H(\mathbf{k}, \omega)$ is the "helicity spectrum" of the turbulence, i.e. the Fourier transform of the quantity

$$\langle \mathbf{u}(\mathbf{x}, t) \cdot \omega(\mathbf{x}+\mathbf{r}, t+\tau) \rangle. \tag{3.7}$$

This spectrum obviously has the property

$$\mathscr{H} = \langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle = \iint H(\mathbf{k}, \omega) \, d\mathbf{k} \, d\omega. \tag{3.8}$$

We thus have a direct and illuminating relationship between the helicity spectrum function and the leading order term in the important expansion (3.5).

If the turbulence is isotropic (and here we use this term in the weak sense, to indicate invariant under rotations of the frame of reference, but not necessarily under parity transformations) then the tensors α_{ij}, β_{ijk} etc. must also be isotropic, i.e.

$$\alpha_{ij} = \alpha \beta_{ij}, \qquad \beta_{ijk} = -\beta \varepsilon_{ijk}, \ldots, \tag{3.9}$$

where α is a pseudo-scalar and β a pure scalar. In this case, the helicity spectrum function is a function only of wave-number magnitude k and ω, and the expression (3.6) yields

$$\alpha = -\frac{\eta}{3} \iint \frac{k^2 H(k, \omega)}{\omega^2 + \eta^2 k^4} \cdot 4\pi k^2 \, dk \, d\omega. \tag{3.10}$$

Similarly, β may be expressed as a weighted integral of the energy spectrum function $E(k, \omega)$.

The mean field equation (3.3) now takes the form

$$\frac{\partial \mathbf{B}_0}{\partial t} = \alpha \nabla \wedge \mathbf{B}_0 + (\eta + \beta) \nabla^2 \mathbf{B}_0, \tag{3.11}$$

where we have assumed that α and β are uniform and constant, as appropriate for turbulence that is homogeneous and statistically stationary. It is easy to see that this equation admits unstable solutions. For example, consider field structures satisfying the "force-free" condition

$$\nabla \wedge \mathbf{B}_0 = K \mathbf{B}_0 \tag{3.12}$$

where K is constant; then obviously (3.11) becomes

$$\frac{\partial \mathbf{B}_0}{\partial t} = [\alpha K - (\eta + \beta) K^2] \mathbf{B}_0 \tag{3.13}$$

and we have exponentially growing behavior if

$$|\alpha K| > (\eta + \beta) K^2, \tag{3.14}$$

i.e. if the scale K^{-1} of the mean field $\mathbf{B}_0$ is sufficiently large. This, in essence, is the "α-effect" of mean-field electrodynamics.

It is important to note that the growing mean-field structure satisfying (3.12) is itself endowed with helicity, and indeed maximal helicity, since it is a Beltrami field. The associated Lorentz force is zero (hence the term "force-free"), and so in this idealized situation no large-scale mean flow is generated. There is nevertheless a saturation effect associated with the fact that the growing mean field ultimately tends to damp out the small-scale turbulence (Moffatt 1970, 1972; Soward 1975).

In astrophysical or planetary contexts, the structure of the growing field is constrained by the global geometry of the fluid domain, and the force-free condition (3.12) cannot then be satisfied. In these circumstances, the Lorentz force associated with the mean field is nonzero, and a large-scale mean flow is generated. This provides an alternative saturation mechanism (Malkus & Proctor 1975). Whichever saturation mechanism operates, the growth of the magnetic field is ultimately limited by the energy that is available to the turbulent flow, for transfer to the magnetic field. The magnetic field may be regarded, in somewhat philosophical terms, as the means by which the energy of the system is most conveniently dissipated.

The important point in relation to the present discussion is that the presence of nonzero mean helicity in the background turbulence is what is needed to provide a nonzero α-effect, and hence to make the medium unstable to the growth of a large-scale magnetic field. Although this effect was discovered 25 years ago, its many ramifications, and particularly its nature in the much more difficult situation of large magnetic Reynolds number (more relevant in astrophysical contexts), are still the subject of continuing research and controversy.

4. HELICITY AS A TOPOLOGICAL CONSTRAINT IN RELAXATION TO EQUILIBRIUM

Helicity also plays a crucial role in the problem of relaxation to magnetostatic equilibrium—a problem of central importance in the context of thermonuclear fusion plasmas. Here again, the initial step was taken by Woltjer (1958), who showed that if the magnetic energy associated with a magnetic field in a finite system is minimized subject to the single constraint that the total magnetic helicity is prescribed, then the minimizing magnetic field is force-free, i.e.

$$\nabla \wedge \mathbf{B} = \lambda \mathbf{B}, \tag{4.1}$$

where λ is constant. Thus, just as in the dynamo context, a Beltrami field is in some sense preferred.

This result was put to very good effect by J. B. Taylor (1974), who considered the process of relaxation of a magnetic field to magnetostatic equilibrium in a turbulent plasma. Taylor recognized that there is a helicity invariant corresponding to every magnetic surface (on which $\mathbf{B} \cdot \mathbf{n} = 0$) in the flow, but argued that only the global magnetic helicity (integrated over the whole fluid domain) could be expected to be invariant during the relaxation process. This principle, if accepted, immediately leads to Woltjer's variational problem, and to the emergence of force-free states of the form (4.1). For a plasma contained in a cylindrical domain, solutions of (4.1) in terms of Bessel functions are easily obtained and in particular, solutions may be found exhibiting an axial magnetic field that reverses sign at the outer boundary of the cylindrical cross-section—a behavior characteristic of the reversed field pinch (RFP). This theory is therefore a very attractive candidate as an explanation for the gross properties of the RFP.

