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On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part 2. Stability criteria for two-dimensional flows

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The techniques developed in Part 1 of the present series are here applied to two-dimensional solutions of the equations governing the magnetohydrodynamics of ideal incompressible fluids. We first demonstrate an isomorphism between such flows and the flow of a stratified fluid subjected to a field of force that we describe as 'pseudo-gravitational'. We then construct a general Casimir as an integral of an arbitrary function of two conserved fields, namely the vector potential of the magnetic field, and the analogous potential of the 'modified vorticity field', the additional frozen field introduced in Part 1. Using this Casimir, a linear stability criterion is obtained by standard techniques. In §4, the (Arnold) techniques of nonlinear stability are developed, and bounds are placed on the second variation of the sum of the energy and the Casimir of the problem. This leads to criteria for nonlinear (Lyapunov) stability of the MHD flows considered. The appropriate norm is a sum of the magnetic and kinetic energies and the mean-square vector potential of the magnetic field.

1. Introduction

In this paper, we develop the approach initiated in Part 1 (Vladimirov & Moffatt 1995), in which new variational principles for magnetohydrodynamic (MHD) flows of an ideal incompressible fluid were established. For such flows, the magnetic field $\boldsymbol{h}(\boldsymbol{x},t)$ is frozen in the fluid, but the vorticity field $\boldsymbol{\omega}(\boldsymbol{x},t)$ is not frozen since the Lorentz force is in general rotational. However, in Part 1 we identified a 'modified vorticity field' $\boldsymbol{w}(\boldsymbol{x},t)$ (see (1.8) below) which *is* frozen in the fluid, and which reduces to $\boldsymbol{\omega}$ when $\boldsymbol{h} \equiv 0$. The existence of this additional frozen-in field has consequencies for the construction of Casimirs, the integral invariants which play an essential role in the derivation of sufficient conditions for stability (or 'stability criteria') for steady solutions $\{\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{H}(\boldsymbol{x})\}$ of the governing equations.

We specialize here to two-dimensional flows (invariant under translations in the z-direction), and we demonstrate first certain helpful analogies (or more accurately isomorphisms) between such flows and flows of a stratified fluid in the Boussinesq approximation. Analogies between stratified and rotating flows can be traced back to Rayleigh (1916) and have been placed on a firm basis by Vladimirov (1985*a*, *b*). The fact that an analogy may exist in MHD situations of the type to be considered here was already noted by Howard & Gupta (1962), but its precise nature has not been previously revealed. It was in fact through consideration of the analogy that the frozen field $\boldsymbol{w}$ was discovered in Part 1; we exploit the analogy further in the present paper.

In the two-dimensional situation, we first show how the presence of two frozen fields, $\boldsymbol{h}$ and $\boldsymbol{w}$, leads to determination of an appropriate Casimir as an integral over the fluid domain of a function of two conserved scalar fields (associated with $\boldsymbol{h}$ and $\boldsymbol{w}$). We then follow the procedures of Arnold (1965, 1966) to obtain stability criteria. The linear criteria obtained in §3 are equivalent to criteria obtained previously by Holm *et al.* (1985). We then consider nonlinear stability of the steady state, i.e. Lyapunov stability with respect to a norm based on the total energy and the mean-square vector potential of the perturbed magnetic field. Here we go beyond the treatment of Holm *et al.* (1985) which we believe to be incorrect (see footnote on p. 195). We consider 'isomagnetic' perturbations under which the field is a 'frozen-field' perturbation of the steady state, and we obtain conditions (Criterion 4.1) under which the norm of the perturbation remains bounded by a constant multiple of its initial value for all time. We then extend the proof to cover arbitrary two-dimensional perturbations (Criteria 4.2 and 4.3); the difficulty here centres on the problem of appropriate continuation of functions describing the steady state beyond their initial range of definition; this difficulty is addressed in detail and is successfully overcome.

We conclude this introduction with a statement of the governing equations. We suppose that the fluid is incompressible, homogeneous and ideal, i.e. inviscid and perfectly conducting, and that it is contained in a domain $\mathscr{D}$ with fixed boundary $\partial\mathscr{D}$. (We shall in general suppose that $\mathscr{D}$ is bounded, but the theory may be easily modified to deal with the case of an unbounded domain.) Let $\boldsymbol{u}(\boldsymbol{x},t)$ be the velocity field, $\boldsymbol{h}(\boldsymbol{x},t)$ the magnetic field (in Alfvén velocity units), $p(\boldsymbol{x},t)$ the pressure field (divided by density), and $\boldsymbol{j} = \nabla\wedge\boldsymbol{h}$ the current density in the fluid. Then the equations governing the evolution of these fields are

$$\mathrm{D}\boldsymbol{u} \equiv \left(\partial/\partial t + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} = -\nabla p + \boldsymbol{j}\wedge\boldsymbol{h}\,, \tag{1.1}$$

$$\mathrm{L}\boldsymbol{h} \equiv \partial\boldsymbol{h}/\partial t - \nabla\wedge(\boldsymbol{u}\wedge\boldsymbol{h}) = 0\,, \tag{1.2}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{h} = 0\,. \tag{1.3}$$

Here D is the material derivative, and L the Lie derivative governing the evolution of a convected solenoidal field; the equation $\mathrm{L}\boldsymbol{h} = 0$ implies that $\boldsymbol{h}$ is frozen in the fluid, its flux through any material circuit being conserved.

We suppose that, consistent with (1.2), the field $\boldsymbol{h}$ is confined for all time to the fluid region; the boundary conditions are then

$$\boldsymbol{n}\cdot\boldsymbol{u} = 0\,, \quad \boldsymbol{n}\cdot\boldsymbol{h} = 0 \quad \text{on } \partial\mathscr{D}\,. \tag{1.4}$$

We suppose further that at $t = 0$, the fields $\boldsymbol{u}$ and $\boldsymbol{h}$ are smooth and satisfy (1.3), but are otherwise arbitrary.

Equation (1.1) has the consequence that

$$\mathrm{L}\boldsymbol{\omega} = \nabla\wedge(\boldsymbol{j}\wedge\boldsymbol{h}) , \tag{1.5}$$

where $\boldsymbol{\omega} = \nabla\wedge\boldsymbol{u}$ is the vorticity field. This means that vortex lines are *not* frozen in the fluid when the Lorentz force $\boldsymbol{j}\wedge\boldsymbol{h}$ is rotational. However, as shown in Part 1, a 'generalized' vorticity field $\boldsymbol{w}$ may be defined as follows: let $\boldsymbol{g}(\boldsymbol{x},t)$ be an arbitrary solenoidal field satisfying

$$\nabla\wedge(\boldsymbol{g}\wedge\boldsymbol{h}) = 0 , \tag{1.6}$$

and let $\boldsymbol{m}(\boldsymbol{x},t)$ be defined by

$$\mathrm{L}\boldsymbol{m} = \boldsymbol{j}+\boldsymbol{g} , \quad \nabla\cdot\boldsymbol{m} = 0 . \tag{1.7}$$

Then the field $\boldsymbol{w}$ defined by

$$\boldsymbol{w} = \boldsymbol{\omega} + \nabla\wedge(\boldsymbol{h}\wedge\boldsymbol{m}) \tag{1.8}$$

satisfies

$$\mathrm{L}\boldsymbol{w} = 0 , \tag{1.9}$$

and provides the appropriate frozen-field generalization of $\boldsymbol{\omega}$ for ideal MHD flow. Note that, since $\boldsymbol{h}$ and $\boldsymbol{w}$ are now two independent frozen-in fields, it follows that $\nabla\wedge(\boldsymbol{h}\wedge\boldsymbol{w})$ is also frozen-in (Tur & Yanovsky 1993), and by iteration, an infinite family of such derived frozen-in fields may be constructed.

