



DIRECT LYAPUNOV METHOD IN NONLINEAR STABILITY OF COMPLEX FLOWS.

V.A.VLADIMIROV¹ and H.K.MOFFATT²

- ¹ Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.
² Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

EXTENDED SUMMARY

INTRODUCTION: Analytical studies of the problem of hydrodynamic stability may be divided into two general classes. The first is based on an explicit solution of the equations of motion. The spectral or normal-mode method being the most celebrated example. The second class, in contrast, avoids a detailed solution of the equations of motion and proceeds by the construction of some functional of the total problem. An important subset of this class is what is often referred to as the *Direct Lyapunov Method (DLM)*, following the pioneering work by Lyapunov (1892) on finite-dimensional mechanical systems. In the past ten years there have been considerable advances in the application of the *DLM* to hydrodynamic stability problems along two distinct lines. The first *the Arnold Stability (AS)* consists of sufficient conditions for stability and represents now a classical and well developed field of research (see Fjørtoft 1950, Arnold 1965a,b, Holm et al. 1985, McIntyre & Shepherd 1987, Vladimirov 1987). The second *the Virial Instability (VI)* consists of sufficient conditions for instability and is a new promising method of the theory of hydrodynamic stability (see Vladimirov 1990, Vladimirov & Rumyantsev 1989,90, Vladimirov & Belov 1988). The main advantage of both *AS* and *VI* methods is the obtaining of stability/instability criteria in a general analytical form for wide classes of steady flows or equilibria. Both approaches are especially useful because they allow one not only to prove stability or instability but also to obtain *a priori* estimates of solutions: lower and/or upper bounds for 'stable' or 'growing' perturbations. During the last few years the main research in this area has been concentrated in expansion of the *DLM* (mainly *AS* approach) to more complex systems than has been done earlier. The main purpose of this paper is to review recent results by the authors and their collaborators in the area of *DLM* in Hydrodynamics. There are three connected directions in this review:

1. **STABILITY OF A SOLID IN A ROTATIONAL FLOW OF AN IDEAL INCOMPRESSIBLE FLUID:** The stability of an equilibrium of a two-dimensional solid in a two-dimensional (planar) steady rotational flow of an ideal incompressible fluid is studied by Arnold's method. The body is under the action of both hydrodynamic pressure forces and an external potential force (acting only on the body). We establish a new energy-type variational principle for such an equilibrium, which is a generalization of a well-known result of Arnold (1965a, 1966). For our Energy-Casimir functional \mathcal{I} we

show that any steady motion of the system 'solid + fluid' represents a stationary point $\delta\mathcal{I} = 0$. The second variation $\delta^2\mathcal{I}$ is calculated for the general case of a solid placed in an arbitrary rotational flow. The decomposition $\delta^2\mathcal{I} = \delta^2\mathcal{I}_{fluid} + \delta^2\mathcal{I}_{cross} + \delta^2\mathcal{I}_{solid}$ is valid, where $\delta^2\mathcal{I}_{fluid}$ is the volume integral corresponding to the fixed boundaries (Arnold's case); $\delta^2\mathcal{I}_{solid}$ corresponds to the motions of the solid itself, and $\delta^2\mathcal{I}_{cross}$ is a surface integral depending on variation fields of both fluid and solid. Two methods to reduce $\delta^2\mathcal{I}$ to positive definite form are presented. First one is based on conservation of the angular momentum for axisymmetric flows. In the second method we use the positive definite volume integral $\delta^2\mathcal{I}_{fluid}$ to make $\delta^2\mathcal{I}_{fluid} + \delta^2\mathcal{I}_{cross} > 0$, so the existence of a symmetry is not required. The general results are applied to obtain the sufficient conditions for linear stability in two particular problems, namely: the stability of the flow with circular streamlines between two cylinders where the inner one represents the moving solid, and the stability of the steady rotation of an arbitrary solid inside the circular domain filled with a fluid.

2. STABILITY OF STEADY FLOWS OF BUBBLY FLUIDS: The Arnold stability and virial instability approaches are developed for two-fluid models of a bubbly fluid. As a first step a new form of Hamiltonian variational principle is introduced to obtain a general form of the equations of motion. Then the Energy-Casimir variational principle for general steady two-dimensional solutions of these equation is formulated. The second variation of energy is used to obtain stability conditions for shear flows. The effect of virtual-mass on the stability of the simplest steady flows is studied. It is shown that even for small values of volume fraction α of bubbles it is necessary to take into account quadratic in α terms in the expansion of the virtual-mass coefficient to provide the stability of the simplest steady solutions. Instability of two-fluid equilibria under the saddle point condition of the energy functional is proven by the virial method. In all cases *a priori* lower and/or upper bounds of perturbations are presented.

3. STABILITY OF MHD FLOWS: A new variational principles for magnetohydrodynamic (MHD) flows of an ideal incompressible fluid are established. For such flows, the magnetic field $\mathbf{h}(\mathbf{x}, t)$ is frozen in the fluid, but the vorticity field is not frozen since the Lorentz force is in general rotational. We constructed a 'modified vorticity field' $\mathbf{w}(\mathbf{x}, t)$ which is frozen in the fluid, and which reduces to vorticity field when $\mathbf{h} \equiv 0$. The existence of this additional frozen-in field has consequences for the construction of Casimirs, the integral invariants which play an essential role in the derivation of sufficient conditions for stability (or 'stability criteria') for steady solutions $\{\mathbf{U}(\mathbf{x}), \mathbf{H}(\mathbf{x})\}$ of the governing equations. Two alternative energy-type variational principles for three dimensional steady MHD flow are established. Both involve a functional R which is the sum of the total energy and another considered functional, the volume integral of a function Φ of Lagrangian coordinates. It is shown that the first variation $\delta^1 R$ vanishes if Φ is suitably chosen (as minus a generalised Bernoulli integral). Expressions for the second variation $\delta^2 R$ are presented. As a second step we consider here flows with simple symmetries (two-dimensional and axisymmetric), and we demonstrate first certain helpful analogies (or more accurately isomorphisms) between such flows and flows of a stratified fluid in the Boussinesq approximation.

Analogies between stratified and rotating flows are known (see Rayleigh 1916, Veronis 1970) and have been placed on a rigorous basis and exploited Vladimirov (1985a,b). The fact that analogies may exist in MHD situation of the type to be considered here was already noted by Howard & Gupta (1962), but the precise nature of the analogy has not been previously revealed. We use the analogy in present paper and in fact it was through consideration of the analogy that the frozen field \mathbf{w} was discovered in Vladimirov & Moffatt (1995). Using this analogy we shall show how the presence of two frozen fields, \mathbf{h} and \mathbf{w} , leads to determination of an appropriate Casimir as an integral over the fluid domain of a function of two conserved scalar fields. We then follow the prescription of Arnold (1965a,b) to obtain stability criteria.

CONCLUSIONS: We have shown that the Direct Lyapunov Method is an effective approach to solve nonlinear stability problems for complex flows. A discussion of further interesting directions of research is presented.

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