Many attempts have been made to provide a justification for Taylor's hypothesis, but none appear totally satisfactory (Taylor 1986). The problem is that the finite magnetic diffusivity processes that are responsible for changing any of the "subhelicities" associated with subdomains of the fluid are equally responsible for changing the global helicity. Conversely, if global helicity is conserved, then it is hard to see why thc subhelicities are not conserved also. If, in Woltjer's variational problem, a Lagrange multiplicr is introduced for each of the subhelicities, then a force-free state of the form (4.1) again emerges, but now with λ as a function of position, constant on each field line (i.e. on each magnetic surface). It may be noted that the cylindrical force-free configurations considered by Taylor do have magnetic surfaces (the cylinders r = constant), so that in this context, λ may be any function of r. This allows more freedom than one would wish in a predictive theory.

Even this generalized variational problem doces not take account of the additional constraint of mass conservation. If this constraint is somehow included in the formulation, and if the total energy of the system (magnetic energy plus internal energy) is minimized, then the field relaxes to a magnetostatic equilibrium satisfying

$$\mathbf{j} \wedge \mathbf{B} = \nabla p, \tag{4.2}$$

where $\mathbf{j}$ = curl $\mathbf{B}$, and p is the pressure field in the plasma.

At this point, the theory becomes of great potential interest in the totally different context of classical inviscid fluid mechanics. For there is a well-known analogy between Equation (4.2) describing magnetostatic equilibrium and the equation

$$\mathbf{u} \wedge \boldsymbol{\omega} = \nabla s \tag{4.3}$$

describing steady solutions of the Euler equations in an inviscid incompressible fluid. The analogy here is between the variables **u** and **B** (both solenoidal), **j** and $\boldsymbol{\omega}$, and p and $s_0 - s$, where s_0 is constant. This analogy is exact, and if any solution, or family of solutions, of (4.2) can be found by any means, then there immediately corresponds a solution or family of solutions of Equation (4.3), provided the boundary conditions are compatible.

This correspondence has been exploited in a series of papers by Moffatt (1985, 1986a,b, 1988). The magnetic relaxation problem is considered in a fluid that is considered incompressible and viscous, but perfectly conducting. The perfect conductivity ensures that the topology of the magnetic field is conserved during the relaxation process; this means that not only are all the helicity invariants conserved, but also any more subtle topological properties are guaranteed to be conserved because **B** satisfies the frozen-field equation. Viscosity however allows a mechanism for energy dissipation, and the energy of the system decreases to the lowest level that is available to it, allowing for all the topological constraints. In the case of a knotted flux tube of volume V carrying flux ϕ, this minimum energy has the form $m(\gamma)\phi^2 V^{-1/3}$, where $m(\gamma)$ is a function determined solely by the topology of the knot (Moffatt 1990b).

The helicity plays an explicit role in this process, as first pointed out by Arnold (1974). By virtue of the Schwartz inequality

$$\left|\int \mathbf{A} \cdot \mathbf{B}\, dV\right| \leq \left\{\int \mathbf{A}^2\, dV \int \mathbf{B}^2\, dV\right\}^{1/2} \tag{4.4}$$

and the Poincaré inequality

$$\int \mathbf{B}^2\, dV \geq q_0^2 \int \mathbf{A}^2\, dV, \tag{4.5}$$

where q_0 is a positive constant dependent on the geometry and size of the fluid domain, the magnetic energy is bounded below according to

$$\int \mathbf{B}^2\, dV > q_0 \left|\int \mathbf{A} \cdot \mathbf{B}\, dV\right|. \tag{4.6}$$

Thus, if the magnetic helicity is nonzero, then the magnetic energy is definitely bounded away from zero. Actually, this result is much deeper than would appear from this simple presentation. It is the nontrivial topology implied by the nonzero helicity that guarantees that the magnetic energy cannot relax to zero. As shown by Freedman (1988), as long as the topology of the field is nontrivial, then, even if the global magnetic helicity is zero, the magnetic energy is still bounded away from zero. There are interesting examples of magnetic field topology (e.g. the Borromcan ring topology) for which the helicity is zero (magnetic field lines being pairwise unlinked) but for which the minimum magnetic energy is definitely nonzero. Such situations are covered by Freedman's theorem.

5. STRUCTURE OF TURBULENCE AND SUPPRESSION OF NONLINEARITY

The existence of a large family of steady solutions of the Eulcr equations (or "Euler flows") as determined by the above procedure, makes an attractive starting point for consideration of the problem of turbulence. Euler flows may be regarded as fixed points of the (infinite dimensional) dynamical system controlling the evolution of turbulence, and therefore provide a natural focus of investigation. Even if—as seems likely—these flows are in general unstable, nevertheless their structure, and the connections between them (heteroclinic orbits), may provide useful insights concerning the structure of turbulence.

An Euler flow, satisfying Equation (4.3), is characterized by a family of "stream vortex" surfaces s = constant, among which may be embedded vortex sheets, across which s is discontinuous. The flow may also contain subdomains in which s is identically zero, and $\boldsymbol{\omega} = \lambda \mathbf{u}$, where λ is constant on streamlines. Within these subdomains, there may exist smaller subdomains within which λ is constant, and in these subdomains, the field has maximal helicity (i.e. the Schwartz inequality (4.4) applied to the fields $\mathbf{u}$ and $\boldsymbol{\omega}$ now becomes an equality). The vortex sheets are necessarily separated from these regions of maximal helicity, and this has led to the speculation (Moffatt 1985) that there should be an anticorrelation between helicity and energy dissipation. This suggestion has stimulated a number of experimental and computational investigations, some of which will be described in the following section.

The vortex sheets in the above scenario are of course prone to Kelvin-Helmholtz instability, and have a natural tendency to wind up into double spiral structures. These structures, characterized by accumulation points of tangential discontinuities of velocity, can yield a power law spectrum $k^{-\mu}$, with μ a fraction between 1 and 2. If the spiral structure is just right, then the Kolmogorov spectrum ($\mu = \frac{5}{3}$) emerges (Moffatt 1984). A two-dimensional analysis of this type of process has been developed by Gilbert (1988). The same techniques have been recently applied (Moffatt 1991) using the spiral structure computed by Krasny (1986) to show that the Kolmogorov spectrum can indeed emerge from the Kelvin-Helmholtz instability. Viscous dissipation of course irons out the singularity at the center of the spiral, and provides an exponential cutoff to the spectrum.