The global invariants of the system (1.1)–(1.4) are the total energy

$$\mathscr{E}_{\mathrm{tot}} = \tfrac{1}{2}\int_{\mathscr{D}}(\boldsymbol{u}^2+\boldsymbol{h}^2)\mathrm{d}\tau , \tag{1.10}$$

the magnetic helicity

$$\mathscr{H}_M = \int_{\mathscr{D}}(\boldsymbol{h}\cdot\mathrm{curl}^{-1}\boldsymbol{h})\mathrm{d}\tau , \tag{1.11}$$

the cross-helicity

$$\mathscr{H}_C = \int_{\mathscr{D}}\boldsymbol{u}\cdot\boldsymbol{h}\,\mathrm{d}\tau = \int(\boldsymbol{h}\cdot\mathrm{curl}^{-1}\boldsymbol{w})\mathrm{d}\tau , \tag{1.12}$$

and the 'generalized' helicity

$$\mathscr{H}_W = \int_{\mathscr{D}}\boldsymbol{w}\cdot\mathrm{curl}^{-1}\boldsymbol{w}\,\mathrm{d}\tau . \tag{1.13}$$

The three helicities are all topological in character (Moffatt 1969) and are automatically conserved under the procedures developed in the following sections.

2. Isomorphism between two-dimensional MHD and stratified flow

Suppose first that $\boldsymbol{u}$ and $\boldsymbol{h}$ are invariant under translations in a fixed direction, which we may take to be the direction Oz of a Cartesian coordinate system $Oxyz$. The domain $\mathscr{D}$ is then cylindrical with unit normal on $\partial\mathscr{D}$

$$\boldsymbol{n} = (n_1, n_2, 0) ; \tag{2.1}$$

the fields $\boldsymbol{u}$ and $\boldsymbol{h}$ may be decomposed in the form

$$\boldsymbol{u}(x,y,t) = \boldsymbol{v} + u_3\boldsymbol{e}_z , \quad \boldsymbol{h}(x,y,t) = \boldsymbol{b} + h_3\boldsymbol{e}_z , \tag{2.2}$$

where

$$\boldsymbol{v} = \nabla\wedge(\psi \boldsymbol{e}_z) , \quad \boldsymbol{b} = \nabla\wedge(\rho \boldsymbol{e}_z) . \tag{2.3}$$

Here, $\psi(x, y, t)$ is a streamfunction for the (x, y) components of $\boldsymbol{u}$, and $\rho(x, y, t)$ is a flux function for the (x, y) components of $\boldsymbol{h}$. The z-components of vorticity and current are given by

$$q \equiv \omega_z = -\nabla^2\psi , \quad \Phi \equiv j_z = -\nabla^2\rho . \tag{2.4}$$

The choice of notation here is deliberate, for reasons that will be apparent below.

The z-components of (1.1) and (1.2) now become

$$\mathrm{D}u_3 = -\{\rho, h_3\} , \quad \mathrm{D}h_3 = -\{\rho, u_3\} , \tag{2.5}$$

where we use the notation

$$\{a, b\} = \partial(a, b)/\partial(x, y) . \tag{2.6}$$

Defining $p^* = p + \frac{1}{2}h_3^2 - \rho\Phi$, the (x, y) components may be written

$$\mathrm{D}\boldsymbol{v} = -\nabla p^* - \rho\nabla\Phi , \quad \mathrm{D}\rho = 0 , \tag{2.7}$$

and the boundary conditions (2.4) become

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 , \quad \rho = \text{const. on } \partial\mathscr{D} . \tag{2.8}$$

Note that the fields (u_3, h_3) do not appear in (2.7) which may therefore be considered independently of (2.5). When a solution $\{\boldsymbol{v}, \rho\}$ of (2.7) is known, the corresponding evolution of (u_3, h_3) is then given by (2.5).

Equations (2.7) with boundary conditions (2.8) are evidently identical with the equations and boundary conditions governing two-dimensional flow of an analogue Boussinesq fluid with density $\rho(x, y, t)$ in a 'pseudo-gravitational' field of potential Φ. Only the dependence of Φ on ρ (equation. (2.4)) looks unusual: in a self-gravitating fluid, we would have $\nabla^2\Phi = -\rho$; here, by contrast, we have

$$\nabla^{-2}\Phi = -\rho \tag{2.9}$$

and this is why we use the term 'pseudo-gravitational'. The main point however is that MHD flows with translational invariance are isomorphic to flows of a Boussinesq fluid with body-force potential $\Phi = -\nabla^2\rho$. Stability techniques that are familiar in the stratified flow context can be readily adapted to MHD problems, as we shall now demonstrate.

3. Linear stability criteria

We now consider a steady solution of (2.7) in the form

$$\boldsymbol{v} = \boldsymbol{V}(x, y) = \nabla\wedge\left(\Psi(x, y)\boldsymbol{e}_z\right) , \quad p^* = P(x, y) , \quad \rho = A(\Psi) , \tag{3.1}$$

where the functional form of ρ is implied by $\mathrm{D}\rho = 0$. Capital letters will be used throughout for properties of the steady state whose stability is to be investigated. In this state,

$$\boldsymbol{b} = \boldsymbol{B} = A'(\Psi)\boldsymbol{V}. \tag{3.2}$$

The curl of (2.7) simplifies to the equation

$$(\boldsymbol{V}\cdot\nabla)(Q - A'(\Psi)J) = 0 , \tag{3.3}$$

where

$$Q = -\nabla^2 \Psi \ , \quad J = -\nabla^2 A \ . \tag{3.4}$$

Hence we have the generalized Grad–Shafranov equation (see e.g. Biskamp 1993)

$$-\nabla^2 \Psi + A'(\Psi)\nabla^2 A = G(\Psi) \tag{3.5}$$

for some function $G(\Psi)$.

Now consider (1.6)–(1.9). These are satisfied by taking

$$\boldsymbol{g} = (0, 0, g(\rho)) \ , \quad \boldsymbol{m} = (0, 0, m) \ , \quad \boldsymbol{w} = (0, 0, w) \ , \tag{3.6}$$

where

$$\mathrm{D}m = -\nabla^2 \rho + g(\rho) \tag{3.7}$$

and

$$\mathrm{D}w = 0 \ , \quad w = q + \{\rho, m\} \ . \tag{3.8}$$

In the steady state, (3.7) implies that

$$w = W(\Psi) \ , \quad W(\Psi) = -\nabla^2 \Psi + \Big(\nabla^2 A - g(A)\Big) A'(\Psi) \ , \tag{3.9}$$

and comparison with (3.5) shows that

$$W(\Psi) + g\Big(A(\Psi)\Big) A'(\Psi) = G(\Psi) \ . \tag{3.10}$$

Also, from (3.7), $m = M(x, y)$ where

$$\boldsymbol{V} \cdot \nabla M = -\nabla^2 A + g(A) \ . \tag{3.11}$$

The function $g(A)$ may be chosen to ensure that M is single-valued in $\mathscr{D}$: if $\mathscr{D}$ is bounded, then the streamlines Ψ = const. are closed curves in $\mathscr{D}$, and we define

$$g(A(\Psi)) \equiv \oint_{\Psi = \mathrm{const.}} |\boldsymbol{V}|^{-1} \nabla^2 \rho \, \mathrm{d}\ell \Big/ \oint_{\Psi = \mathrm{const.}} |\boldsymbol{V}|^{-1} \mathrm{d}\ell . \tag{3.12}$$

(If $\mathscr{D}$ is unbounded, then on any streamlines that are not closed, we simply choose g so that $g(A)$ is a smooth function of A; if no streamlines are closed, we take $g \equiv 0$.)

Under general unsteady evolution, both ρ and w are conserved fields ($\mathrm{D}\rho = \mathrm{D}w = 0$); hence the appropriate Casimir is

$$\mathscr{C} = \int_{\mathscr{D}} F(\rho, w) \, \mathrm{d}\tau \ , \tag{3.13}$$

where $\mathrm{d}\tau = \mathrm{d}x\mathrm{d}y$, and $F(\rho, w)$ is an arbitrary smooth function of ρ and w. If $\mathscr{D}$ is not simply connected, i.e. $\partial\mathscr{D}$ consists of separate closed components (external or internal) $\partial\mathscr{D}_i$ ($i = 1, 2, ..., n$), then we must also take account of the 'circulational invariant'

$$\Gamma \equiv \sum_{i=1}^{s} \gamma_i \Gamma_i \ , \quad \Gamma_i = \oint_{\partial\mathscr{D}_i} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{\ell} \ , \tag{3.14}$$

where the γ_i are arbitrary constants. Each Γ_i is constant because $j_z = \boldsymbol{e}_z \wedge \boldsymbol{b}$ is parallel to $\nabla\rho$, i.e. to $\boldsymbol{n}$, on each $\partial\mathscr{D}_i$.

Consider now the functional

$$\mathscr{R}\{\psi, \rho, m\} = \mathscr{E} + \mathscr{C} + \Gamma, \tag{3.15}$$

where

$$\mathscr{E} = \tfrac{1}{2}\int_{\mathscr{D}} \left(\boldsymbol{v}^2 + \boldsymbol{b}^2\right) \mathrm{d}\tau \tag{3.16}$$

is the conserved energy of the fields $\{\boldsymbol{v}, \boldsymbol{b}\}$. The dependence of $\mathscr{R}$ on m enters through $\mathscr{C}$. Let $\delta\psi, \delta\rho, \delta m$ be independent variations, and let

$$\delta\boldsymbol{v} = \nabla \wedge (\delta\psi \boldsymbol{e}_z) , \quad \delta q = -\nabla^2 \delta\psi . \tag{3.17}$$

From (3.8b), the first- and second-order variations of w are then

$$\delta^1 w = \delta q + \{A, \delta m\} + \{\delta\rho, M\} , \quad \delta^2 w = \{\delta\rho, \delta m\} . \tag{3.18}$$

The first variation of $\mathscr{R}$ is easily calculated:

$$\delta^1 \mathscr{R} = \int \{\nabla\Psi \cdot \delta\nabla\psi + \nabla A \cdot \nabla\delta\rho + F_A \delta\rho + F_W \delta^1 w\} \,\mathrm{d}\tau + \sum_{i=1}^{s} \gamma_i \oint_{\partial\mathscr{D}_i} \delta\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{\ell}, \tag{3.19}$$

where $F_A = \partial F(A, W)/\partial A$, $F_W = \partial F(A, W)/\partial W$. From here on, F and its derivatives are always evaluated at $\rho = A$, $w = W$. By standard manipulations using (3.11) and (3.18a), and the boundary condition

$$\delta\rho = 0 \quad \text{on} \quad \partial\mathscr{D} \tag{3.20}$$

(3.19) converts to

$$\begin{aligned}\delta^1 \mathscr{R} = &\int \left[\nabla\left(F_W + \Psi\right) \cdot \nabla\delta\psi + \left(F_A - g(A)\right)\delta\rho + \left(F_W + \Psi\right)\{\delta\rho, M\}\right] \mathrm{d}\tau \\ &+ \oint_{\partial\mathscr{D}} F_W \,\delta\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{\ell} + \sum_{i=1}^{s} \gamma_i \oint_{\partial\mathscr{D}_i} \delta\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{\ell} .\end{aligned} \tag{3.21}$$

We now choose $F(A, W)$ and γ_i so that

$$\delta^1 \mathscr{R} = 0 . \tag{3.22}$$

A natural choice is given by

$$F_W = -\Psi , \quad F_A = g(A) , \quad \gamma_i = \Psi_i , \tag{3.23}$$

where Ψ_i is the constant value of Ψ on $\partial\mathscr{D}_i$. These equations obviously do not determine $F(A, W)$ uniquely; the remaining freedom in F will be used later. Note that, by differentiating (3.23a, b) with respect to Ψ, we obtain

$$\begin{pmatrix} F_{WW} & F_{AW} \\ F_{AW} & F_{AA} \end{pmatrix} \begin{pmatrix} W'(\Psi) \\ A'(\Psi) \end{pmatrix} = \begin{pmatrix} -1 \\ g'(A)A'(\Psi) \end{pmatrix} . \tag{3.24}$$

Consider now the second variation of $\mathscr{R}$, given by

$$\delta^2 \mathscr{R} = \tfrac{1}{2} \int_{\mathscr{D}} \{(\delta\boldsymbol{v})^2 + (\nabla\delta\rho)^2 + F_{WW}(\delta^1 w)^2 + 2F_{AW}\delta\rho\delta^1 w + F_{AA}(\delta\rho)^2 + 2F_W \delta^2 w\} \,\mathrm{d}\tau. \tag{3.25}$$

By standard manipulations (see Appendix A), this may be expressed in the form

$$\delta^2 \mathscr{R} = \tfrac{1}{2} \int \left\{ \left(\nabla(\delta\psi - \sigma\delta\rho)\right)^2 + (1 - \sigma^2)(\nabla\delta\rho)^2 + X(\delta\rho)^2 + F_{WW}\left(\delta^1 w - \frac{\mathrm{d}W}{\mathrm{d}A}\delta\rho\right)^2 \right\} \mathrm{d}\tau, \tag{3.26}$$

where

$$\sigma = \Psi'(A) \tag{3.27}$$

and

$$X = \frac{\nabla\Psi \cdot \nabla Q - \nabla A \cdot \nabla J}{(\nabla A)^2} + \sigma\nabla^2\sigma\ . \tag{3.28}$$

(Recall the definition (3.4) of Q and J.)

Now according to the general theory of Arnold (1965, 1966), if $\mathscr{R}$ is extremal (maximum or minimum) for all admissible variations $(\delta\psi, \delta\rho, \delta m)$ then the system considered is linearly stable. Here, $\mathscr{R}$ is minimal at (Ψ, A, M) if $\delta^2\mathscr{R}$ is positive-definite, and this is certainly true provided

$$F_{WW} \geqslant 0\ , \quad \sigma^2 \leqslant 1\ , \quad X \geqslant 0\ , \tag{3.29}$$

throughout $\mathscr{D}$. We satisfy the first of these by using the remaining freedom in the choice of $F(A, W)$: let

$$F(A, W) = F_0(W) + WF_1(A) + F_2(A)\ , \tag{3.30}$$

where F_0, F_1, F_2 are smooth functions such that

$$F_0''(W) \geqslant 0\ , \quad F_1(A) = -\Psi - F_0'(W)\ , \quad F_2(A) = g(A) - WF_1'(A)\ . \tag{3.31}$$

(Recall that $\Psi = \Psi(A)$, $W = W(A)$.) Then (3.23*a, b*) are satisfied. We thus obtain the following stability criterion:

CRITERION 3.1. *The steady flow (3.1) is linearly stable provided*

$$\left.\begin{aligned} &\left(\Psi'(A)\right)^2 \leqslant 1 \ \ (\textit{or equivalently } V^2 \leqslant \boldsymbol{B}^2), \\ &\frac{\nabla\Psi \cdot \nabla Q - \nabla A \cdot \nabla J}{(\nabla A)^2} + \Psi'(A)\nabla^2\,\Psi'(A) \geqslant 0 \end{aligned}\right\} \tag{3.32}$$

throughout $\mathscr{D}$.