By this mechanism, vortex sheets may be expected to wind up into vortex filaments having rather complex cross-sectional spiral structure, a conclusion that is not inconsistent with numerical simulation of three-dimensional turbulence at moderate Reynolds number (Kerr 1985, She et al 1990).

The absolutely stationary Euler flows may in fact be far too restrictive as a realistic starting point for analysis of turbulent structure. An alternative starting point is to regard turbulence as a "sea of interacting vortons," an idea that is related to the early suggestion of Synge & Lin (1943) that turbulencc might be approached through consideration of a random distribution of spherical vortices. The word "vorton" is used here to mean any rotational solution of the Euler equations that propagates with its self-induced velocity, without change of structure (Moffatt 1986b, 1988). The relaxation procedure described above can be adapted to establish the existence of an extremely wide family of such Euler flows both with and without helicity. Hill's spherical vortex is the prototype of the nonhelical vorton, while the "gyrostatic" vortex of Hicks (1899) is the prototype for the vorton with helicity.

We now picture a random sea of such vortons with nonlinear interactions occurring in regions of "grazing incidence" where high shear may be expected. If we define

$$(\mathbf{u} \wedge \boldsymbol{\omega})_S = \mathbf{u} \wedge \boldsymbol{\omega} - \nabla s \tag{5.1}$$

as the "solenoidal projection" of the Lamb vector $\mathbf{u} \wedge \boldsymbol{\omega}$, then a measure of "Eulerization" of the flow is given by the ratio

$$Z = \langle (\mathbf{u} \wedge \boldsymbol{\omega})_S^2 \rangle / \langle \mathbf{u}^2 \rangle \langle \boldsymbol{\omega}^2 \rangle, \tag{5.2}$$

as proposed by Kraichnan & Panda (1988). In computations of decaying homogeneous turbulence, Kraichnan & Panda found that the value of Z is significantly depressed below the value that it would have for a random velocity field with Gaussian distribution, the asymptotic valuc bcing 57% of the Gaussian value. This result is understandable in terms of the interacting vorton model, because $(\mathbf{u} \wedge \boldsymbol{\omega})_S$ is effectively zero within each vorton, so that contributions to Z come only from the regions of space in which these vortons are interacting. These are the regions in which shear instability and consequent spiral wind-up is occurring (Moffatt 1990a).

Such ideas are of course extremely speculative in character, but nevertheless act as a useful stimulus for experiments, both laboratory and computational. In the following sections we turn to the practical considerations involved in the experimental detection of effects associated with helicity in turbulent flow.

6. OBSERVATIONAL AND EXPERIMENTAL ASPECTS

6.1 *Helical Structures*

It is clear, at least intuitively, that a vortex having a nonzero axial component of velocity is characterized by nonzero helicity, i.e. is a helical structure. There is a great variety of structures of this kind, e.g. Taylor-Görtler vortices, leading edge and trailing vortices shed from wings and slender bodies, streamwise vortices in boundary layers and free shear flows, Langmuir circulations in the ocean, and analogous structures in the atmosphere—tornados and rotating storms [for references see, e.g. Tsinober & Levich (1983), and in geophysical contexts Lilly (1982, 1986), Etling (1985), and Hauf (1985)].

6.2 *Generation of Helicity in Initially Helicity-Free Flows*

Batchelor & Gill (1962) discovered that the most unstable mode (according to linear stability analysis) of a round axisymmetric jet is the mode with azimuthal wave number $m = \pm 1$; in other words the most unstable perturbation is a helical (spinning) mode lacking axial symmetry. Similar results were obtained subsequently in a number of papers: Kambe (1969), Lessen & Singh (1973), Lopez & Kurzweg (1979), Morris (1976), Plaschko (1979), and Strange & Crighton (1983).

Nonlinear analysis by Goldshtik et al (1983, 1985) revealed that the nonaxisymmetric nature of the most unstable modes can result in a steady rotation (clockwise or counterclockwise depending on the initial perturbation) of the initially nonrotating flow. This means that nonzero azimuthal velocity and axial vorticity—i.e. nonzero helicity—are generated in the initially helicity-free flow.

There is some experimental evidence (mostly qualitative) that helical modes are at least as unstable as their axisymmetric counterparts (Crow & Champagne 1971, Browand & Laufer 1977, Chan 1977, Dimotakis et al 1983). There is also evidence (both theoretical and experimental) that helical modes play an important role in wakes past axisymmetric bodies like spheres, circular disks, and other bodies of revolution (Achenbach

1974, Scholz 1986, Monkewitz 1988 and references therein). Helical modes and structures have been found in transitional and turbulent pipe flows (Sabot & Comte-Bellot 1976, Bandyopadhyay 1986, Stahl 1986).

No experimental evidence has so far been reported on the existence and/or development of self-induced rotation in round jets, wakes, and pipes. This has however been observed in some other axisymmetric flows (Kawakubo et al 1978 and Shingubara et al 1988, in flow around a sink; Torrance 1979, in a hot plume in a stratified environment; and Bojarevičs & Shcherbinin 1983, in an axisymmetric meridional flow due to an electric current source).

The spontaneous emergence of self-induced rotation is a special example of the breaking of reflectional symmetry, which is characterized by generation of nonzero helicity in an initially helicity-free flow.

6.3 *Quantitative Measurements of Helicity*

The detection of a lack of reflectional symmetry in a flow does not require direct measurements of the helicity field; it is sufficient to measure pseudoscalar correlations like $\langle \mathbf{u}(\mathbf{x}) \wedge \mathbf{u}(\mathbf{x}+\mathbf{r})\rangle \cdot \mathbf{r}$, which is closely related to the helicity of the flow (see below).