The first condition means that the flow must be *sub-Alfvénic*; the second places a constraint on the degree of misalignment of the fields (Ψ, Q) and (A, J). Note that, if the unperturbed state is magnetostatic (i.e. $\Psi = 0$), then (3.32) reduces to

$$\mathrm{d}J/\mathrm{d}A \leqslant 0\ . \tag{3.33}$$

As indicated in the introduction, the criterion (3.32) is equivalent to that obtained by Holm *et al.* (1985, p. 41) who based their treatment on the use of a Casimir

$$\tilde{\mathscr{C}} = \int_{\mathscr{D}} \left(qF_1(\rho) + F_2(\rho)\right) \mathrm{d}\tau\ , \tag{3.34}$$

where q is the vorticity field. Since $\mathrm{D}q \neq 0$, there is little *a priori* reason for this choice. However, it happens that

$$\int_{\mathscr{D}} \{\rho, m\} F_1(\rho)\, \mathrm{d}\tau = \int_{\mathscr{D}} \{N(\rho), m\}\, \mathrm{d}\tau\ , \tag{3.35}$$

where $N'(\rho) = F_1(\rho)$, and the latter integral vanishes by an application of Green's theorem in the plane, using $\rho = \text{const.}$ on $\partial\mathscr{D}$. Hence, in fact, using (3.8*b*),

$$\tilde{\mathscr{C}} = \int_{\mathscr{D}} \left(wF_1(\rho) + F_2(\rho)\right) \mathrm{d}\tau\ , \tag{3.36}$$

and the invariance of $\tilde{\mathscr{C}}$ follows from $\mathrm{D}\rho = \mathrm{D}w = 0$.

We conclude this section with two simple examples that may help to clarify the conditions (3.32).

Example (i): plane parallel field and flow

Suppose that $A = A(y)$, $\Psi = \Psi(y)$ for $0 \leqslant y \leqslant L$, so that

$$V = V(y)\boldsymbol{e}_x\ , \quad \boldsymbol{B} = B(y)\boldsymbol{e}_x \tag{3.37}$$

with $V(y) = \mathrm{d}\Psi/\mathrm{d}y, B(y) = \mathrm{d}A/\mathrm{d}y$. Then the conditions (3.32) reduce to

$$\left.\begin{aligned} &B^2 \geqslant V^2, \\ &\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}y}\left\{\left(1-\frac{V^2}{B^2}\right)\frac{\mathrm{d}B}{\mathrm{d}y}\right\} \geqslant 0 \end{aligned}\right\} \tag{3.38}$$

for $0 \leqslant y \leqslant L$. These conditions may be considerably strengthened by noting that, under Gallilean transformation $x' = x - V_0 t$, $t' = t$, V transforms to $V - V_0$ while B is invariant. If the state $(V(y), B(y))$ is stable in any frame of reference, then it is stable in all frames related by such Gallilean transformation. Hence the state is stable if there exists a value of V_0 such that the inequalities

$$\left.\begin{aligned} &\min_{0\leqslant y\leqslant L}\left(B^2-(V-V_0)^2\right) \geqslant 0, \\ &\min_{0\leqslant y\leqslant L}\left[\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}y}\left\{\left(1-\frac{(V-V_0)^2}{B^2}\right)\frac{\mathrm{d}B}{\mathrm{d}y}\right\}\right] \geqslant 0 \end{aligned}\right\} \tag{3.39}$$

are both satisfied. For example, if $V(y) = y^2$, $B(y) = y$ for $0 < y < 1$, then (3.38) is not satisfied, but (3.39) *is* satisfied if $V_0^2 > 1$. This state is therefore stable.

Example (ii): flow with circular streamlines

Suppose that, with polar coordinates (r, θ),

$$A = A(r)\ , \quad \Psi = \Psi(r) \quad (a \leqslant r \leqslant b) \tag{3.40}$$

so that

$$V = V(r)\boldsymbol{e}_\theta\ , \quad \boldsymbol{B} = B(r)\boldsymbol{e}_\theta\ , \tag{3.41}$$

with $V(r) = -\Psi'(r)$, $B(r) = -A'(r)$. In this case, the conditions (3.32) reduce to

$$\left.\begin{aligned} &B^2 \geqslant V^2, \\ &\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\left(1-\frac{V^2}{B^2}\right)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rB)\right\} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V^2}{B^2}\right) \geqslant 0 \end{aligned}\right\} \tag{3.42}$$

for $a \leqslant r \leqslant b$. Note that for the case in which the field $B(r)$ is produced by a current confined to the region $r < a$, we have $rB =$ const. and (3.42*b*) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}r}(rV)^2 \geqslant 0\ . \tag{3.43}$$

Remarkably, this is precisely Rayleigh's (1916) criterion for the stability of axisymmetric flow to axisymmetric perturbations. Here, we are concerned with MHD flow subjected to plane non-axisymmetric perturbations; the fact that the same criterion emerges is pure coincidence!

Again, the criterion (3.42) may be improved by considering a frame of reference rotating with angular velocity Ω_0 say. In this frame, $V(r)$ is replaced by $V(r) - r\Omega_0$, but $B(r)$ is unchanged. The state (3.41) is thus linearly stable to pertubations in the plane of the flow provided there exists any value of Ω_0 such that the inequalities

$$\left.\begin{aligned} &B^2 \geqslant \left(V - r\Omega_0\right)^2, \\ &\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\left(1 - \frac{(V - r\Omega_0)^2}{B^2}\right)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rB)\right\} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{(V - r\Omega_0)^2}{B^2}\right) \geqslant 0 \end{aligned}\right\} \tag{3.44}$$

are simultaneously satisfied for $a \leqslant r \leqslant b$.

4. Nonlinear stability criteria

Consider now a *finite-amplitude* (but still two-dimensional) perturbation of the steady solution (3.1), given by

$$\psi = \Psi(\mathbf{x}) + \tilde{\psi}(\mathbf{x},t), \quad \mathbf{v} = \mathbf{V}(\mathbf{x}) + \tilde{\mathbf{v}}(\mathbf{x},t), \quad \rho = A(\mathbf{x}) + \tilde{\rho}(\mathbf{x},t), \quad \mathbf{x} \equiv (x,y) \in \mathscr{D}\,, \tag{4.1}$$

with

$$\tilde{\mathbf{v}} \cdot \mathbf{n} = 0\,, \quad \tilde{\rho} = 0 \;\text{ on }\; \partial\mathscr{D}\,, \tag{4.2}$$

so that the constant values of ρ on $\partial\mathscr{D}_i$ $(i = 1, ..., s)$ are unchanged by the perturbation. Let $A^- \equiv \min_{\mathscr{D}} A(\mathbf{x})$, $A^+ \equiv \max_{\mathscr{D}} A(\mathbf{x})$, and let $\mathscr{A}$ be the closed interval $[A^-, A^+]$. Let us introduce the notation

$$\mathbf{v} = (v_1, v_2, v_3, v_4, v_5) \equiv (\tilde{\psi}_x, \tilde{\psi}_y, \tilde{\rho}_x, \tilde{\rho}_y, \tilde{\rho})\,. \tag{4.3}$$

To measure the deviation of the perturbed solution (4.1) from the unperturbed one (3.1) we shall exploit the norm (or, more accurately, seminorm) given by

$$\|\mathbf{v}\|^2 \equiv \int_{\mathscr{D}} v_i v_i \mathrm{d}\tau = \int_{\mathscr{D}} \left\{\left(\nabla\tilde{\psi}\right)^2 + \left(\nabla\tilde{\rho}\right)^2 + \tilde{\rho}^2\right\}\mathrm{d}\tau\,. \tag{4.4}$$

We adopt the standard Lyapunov definition of stability: the steady state (3.1) is stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{v}(0)\| < \delta \Rightarrow \|\mathbf{v}(t)\| < \epsilon$.†