Simple as it seems, the main difficulty from the experimental point of view is that correlations of this kind are nonzero only for $r \neq 0$. For small values of r, this is therefore equivalent to measuring velocity derivatives (along with velocity as well). For large values of r the correlations become too small to be measured properly. There may exist an intermediate range of r at which these correlations can be measured satisfactorily. As yet, however, there have been no systematic attempts to do this. Instead, all attempts have been directed towards direct measurement of helicity density $u_i\omega_i$ or part of it, e.g. $u_1\omega_1$.

METHODS These attempts have been based on two recently developed methods of vorticity measurement (for a review of such methods, see Foss & Wallace 1989).

The idea underlying the first method comes from MHD and is based on the equation div $\mathbf{j} = 0$ and Ohm's law $\mathbf{j} = \sigma(-\nabla\phi + \mathbf{u} \times \mathbf{B})$, where $\mathbf{j}$ is electric current, ϕ the potential of the electric field, $\mathbf{u}$ the velocity, and $\mathbf{B}$ the magnetic field intensity. It follows that $\nabla^2\phi = \mathbf{B} \cdot \boldsymbol{\omega}$ (the term $\mathbf{u} \cdot$ curl $\mathbf{B}$ is several orders smaller). This enables one to obtain the component of vorticity ω_B parallel to the magnetic field, by measuring $\nabla^2\phi$ through the use of a seven-electrode probe, which realizes a central-difference approximation of the Laplacian. This idea was suggested by Grossman et al (1958) and has been implemented in turbulent grid flow of salted water on a small-scale facility (with test section of 5 cm $\times$ 5 cm cross-section) by Tsinober et al (1987). The method has the principal advantage of being an absolute one, i.e. it does not require any calibration; it provides the quantity $u_1\omega_1$ via simultaneous measurements of u_1 and ω_1 at the same location (Kit et al 1988, Dracos et al 1990, Kholmyansky et al 1990). This has been done with a probe consisting of the seven-electrode probe as described in Tsinober et al (1987) (for details of the manufacture, see Kholmyansky et al 1991) assembled together with a commercial 55R15 DANTEC hot film probe. By this method vorticity and velocity are measured independently (unlike the air flow measurements with multi-hot-wire techniques described below). The experiments were carried out in a salt water tunnel ($\simeq 1\%$ concentration) in the presence of a longitudinal magnetic field (1 Tesla). The use of salt water (low electric conductivity) ensures that the dynamic interaction of the flow and the magnetic field is entirely negligible, since in these conditions the electromagnetic force is several orders of magnitude smaller than both the viscous and inertial forces.

The second method is based on using multi-sensor hot-wire probes and appropriate calibration procedures to measure all the nine velocity gradients along with the three components of the velocity vector. This technique was first suggested by Balint et al (1987) (for further references see Foss & Wallace 1989). They used a nine-hot-wire probe consisting of three arrays each having three hot-wire sensors with a common prong (i.e. common resistance) in each array; and a calibration procedure based on certain restrictive assumptions about the symmetry of each array and the flow around the probe, using the effective velocity approach for implementation of King's law for multi-hot-wire probes. This, in turn, resulted in extremely tight requirements on the geometrical precision of the probe and its alignment in the flow. The technique has been used to measure helicity density and some related quantities in turbulent flow past a grid (Kit et al 1987, 1988; Wallace & Balint 1990), and in turbulent boundary layers and mixing layers (Wallace & Balint 1990). The technique has been improved by manufacturing a twelve-hot-wire probe consisting of three arrays each having four-hot-wire sensors without common prongs and a fully three-dimensional calibration procedure in velocity, pitch, and yaw (Dracos et al 1990; Tsinober et al 1990, 1991). This resulted in considerable reduction of errors resulting from crosstalking due to common resistance and the requirements of very precise alignment of the probe with the mean flow and extreme precision of its construction. This improved technique has been used to measure helicity density and some other quantities related to the field of velocity derivatives in turbulent grid flow and in the outer part of a turbulent boundary layer.

RESULTS One of the most significant results obtained for turbulent grid flows in the above papers (Kit et al 1987, 1988; Tsinober et al 1988; Dracos et al 1990; Kholmyansky et al 1991) is that the correlation coefficients between u_1 and ω_1, and between $\mathbf{u}$ and $\boldsymbol{\omega}$ are different from zero. When the initial disturbance was not controlled, the sign of $\langle u_1\omega_1\rangle$ was negative (salt water flow experiments) while the sign of $\langle u_i\omega_i\rangle$ was positive (air flow experiments). A special sct of experiments in salt water flow with controlled sign of $\langle u_1\omega_1\rangle$ was performed by Kholmyansky et al (1991). They used a grid with circular mesh and installed, in the grid holes, propellers producing rotational perturbations that were clockwise, anticlockwise, or both. In all cases when a particular sign of mean helicity (more precisely $\langle u_1\omega_1\rangle$) was expected (i.e. corresponding propellers installed) it was indeed observed. These results were also important as a check (at least qualitative) of reliability. It is noteworthy that the normalized quantity $\langle u_1\omega_1\rangle/\langle u_1^2\rangle^{1/2}\langle\omega_1^2\rangle^{1/2}$ increases downstream from the grid in most of the experiments and it is of special importance that the same behavior is observed also in a flow past a grid with empty holes.

In all cases the main contribution to $\langle u_1\omega_1\rangle$ comes from the largest scales as can be seen from the one-dimensional spectra for different cases shown in Figure 3. These scales are typically of the order of the diameter (~ 30 cm) of the test section. Although, at present, it is not understood why most of the mean helicity generated is in the large scales, there is a clear indication that a small disturbance in helicity is picked up and amplified by the turbulent flow past the grid. It should be borne in mind that the typical values of $\langle u_1\omega_1\rangle/\langle u_1^2\rangle^{1/2}\langle\omega_1^2\rangle^{1/2}$ or $\langle u_i\omega_i\rangle/\langle u^2\rangle^{1/2}\langle\omega^2\rangle^{1/2}$ were 0.05–0.1 (i.e. many of these values were within the experimental error), thereby demonstrating the particular importance of the experiments with controlled sign of mean helicity. The above results provide a clear indication that turbulent grid flow lacks reflectional symmetry, even when there is no deliberate injection of helicity.

The experiments in air flow allowed measurement of the full helicity density $u_i\omega_i$ and consequently the instantaneous angle θ between $\mathbf{u}$ and $\boldsymbol{\omega}$. The probability density function of the cosine of this angle was larger at $\theta = 0$ than at $\theta = \pi$ (Figure 4), at least in some cases, which is consistent with a positive value of $\langle u_i\omega_i\rangle$ in these experiments, and which also indicates some tendency of vorticity and velocity to be aligned. The positiveness of $\langle u_i\omega_i\rangle$ was checked by calculation of correlations of the type $\langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle$ with $i \neq j$ and $\mathbf{r} \neq 0$. Some of these correlations are nonzero in isotropic flow only if the latter lacks reflectional symmetry:

$$\varepsilon_{ijk}\langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle = g(r)r_k \tag{6.1}$$

where

$$g(0) = \tfrac{1}{6}\langle\mathbf{u}\cdot\boldsymbol{\omega}\rangle. \tag{6.2}$$

The correlations on the left-hand side of (6.1) were calculated using data

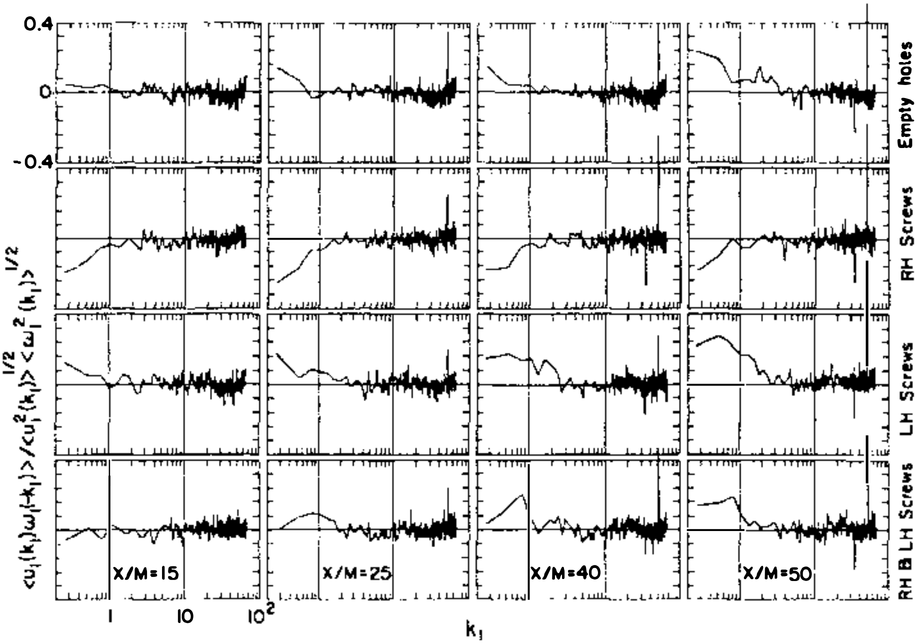

Figure 3 One-dimensional normalized helicity $(u_1\omega_1)$ spectra in salt water flow past grids (from Kholmyansky et al 1990).

for three different arrays of the probe and their values were consistent with the right-hand side of (6.1) assuming that $g(r) \sim g(0)$ for small r.

Some of the above results (for air flow only) have been questioned by Wallace & Balint (1990). In particular they explained the observation of nonzero mean helicity and the asymmetry in the probability distributions of the cosine of the angle between velocity and vorticity in turbulent grid flows as the result of "very small angle misalignments of the arrays with the mean flow.") This discrepancy may be due to a shortcoming of the method, as explained in Tsinober et al (1990, 1991) (resulting from the use of common prongs in the probe and a calibration procedure requiring very precise alignment of the probe with the mean flow and high precision of the probe construction).

6.4 *Application of Helicity Density as a Pseudoscalar Quantity for Characterizing Complex Three-Dimensional Flows*

Levy et al (1990) have very effectively used helicity density and normalized helicity density (i.e. cosine of the angle between the velocity and vorticity

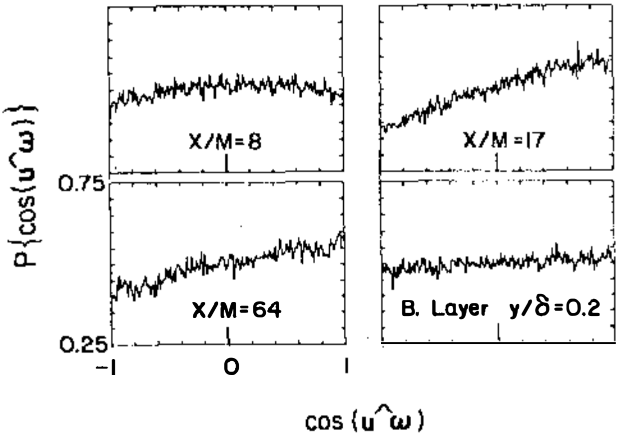

Figure 4 Probability density functions of the cosine of the angle between velocity and vorticity vectors at three distances (in mesh units) from the grid in air flows, and in a boundary layer at $y/\delta = 0.2$ (from Tsinober et al 1991).

vectors) for characterizing flow past an axisymmetric body at nonzero anglc of attack. In particular they uscd the fact that $\mathbf{u}\cdot\omega/u\omega$
in the vicinity of the concentrated vortex core axis, so that by mapping the relative helicity it was possible to locate the vortex core and its axis (see also Melander & Hussain 1990). Another interesting aspect is that helicity density changes sign across a separation or reattachment line, a useful property in locating these lines. Finally mapping of helicity density allows one to locate secondary vortices (see Figure 5).