For the subsequent analysis it is convenient to define the following functions:

$$\alpha(\mathbf{x}, a) \equiv -Q(\mathbf{x})\Psi''(a) + \tilde{G}'(a)\,, \quad \mathbf{x} \in \mathscr{D}\,,\; a \in \mathscr{A}\,; \tag{4.5a}$$

$$\beta(a) \equiv -\Psi''(a)\,, \quad \sigma(a) \equiv \Psi'(a)\,, \quad a \in \mathscr{A}\,; \tag{4.5b}$$

$$\mu(\lambda; \beta, \sigma) \equiv \frac{(1-\lambda)\beta^2}{(1-\lambda)^2 - \sigma^2}\,. \tag{4.5c}$$

where $\tilde{G}(a) \equiv \Psi'(a)G(\Psi(a))$, with $G(\Psi)$ given by (3.5). It is also useful to introduce notations, related to these functions:

$$\alpha^-(\mathbf{x}) \equiv \min_{a\in\mathscr{A}}\left\{\alpha(\mathbf{x}, a)\right\}, \; \alpha^+(\mathbf{x}) \equiv \max_{a\in\mathscr{A}}\left\{\alpha(\mathbf{x}, a)\right\}, \; \mathbf{x} \in \mathscr{D}\,, \tag{4.5d}$$

$$\sigma_0^2 \equiv \max_{a\in\mathscr{A}}\left(\Psi'(a)\right)^2\,, \quad \beta_0^2 \equiv \max_{a\in\mathscr{A}}\left(\Psi''(a)\right)^2\,, \tag{4.5e}$$

† This nonlinear stability problem has been considered (among many other cases) by Holm *et al.* (1985, p. 42). We believe however that their treatment of the inequalities required to obtain convexity estimates is incorrect, and that the resulting stability theorem (p. 44) is therefore invalid. However, the error does not affect the validity of the nonlinear stability criteria obtained in two particular cases (Alfven solutions for which $\mathbf{V}(\mathbf{x}) \equiv \mathbf{B}(\mathbf{x})$, and magnetostatic equilibria for which $\mathbf{V}(\mathbf{x}) \equiv 0$).

4.1. *Isomagnetic Perturbations*

Consider first a particular class of finite-amplitude perturbations with general initial data for the streamfunction $\tilde{\psi}(\boldsymbol{x},0)$ and with initial data for the flux function (density) from the interval $\mathscr{A}$:

$$A^- \leqslant \rho(\boldsymbol{x},0) \leqslant A^+ . \tag{4.6}$$

Note that if (3.48) is satisfied then, according to (2.7*b*),

$$A^- \leqslant \rho(\boldsymbol{x},t) \leqslant A^+ \tag{4.7}$$

for all $t > 0$. Such perturbations may be imagined as obtained at the initial instant $t = 0$ by displacement of fluid particles from their position in the unperturbed state (3.1), the value of the flux function ρ for each fluid particle being unchanged. For this class of 'isomagnetic' perturbations we shall obtain the following nonlinear stability criterion:

CRITERION 4.1. *Suppose that: (i) the function $A(\Psi)$ defined by equation (3.2) is invertible and the inverse function $\Psi(A)$ is twice continuously differentiable for all $A \in \mathscr{A}$; (ii) the function $\tilde{G}(A) \equiv \Psi'(A)G(\Psi(A))$ (where $G(\Psi)$ is given by (3.5)) is continuously differentiable for all $A \in \mathscr{A}$; (iii) there exist constants ϵ^-, ϵ^+ such that, for $a \in \mathscr{A}$, $\boldsymbol{x} \in \mathscr{D}$,*

$$0 < \epsilon^- < 1, \epsilon^+ > 2 - \epsilon^- ; \quad \left|\Psi'(a)\right| < 1 - \epsilon^- ; \tag{4.8a}$$

$$\alpha^-(\boldsymbol{x}) - \mu(\epsilon^-, \beta_0, \sigma_0)\Big(\boldsymbol{B}(\boldsymbol{x})\Big)^2 > \epsilon^-, \quad \alpha^+(\boldsymbol{x}) - \mu(\epsilon^+, \beta_0, \sigma_0)\Big(\boldsymbol{B}(\boldsymbol{x})\Big)^2 < \epsilon^+ . \tag{4.8b}$$

Then the steady state (3.1) is nonlinearly stable to perturbations with initial data satisfying (4.6). Moreover, the following a priori estimate holds true:

$$\epsilon^- \|\boldsymbol{v}(t)\| \leqslant \epsilon^+ \|\boldsymbol{v}(0)\| . \tag{4.9}$$

Proof. Following the prescription of Arnold (1966) we decompose the conserved functional $\mathscr{R} = \mathscr{E} + \tilde{\mathscr{C}} + \Gamma$ in the form†

$$\mathscr{R} = \mathscr{R}_0 + \mathscr{R}_1 + \mathscr{R}_2 , \tag{4.10}$$

$$\mathscr{R}_0 = \int_{\mathscr{D}} \left\{ \tfrac{1}{2}\Big(\nabla\Psi\Big)^2 + \tfrac{1}{2}\Big(\nabla A\Big)^2 + QF_1(A) + F_2(A) \right\} \mathrm{d}\tau + \sum_{i=1}^{s} \gamma_i \oint_{\partial\mathscr{D}_i} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{\ell} , \tag{4.11}$$

$$\mathscr{R}_1 = \int_{\mathscr{D}} \left\{ \nabla\Psi \cdot \nabla\tilde{\psi} + \nabla A \cdot \nabla\tilde{\rho} + \Big[QF_1'(A) + F_2'(A)\Big]\tilde{\rho} - -F_1(A)\nabla^2\tilde{\psi} \right\} \mathrm{d}\tau$$
$$+ \sum_{i=1}^{s} \gamma_i \oint_{\partial\mathscr{D}_i} \tilde{\boldsymbol{v}} \cdot \mathrm{d}\boldsymbol{\ell} , \tag{4.12}$$

$$\mathscr{R}_2 = \int_{\mathscr{D}} \left\{ \tfrac{1}{2}\Big(\nabla\tilde{\psi}\Big)^2 + \tfrac{1}{2}\Big(\nabla\tilde{\rho}\Big)^2 + Q\hat{F}_1(\tilde{\rho}) + \hat{F}_2(\tilde{\rho}) - \Big(F_1(A+\tilde{\rho}) - F_1(A)\Big)\nabla^2\tilde{\psi} \right\} \mathrm{d}\tau , \tag{4.13}$$

with

$$\hat{F}_\alpha(\tilde{\rho}) \equiv F_\alpha(A+\tilde{\rho}) - F_\alpha(A) - F_\alpha'(A)\tilde{\rho} \;\; (\alpha = 1,2) . \tag{4.14}$$

† The functional $\mathscr{R}$ used here is the same as that introduced by Holm *et al.* (1985) (see §3 above).

We choose the arbitrary functions $F_1(\rho)$, $F_2(\rho)$ and the constants γ_i such that

$$F_1(A) = -\Psi(A)\ , \quad F_2'(A) = \Psi'(A)G\Big(\Psi(A)\Big) \equiv \tilde{G}(A)\ , \quad \gamma_i = \Psi_i\ , \tag{4.15}$$

where $G(\Psi)$ is given by (3.5). With this choice the functional $\mathscr{R}_1$ (corresponding to the first variation of $\mathscr{R}$) vanishes in the steady state (3.1). Since $\mathscr{R}_0$ does not depend on time we conclude that $\mathscr{R}_2$ is an invariant of the exact nonlinear problem (2.5)–(2.8).

Let us now transform $\mathscr{R}_2$ to a form that is convenient for the subsequent stability analysis. Integrating the last term in (4.11) by parts and using (4.2*b*) we obtain

$$-\int_{\mathscr{D}}\Big(F_1(A+\tilde{\rho})-F_1(A)\Big)\nabla^2\tilde{\psi}\mathrm{d}\tau = \int_{\mathscr{D}}\Big\{\Big(F_1'(A+\tilde{\rho})-F_1'(A)\Big)\nabla A\cdot\nabla\tilde{\psi}+F_1'(A+\tilde{\rho})\nabla\tilde{\psi}\cdot\nabla\tilde{\rho}\Big\}\mathrm{d}\tau\,.$$

Substituting in (4.13) and using Taylor's formula with remainder in Lagrange's form, we find

$$\mathscr{R}_2 = \int_{\mathscr{D}}\Big\{\tfrac{1}{2}\Big(\nabla\tilde{\psi}\Big)^2+\tfrac{1}{2}\Big(\nabla\tilde{\rho}\Big)^2-\sigma(a_0)\nabla\tilde{\psi}\cdot\nabla\tilde{\rho}+\beta(a_1)\tilde{\rho}\nabla A\cdot\nabla\tilde{\psi}+\tfrac{1}{2}\alpha(\boldsymbol{x},a_2)\tilde{\rho}^2\Big\}\mathrm{d}\tau, \tag{4.16}$$

where

$$a_0 \equiv A+\tilde{\rho}\ , \quad a_1 \equiv A+\theta_1\tilde{\rho}\ , \quad a_2 \equiv A+\theta_2\tilde{\rho}\ , \tag{4.17a}$$

and where we have used the fact that, according to (4.5*a, b*), (4.15),

$$F_1'(a) = -\sigma(a)\ , \quad F_1''(a) = \beta(a)\ , \quad Q(\boldsymbol{x})F_1''(a)+F_2''(a) = \alpha(\boldsymbol{x},a) \tag{4.17b}$$

for all $a \in \mathscr{A}$. In (4.17*a*), θ_1, θ_2 are functions of A and $\tilde{\rho}$ such that

$$0 < \theta_1 < 1\ , \quad 0 < \theta_2 < 1\ .$$

Note that for perturbations with initial data in the range (4.6), $a_0, a_1, a_2 \in \mathscr{A}$. Using the notation (4.3), equation (4.16) may be written in the form

$$2\mathscr{R}_2 = \int_{\mathscr{D}} R_{ik}v_iv_k\ \mathrm{d}\tau \tag{4.18a}$$

where R_{ik} are elements of the symmetric 5×5 matrix

$$\hat{\boldsymbol{R}} \equiv \begin{pmatrix} 1 & 0 & -\sigma & 0 & \beta A_x \\ 0 & 1 & 0 & -\sigma & \beta A_y \\ -\sigma & 0 & 1 & 0 & 0 \\ 0 & -\sigma & 0 & 1 & 0 \\ \beta A_x & \beta A_y & 0 & 0 & \alpha \end{pmatrix}. \tag{4.18b}$$

If the positive constants ϵ^- and ϵ^+ in the conditions (4.8) are such that

$$\epsilon^-\int_{\mathscr{D}} v_iv_i\mathrm{d}\tau \leqslant 2\mathscr{R}_2 \leqslant \epsilon^+\int_{\mathscr{D}} v_iv_i\mathrm{d}\tau, \tag{4.19}$$

then the *a priori* estimate (4.9) and hence the nonlinear stability of the flow (3.1) follow immediately from the fact that $\mathscr{R}_2$ is an invariant of the exact nonlinear problem (2.5)–(2.8). The stability analysis therefore reduces to obtaining upper and lower bounds for $\mathscr{R}_2$. Obviously, the inequalities (4.19) are satisfied provided that the two quadratic forms $(R_{ik}-\epsilon^-\delta_{ik})v_iv_k$ and $(\epsilon^+\delta_{ik}-R_{ik})v_iv_k$ are positive definite. The necessary and sufficient conditions for this are

$$\epsilon^- < 1\ , \quad \sigma^2(a) < (1-\epsilon^-)^2\ , \quad \alpha(\boldsymbol{x},a) > \epsilon^- + \mu\Big(\epsilon^-;\beta,\sigma\Big)\boldsymbol{B}^2\ , \tag{4.20a}$$

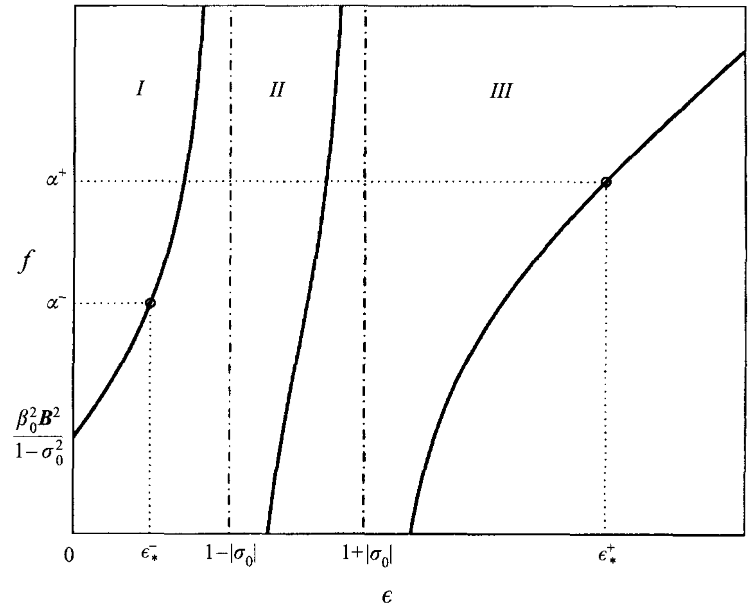

FIGURE 1. The function $f(\epsilon)$ defined by equation (4.26).

and

$$\epsilon^+ > 1\,, \quad \sigma^2(a) < (1-\epsilon^+)^2\,, \quad \alpha(\boldsymbol{x},a) < \epsilon^+ + \mu\Big(\epsilon^+;\beta,\sigma\Big)\boldsymbol{B}^2, \tag{4.20b}$$

where $\mu(\lambda;\beta,\sigma)$ is given by (4.5c).

Let us now show that (4.20a,b) are satisfied under the conditions (4.8a,b). First we take ϵ^+ and ϵ^- such that

$$0 < \epsilon^- < 1-|\sigma_0|\,, \quad 2-\epsilon^- < \epsilon^+ < \infty\,. \tag{4.21}$$

These inequalities define a domain $\mathscr{M}$ in the plane (ϵ^-,ϵ^+). If $\epsilon^-,\epsilon^+ \in \mathscr{M}$ then the inequalities (4.20) are also satisfied and, in addition,

$$0 < \mu(\epsilon^-;\beta,\sigma) \leqslant \mu(\epsilon^-,\beta_0,\sigma_0)\,, \quad 0 > \mu(\epsilon^+;\beta,\sigma) \geqslant \mu(\epsilon^+;\beta_0,\sigma_0)\,. \tag{4.22}$$

Next we suppose that there exist $\epsilon^-,\epsilon^+ \in \mathscr{M}$ such that

$$\epsilon^- + \mu(\epsilon^-;\beta_0,\sigma_0)\boldsymbol{B}^2 < \alpha^-(\boldsymbol{x})\,, \; \epsilon^+ + \mu(\epsilon^+;\beta_0,\sigma_0)\boldsymbol{B}^2 > \alpha^+(\boldsymbol{x}), \tag{4.23}$$

where $\alpha^-(\boldsymbol{x})$, $\alpha^+(\boldsymbol{x})$ are given by (4.5d), and hence

$$\epsilon^- + \mu(\epsilon^-;\beta,\sigma)\boldsymbol{B}^2 < \alpha(\boldsymbol{x},a) < \epsilon^+ + \mu(\epsilon^+;\beta,\sigma)\,. \tag{4.24}$$

Hence we have shown that the conditions (4.21), (4.23) are in fact sufficient for all six inequalities (4.20a,b) to be satisfied. The conditions (4.21), (4.23) are equivalent to the conditions (4.8a,b). Criterion 4.1 is thus established.