6.5 *Possible Relation Between Helicity and the Vortex Breakdown*

The term vortex breakdown is normally used to denote a class of phenomena in concentrated vortices having an axial component of velocity (for recent reviews see Escudier 1988, Leibovich 1991). These phenomena are characterized by qualitative changes of the flow geometry (or topology) leading to formation of organized structures of various kinds depending on flow conditions, well illustrated in the Vogel-Ronnenberg flow in a cylinder with one rotating lid (Figure 6, from Escudier 1984). It is noteworthy that (concentrated) vortex flows with nonzero axial velocity typically possess large helicity. Qualitative changes in the topology of flows resulting from vortex breakdown are therefore presumably closely related to large changes in helicity, which may therefore be an appropriate parameter for the characterization and classification of such flows.

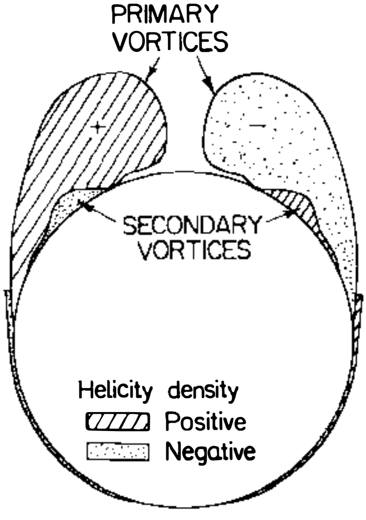

Figure 5 Helicity density in a cross section of a flow around an ogive cylinder at a 40° angle of attack (from Levi 1988).

7. NUMERICAL EXPERIMENTS

The numerical investigation of the influence of helicity on turbulent flows was initiated by André & Lesieur (1977). Using an "eddy-damped quasi-normal markovianized" two-point closure (EDQNM), they showed that homogeneous turbulence with significant mean helicity has a much slower rate of decay than that with zero mean helicity. This result was confirmed by Van (1987 unpublished) and Polifke & Shtilman (1989) via direct numerical simulation.

One of the main objectives in numerical simulation has been determination of the statistics of the angle between the velocity and vorticity vectors. The first investigation of this kind was by Pelz et al (1985), who found that a large probability of alignment exists between velocity and vorticity for the flow that develops from an initial "Taylor-Green" state (see Brachet et al 1983) in which the initial helicity density is identically zero. They also found that, as suggested by the discussion of Section 5 above, this alignment is more pronounced in regions with low dissipation (see also Shtilman et al 1985). Subsequent computations however (Pelz et al 1986, Kerr 1987, Ashurst et al 1987, Rogers & Moin 1987, Shtilman et al 1988, Herring & Metais 1989) of nearly isotropic turbulence (both forced and decaying) have shown that this alignment tendency is not so strong as originally believed: The probability density of the cosine of the angle θ between the two vectors was at best 30% larger at $\cos\theta = \pm 1$ than at

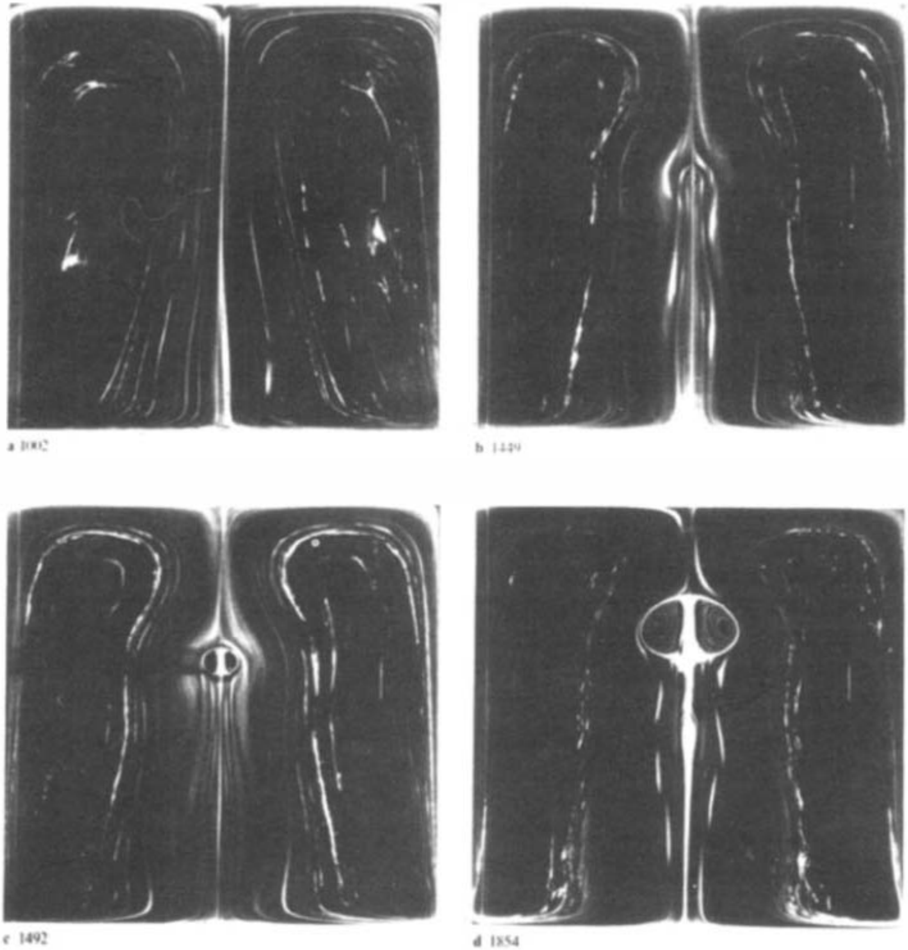

Figure 6 Flow structure in a stationary cylinder with lower rotating lid for $H/R = 2$ and four valucs of Ω^2/ν. H–height, R–radius of the cylinder, Ω–angular velocity of the rotating lid (from Escudier 1984).