Consider now the existence of constants ϵ^-, ϵ^+ satisfying the conditions (4.8) of Criterion 4.1. It is convenient to rewrite the inequalities (4.8b) in the form

$$\alpha^-(\boldsymbol{x}) > f(\epsilon^-), \quad \alpha^+(\boldsymbol{x}) < f(\epsilon^+), \tag{4.25}$$

where

$$f(\epsilon) \equiv \epsilon + \frac{1-\epsilon}{(1-\epsilon)^2-\sigma_0^2}\beta_0^2\boldsymbol{B}^2\,. \tag{4.26}$$

The graph of $f(\epsilon)$ (see figure 1) consists of three branches, denoted by I, II and III. From (4.8*a*) it follows that

$$0 < \epsilon^- < 1 - |\sigma_0| , \quad \epsilon^+ > 1 + |\sigma_0| , \tag{4.27}$$

so that ϵ^- lies on I and ϵ^+ on III. The condition (4.25*a*) is satisfied for any $\epsilon < \epsilon_*^-$ and (4.25*b*) is valid for $\epsilon > \epsilon_*^+$, where ϵ_*^- and ϵ_*^+ are points on the first and third branches of the graph corresponding to $f(\epsilon_*^-) = \alpha^-$ and $f(\epsilon_*^+) = \alpha^+$ respectively. Figure 1 also shows that a constant ϵ^-, satisfying the conditions (4.8), does exist provided

$$\alpha^-(\boldsymbol{x}) > f(0) = \frac{\beta_0^2}{1-\sigma_0^2}\Big(\boldsymbol{B}(\boldsymbol{x})\Big)^2 \quad \text{for all } \boldsymbol{x} \in \mathscr{D} , \tag{4.28}$$

while a constant ϵ^+ always exists for any given ϵ^- and α^+. We can now formulate the following:

COROLLARY. *The steady state (3.1) is nonlinearly stable to perturbations with initial data (4.6) provided*

$$\lambda\Big(\boldsymbol{V}(\boldsymbol{x})\Big)^2 < \Big(\boldsymbol{B}(\boldsymbol{x})\Big)^2 < \frac{1-\sigma_0^2}{\beta_0^2}\alpha^-(\boldsymbol{x}) \tag{4.29}$$

for any constant $\lambda > 1$, where $\alpha^-(\boldsymbol{x})$, β_0, σ_0 are given by (4.5d, e).

The criterion in this form admits comparison with the linear stability criterion (3.1) involving the inequalities (3.32).

4.2. *General perturbations*

The inequalities (4.8) or (4.29) give sufficient conditions for nonlinear stability of (3.1) only with respect to perturbations with initial data satisfying (4.6). We consider now a general situation when the perturbations are quite arbitrary (without the restriction (4.6) on the initial data). Let α_0^- and α_0^+ be the minimum and maximum values of the function $\alpha(\boldsymbol{x}, a)$ for all $\boldsymbol{x} \in \mathscr{D}, a \in \mathscr{A}$, i.e.

$$\alpha_0^- \equiv \min_{\boldsymbol{x}\in\mathscr{D},a\in\mathscr{A}} \alpha(\boldsymbol{x},a) = \min_{\boldsymbol{x}\in\mathscr{D}} \alpha^-(\boldsymbol{x}) , \quad \alpha_0^+ \equiv \max_{\boldsymbol{x}\in\mathscr{D},a\in\mathscr{A}} \alpha(\boldsymbol{x},a) = \max_{\boldsymbol{x}\in\mathscr{D}} \alpha^+(\boldsymbol{x}) . \tag{4.30}$$

We shall obtain the following nonlinear stability criterion.

CRITERION 4.2. *Suppose that: (i) the same conditions (as in Criterion 4.1) hold concerning smoothness of functions $\Psi(A)$ and $\tilde{G}(A)$; (ii) there exist constants ϵ^-, ϵ^+ such that, for $a \in A$, $\boldsymbol{x} \in \mathscr{D}$,*

$$0 < \epsilon^- < 1 , \; \epsilon^+ > 2 - \epsilon^- ; \quad \Big|\Psi'(a)\Big| < 1 - \epsilon^- ; \tag{4.31a}$$

$$\alpha_0^- - \mu(\epsilon^-, \beta_0, \sigma_0)\Big(\boldsymbol{B}(\boldsymbol{x})\Big)^2 > \epsilon^- , \; \alpha_0^+ - \mu(\epsilon^+, \beta_0, \sigma_0)\Big(\boldsymbol{B}(\boldsymbol{x})\Big)^2 < \epsilon^+ ; \tag{4.31b}$$

(iii) either the function $|F_1'(a)| = |\Psi'(a)|$, defined for all $a \in \mathscr{A}$, attains its maximum value at some internal point of $\mathscr{A}$ or it attains its maximum value at one of the end points of $\mathscr{A}$ and at that point $\Psi''(a) = 0$, i.e.

$$\textit{either} \quad \max_{a\in\mathscr{A}}\Big|\Psi'(a)\Big| = \Big|\Psi'(a^*)\Big| , \quad A^- < a^* < A^+ , \tag{4.32a}$$

$$\textit{or} \quad \max_{a\in\mathscr{A}}\Big|\Psi'(a)\Big| = \Big|\Psi'(a^*)\Big| , \; \Psi''(a^*) = 0 , \quad a^* = A^- \textit{ or } a^* = A^+ ; \tag{4.32b}$$

then the steady state (3.1) is stable to arbitrary finite-amplitude perturbations and the a priori estimate (4.9) holds true.

Proof. To obtain this criterion it is sufficient to show that the inequalities (4.20a, b) hold provided the conditions (4.32) are satisfied. For *arbitrary* perturbations, however, the quantities a_0, a_1 and a_2, defined by equations (4.17a), may be outside $\mathscr{A}$. The inequalities (4.20a, b) must therefore be satisfied for *all* real a.

The functions $\alpha(\boldsymbol{x}, a)$, $\beta(a)$ and $\sigma(a)$ in (4.20) are considered now as defined by (4.17b), so that, according to (4.15), for $a \in \mathscr{A}$, the new definitions coincide with the old ones (4.5a, b). Initially $F_1(a)$ and $F_2(a)$ were arbitrary and then were defined only for $a \in \mathscr{A}$ by (4.15). Hence, $F_1(a)$ and $F_2(a)$ are still arbitrary for $a \notin \mathscr{A}$. We can therefore extend the definition of $F_1(a)$ and $F_2(a)$ to all real a in any way we need and then extend the definition of $\alpha(\boldsymbol{x}, a)$, $\beta(a)$ and $\sigma(a)$ using equations (4.17b).

It can be shown (see Appendix B) that under the conditions of Criterion 4.2 it is always posible to continue $F_1(a)$ and $F_2(a)$ to all $a \notin \mathscr{A}$ in such a way that, first, they remain twice continuously differentiable and, second, the inequalities

$$\alpha_0^- \leqslant \alpha(\boldsymbol{x}, a) = Q(\boldsymbol{x})F_1''(a) + F_2''(a) \leqslant \alpha_0^+ \,, \tag{4.33a}$$

$$\left|\beta(a)\right| = \left|F_1''(a)\right| \leqslant \max_{\eta\in\mathscr{A}}\left|\Psi''(\eta)\right| = \left|\beta_0\right| \,, \tag{4.33b}$$

$$\left|\sigma(a)\right| = \left|F_1'(a)\right| \leqslant \max_{\eta\in\mathscr{A}}\left|\Psi'(\eta)\right| = \left|\sigma_0\right| \,, \tag{4.33c}$$

remain valid for all $a \notin \mathscr{A}$. Then the proof of Criterion 4.2 reduces effectively to that of Criterion 4.1.

If neither (4.32a) nor (4.32b) is satisfied then the function $|\Psi'(a)|$ attains its maximum value at one of the end points of $\mathscr{A}$ and at that point $\Psi''(a) \neq 0$, i.e.

$$\max_{a\in\mathscr{A}}\left|\Psi'(a)\right| = \left|\Psi'(a^*)\right| \,, \quad \Psi''(a^*) \neq 0 \,, \quad a^* = A^- \text{ or } a^* = A^+ \,. \tag{4.34}$$

In this case it is impossible to make a sufficiently smooth continuation of $F_1(a)$ for all real a such that the inequality (4.33c) holds true. It may be shown, however, that it is possible to continue $F_1(a)$ and $F_2(a)$ to all $a \notin \mathscr{A}$ in such a way that the conditions (4.33a, b) remain satisfied and the conditions

$$\left|\sigma(a)\right| = \left|F_1'(a)\right| \;<\; 1 - \epsilon^* \,, \quad \left|\sigma_0\right| \;<\; 1 - \epsilon^* \;<\; 1 - \epsilon^- \tag{4.35}$$

hold instead of (4.33c). In this situation the following stability criterion may be obtained.