$\cos\theta = 0$ (see also Kraichnan & Panda 1988). Moreover, conditional sampling techniques provided no evidence of correlation between low dissipation and high alignment. As observed by Tsinober (1990), high helicity implies low dissipation (since in the extreme situation, there is no nonlinear energy transfer to small scales in regions where **u** is parallel to $\boldsymbol{\omega}$) but low dissipation does not necessarily imply high helicity (e.g. the flow may be locally an Euler flow, but not a Beltrami flow). Thc conditional sampling referred to above was conditioned on low dissipation; a different result might be obtained if the sampling were based on a condition of high helicity density. An anticorrelation does seem to exist in free shear flows according to direct numerical simulations of Metcalfe (see Hussain 1986), Melander & Hussain (1988), and Hussain et al (1988), which show a distinct separation between regions of high helicity density and large dissipation. On the other hand, Rogers & Moin (1987) have investigated homogeneous shear flow, and fully developed channel flow, and have found no evidence for this anticorrelation.

A difficulty in all investigations of this kind is that, unlike the helicity integrated over a Lagrangian domain, the helicity density is not Galilean invariant. If the "sea of interacting vortons" scenario has any validity—these vortons being the coherent structures of the turbulence—then one should compute the helicity density at any point in the frame of reference moving with the dominant large-scale structure. Techniques for achieving this degree of sophistication have not yet been adequately developed.

Much less is known about the behavior of the overall helicity of these flows. In the case of nearly isotropic flow, with initially nonzero mean helicity (Pelz et al 1986, Polifke & Shtilman 1989) the balance of evidence is that the mean helicity decays monotonically to zero, although Shtilman et al (1988) have found fluctuations in time, and even changes of sign of the mean helicity. When the initial state has zero mean helicity, but is weakly reflectionally asymmetric at higher order, the indications (using Clebsch variables) are that substantial nonzero mean helicity may develop—a result that can be interpreted in terms of statistical instability of turbulent flow with reflectional symmetry to disturbances breaking this symmetry. Claims have been made by Kit et al (1988) and Levich & Shtilman (1988) (on the basis of rather limited computational information) concerning spontaneous symmetry breaking and the buildup of coherence in large scales. The arguments are based on the invariance of the measure of helicity fluctuations (Levich & Tsinober 1983) as described in Section 1 above. The claims have been questioned by Polifke (1991), who points out that a small phase coherence in small scales can be sufficient to break the (adiabatic) invariance of the Levich & Tsinober "I-invariant." In his computations, Polifke observed substantial phase coherence in small scales, i.e. the helicity spectrum was of the same sign for a considerable range of high wave numbers, and of larger magnitude than follows from a quasi-Gaussian approximation (Yamamoto & Hosokawa 1981). However, in some cases the sign of the helicity spectrum at high wave numbers was wrong in relation to the required dissipation of mean helicity. The mean helicity in fact exhibits a behavior very similar to that obtained by Shtilman et al (1988).

8. FURTHER PERSPECTIVES

There are further situations in which helicity, or more generally lack of reflectional symmetry, is of basic importance. These will be briefly described in this concluding section.

8.1 *Skew Diffusion*

The influence of helicity on the diffusion of a passive scalar by turbulence has been considered by Moffatt (1983). If the turbulence acts on a mean gradient **G** of the scalar, then a turbulent flux **F** is generated. A two-scale analysis analogous to that used in mean-field electrodynamics (Section 4 above) yields a linear relationship between **F** and **G**. At leading order this takes the simple form

$$F_i = -\mathscr{D}_{ij}G_j, \tag{8.1}$$

where $\mathscr{D}_{ij}$ is a tensor determined by the statistical properties of the turbulence and the molecular diffusion coefficient of the scalar relative to the fluid. The symmetric part of $\mathscr{D}_{ij}$ provides an eddy diffusivity, which—in the limit of vanishing molecular diffusivity—is just the "diffusion by continuous movements" described by Taylor (1921).

$\mathscr{D}_{ij}$ may however have an antisymmetric part also,

$$\mathscr{D}^{(a)}_{ij} = \varepsilon_{ijk}\mathscr{D}^{(a)}_k. \tag{8.2}$$

This can yield an additional contribution to the heat flux parallel to **G**:

$$\mathbf{F}^{(a)} = \mathbf{D}^{(a)} \wedge \mathbf{G}, \tag{8.3}$$

a phenomenon that may be described as "skew diffusion." The pseudovector $\mathbf{D}^{(a)}$ is nonzero only if the turbulence lacks reflectional symmetry. If first-order smoothing theory, of the type described in Section 3, is used, then $\mathbf{D}^{(a)}$ is related to the helicity spectrum function by

$$\mathbf{D}^{(a)} = -\frac{1}{2}\iint \frac{\omega \mathbf{k} H(\mathbf{k}, \omega)}{k^2(\omega^2+\eta^2 k^4)} \mathbf{d}k\, d\omega. \tag{8.4}$$

This integral is zero if H is symmetric with respect to the sign of the frequency parameter ω; but in general there is no need for such a constraint on H, and the integral is nonzero. The effect can also be associated with a lack of time reversibility in the statistics of the turbulence and in this sense the appearance of skew diffusion is a manifestation of the breaking of the symmetry relations of Onsager (1931a,b).

Even for two-dimensional flow, the skew diffusion effect can appear—although not at the level of first-order smoothing. It is in fact lack of reflectional symmetry at the level of cubic averages that is responsible for the effect. In physical terms, if the eddies in one sense (say anticlockwise) dominate over those in the opposite sense, then again, the turbulence is not statistically invariant with respect to time reversal, and the skew diffusion effect appears, the pseudo-vector $\mathbf{D}^{(a)}$ being proportional to the cube of the Peclet number, when this is small. This type of effect can play a part in the diffusion of scalars by two-dimensional turbulence in the ocean and atmosphere (Garrett 1980).

8.2 *The AKA-Effect*

The close similarity between the induction equation and the vorticity equation

$$\frac{\partial \omega}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \omega) + \nu \nabla^2 \omega \tag{8.5}$$

in a fluid of kinematic viscosity ν suggests that the methods of mean field electrodynamics may be applied, at least in a formal sense, to the vorticity field. It has to be "in a formal sense" because there seems to be no natural division of the vorticity field into a large-scale component and a small-scale component. Nevertheless, the formal manipulations can be carried out, and reveal some interesting structural properties. In this case, the two-scale analysis yields a linear relationship between the Reynolds stress associated with the turbulence and the mean velocity field:

$$\langle u_i u_j \rangle = -\Lambda_{ijk} \langle u_k \rangle - \mathcal{N}_{ijkl} \frac{\partial \langle u_k \rangle}{\partial x_l} + \dots \tag{8.6}$$

(Krause & Rudiger 1974). Here again, the tensor coefficients Λ_{ijk}, $\mathcal{N}_{ijkl}$, etc are determined in principle by the statistical properties of the turbulence. Thc leading order term is not invariant under Galilean transformation, and is therefore zero unless there is some non–Galilean invariant forcing in the turbulence. The second term of the series is more natural, having the structure of an eddy viscosity term, plus possibly certain subsidiary effects.

The possibility of a contribution at leading order has been reopened by Frisch et al (1987), who have described the effect (by analogy with the α-effect) as the "anisotropic kinetic α-effect" or simply AKA-effect. A specific low Reynolds number model is considered, in which the turbulence is driven by a random body-forcc distribution. The leading coefficient Λ_{ijk} can then be calculated explicitly in terms of the statistical properties of the force-field. A particular example of a force-field yielding a nonzero effect is constructed:

$$\mathbf{f} = f_0[\cos(ky+\omega t), \cos(kx-\omega t), \cos(ky+\omega t)+\cos(kx-\omega t)]. \quad (8.7)$$

This is a space-periodic force-field whose pattern moves relative to the fluid with velocity

$$\mathbf{v} = (\omega/k, -\omega/k, 0). \quad (8.8)$$

It is this feature that leads to a breaking of the Galilean invariance and the possibility of a nonzero Λ_{ijk}. In a subsequent paper, Sulem et al (1989) pursued the resulting instability into the nonlinear regime, and showed that the energy is increasingly transferred to the smallest wave numbers available to the system. The large-scale turbulent flow that is generated by this mechanism has nonzero helicity. Note that the force-field (8.7) itself has zero helicity, but it is noninvariant under time-reversals ($t \to -t$), a property closely related to lack of reflectional symmetry.

A similar analogue of the α-effect was found by Moiseev et al (1983) (see also Tseskis & Tsinober 1974) in the context of turbulence in a compressible fluid. Whatever the nature of this effect, the Galilean invariance argument suggests that its origin must still be traced to some input to the problem that is itself non–Galilean invariant [like the force-field (8.7)].

8.3 *Relaxation Under Modified Euler Equation Evolution*

As emphasized at the outset, helicity is an inviscid invariant of the Euler equations by virtue of the fact that the vortex lines are frozen in the fluid. If we modify the Euler equations by replacing the actual fluid velocity U by an arbitrary incompressible differentiable velocity field $\mathbf{v}$, then, by the same token, helicity remains an invariant of the modified equation, and indeed the topology of the vorticity field is conserved by thc modified equation. However, in general, the kinetic energy of the flow is no longer an invariant of the modified equation. Vallis et al (1989) exploited this property in an investigation of relaxation procedures that may lead to steady solutions of the Euler equations for which the topology of the vorticity field (rather than that of the velocity field) is prescribed. In this case, the combination of Schwartz and Poincaré inequalities leads to a lower bound not on the energy of the flow but on its enstrophy:

$$\langle \omega^2 \rangle \geq q_0 |\mathscr{H}|. \quad (8.9)$$

There is no bound on the energy, and therefore no guarantee that the relaxation process leads to a nontrivial steady state.

For the special case of two-dimensional flow however, there is the additional constraint that the enstrophy itself is conserved. The Poincaré inequality then implies the existence of an *upper* bound on the energy. Vallis et al use the choice

$$\mathbf{v} = \mathbf{u} + \alpha \frac{\partial \mathbf{u}}{\partial t}, \tag{8.10}$$

where α is a negative constant, and the associated energy equation

$$\frac{d}{dt}\langle \mathbf{u}^2 \rangle = -\alpha \langle \mathbf{u}^2 \rangle, \tag{8.11}$$

to show that in this case the relaxation process guarantees the existence of a steady solution of the Euler equations with prescribed topology of the contours of constant vorticity. Clearly, this steady state is characterized by maximum kinetic energy with respect to displacements of the fluid that convect the vorticity field ("isovortical perturbations"). Flows constructed in this way are therefore stable (Benjamin 1976).

This type of procedure has been considered from a general Hamiltonian standpoint by Shepherd (1990). It is not as yet clear whether the proposed procedure can be implemented in order to compute new stable solutions on the Euler equations. If this can be done, either by the prescription of Vallis et al, or by some alternative prescription, then the procedure is likely to be of the greatest importance, not only for laminar flow theory, but also in relation to the coherent structures in turbulence.

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