CRITERION 4.3. *Suppose that: (i) the function $A(\Psi)$ defined by equation (3.2) is invertible and the inverse function $\Psi(A)$ is twice continuously differentiable for all $A \in \mathscr{A}$; (ii) the function $\tilde{G}(A) \equiv \Psi'(A)G(\Psi(A))$ (where $G(\Psi)$ is given by (3.5)) is continuously differentiable for all $A \in \mathscr{A}$; (iii) there exist constants ϵ^-, ϵ^+, ϵ^* such that, for $a \in \mathscr{A}$, $\boldsymbol{x} \in \mathscr{D}$,*

$$0 \;<\; \epsilon^* \;<\; \epsilon^- \;<\; 1 \,, \; \epsilon^+ \;>\; 2 - \epsilon^- \;; \; \left|\Psi'(a)\right| \;<\; 1 - \epsilon^* \;; \tag{4.36}$$

$$\alpha_0^- - \mu(\epsilon^-, \beta_0, 1 - \epsilon^*)\Big(\boldsymbol{B}(\boldsymbol{x})\Big)^2 \;>\; \epsilon^- \,, \; \alpha_0^+ - \mu(\epsilon^+, \beta_0, 1 - \epsilon^*)\Big(\boldsymbol{B}(\boldsymbol{x})\Big)^2 \;<\; \epsilon^+ . \tag{4.37}$$

Then the a priori estimate (4.9) holds, and the steady state (3.1) is stable to arbitrary finite amplitude perturbations.

The proof of Criterion 4.3 is analogous to that of Criterion 4.1.

5. Conclusions

In this paper, we have applied the general principles developed in Part 1 with the aim of obtaining linear and nonlinear stability criteria for steady two-dimensional MHD flows of an ideal fluid. In the unperturbed steady state, both velocity and magnetic field are non-zero, and (in §§3 and 4) attention is restricted to the fields which have components only in the x- and y-directions. Moreover, stability is considered with respect to two-dimensional perturbations.

The use of the frozen-in 'modified vorticity field', whose existence was proved in Part 1, has proved useful in constructing an appropriate Casimir. This has been used in §3, by techniques which are now standard, to obtain a linear stability criterion. This criterion was obtained previously (by Holm *et al.* 1985) but the present treatment sheds new light on the problem.

In §4 we have developed procedures first proposed by Arnold (1965, 1966) to obtain nonlinear stability criteria. We have considered first 'isomagnetic' perturbations, i.e. perturbations from the steady state under which the magnetic field is frozen, and the vector potential of a material fluid particle is therefore conserved. This assumption leads to the stability Criterion 4.1, which admits comparison with the earlier linear stability Criterion 3.1. We then considered arbitrary initial perturbations (unconstrained by the isomagnetic condition) which requires continuation of the functions determining the steady state outside their initial domain of the definition. We have successfully overcome this difficulty, and hence obtained the general stability Criterion 4.2.

Much remains to be done in this area, particularly the testing of these stability criteria through computational experiments. This is the subject of a continuing investigation. Similar techniques can also be applied to axisymmetric steady states (Vladimirov, Moffatt & Ilin 1996), and to states involving helical symmetry.

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Appendix A. Derivation of equation (3.26)

From (3.18*b*), (3.23), the last term in (3.25) may be written as

$$\mathscr{I} \equiv 2\int_{\mathscr{D}} \frac{\partial F}{\partial W}\delta^2 w \mathrm{d}\tau = -2\int_{\mathscr{D}} \Psi\{\delta\rho, \delta m\}\mathrm{d}\tau .$$

Applying to this formula the identity

$$\int_{\mathscr{D}} A\{B,C\}\mathrm{d}\tau = \int_{\mathscr{D}} B\{C,A\}\mathrm{d}\tau + \int_{\partial\mathscr{D}} AB\nabla C \cdot \mathrm{d}\boldsymbol{l} \qquad \text{(A 1)}$$

we find

$$\begin{aligned} \mathscr{I} &= -2\int_{\mathscr{D}} \delta\rho\{\delta m, \Psi\} - 2\int_{\partial\mathscr{D}} \Psi\delta\rho\nabla\delta m \cdot \mathrm{d}\boldsymbol{l} \\ &= 2\int_{\mathscr{D}} \frac{\mathrm{d}\Psi}{\mathrm{d}A}\delta\rho\{A, \delta m\}\mathrm{d}\tau \quad \text{using (3.20)} \\ &= 2\int_{\mathscr{D}} \frac{\mathrm{d}\Psi}{\mathrm{d}A}\delta\rho\Big(\delta^1 w - \delta q - \{\delta\rho, M\}\Big)\mathrm{d}\tau \quad \text{using (3.18}a\text{)} \qquad \text{(A 2)} \end{aligned}$$

Note that

$$
\begin{aligned}
\mathscr{I}_1 &\equiv 2\int_{\mathscr{D}} \frac{d\Psi}{dA}\delta\rho\{\delta\rho, M\}d\tau = \int_{\mathscr{D}} \frac{d\Psi}{dA}\{(\delta\rho)^2, M\}d\tau \\
&= \int_{\mathscr{D}} (\delta\rho)^2\Big\{M, \frac{d\Psi}{dA}\Big\}d\tau \quad \text{using (A1), (3.20)} \\
&= \int_{\mathscr{D}} \frac{d^2\Psi}{dA^2}\frac{dA}{d\Psi}\Big(-\nabla^2 A + g(A)\Big)(\delta\rho)^2 d\tau \quad \text{using (3.11).}
\end{aligned}
\tag{A 3}
$$

Substitution of (A2), (A3) in (3.25) results in

$$
\begin{aligned}
\delta^2\mathscr{R} = \tfrac{1}{2}\int \Bigg\{&(\delta \boldsymbol{v})^2 + (\nabla\delta\rho)^2 + F_{WW}(\delta' w)^2 + 2\Big(F_{AW} + \frac{d\Psi}{dA}\Big)\delta\rho\delta' w \\
&+ \Big[F_{AA} - \frac{d^2\Psi}{dA^2}\frac{dA}{d\Psi}\Big(-\nabla^2 A + g(A)\Big)\Big](\delta\rho)^2 + 2\frac{\partial\Psi}{\partial A}\delta\rho\nabla^2\delta\psi\Bigg\}\, d\tau.
\end{aligned}
\tag{A 4}
$$

From (3.9), (3.24), we obtain

$$
F_{WA} = -\frac{\partial\Psi}{\partial A} - F_{WW}\left(\frac{dW}{dA}\right)
\tag{A 5}
$$

$$
\begin{aligned}
F_{AA} - F_{WW}\left(\frac{dW}{dA}\right)^2 = &\frac{\nabla A\cdot\nabla(\nabla^2 A)}{(\nabla A)^2} - \frac{\nabla\Psi\cdot\nabla(\nabla^2\Psi)}{(\nabla A)^2} \\
&-\frac{d^2 A}{d\Psi^2}\left(\frac{d\Psi}{dA}\right)^2\Big(-\nabla^2 A + g(A)\Big)
\end{aligned}
\tag{A 6}
$$

Substituting (A5) into (A4) and using (A6) we find

$$
\begin{aligned}
\delta^2\mathscr{R} = \tfrac{1}{2}\int_{\mathscr{D}} \Bigg[&\Big(\nabla\delta\psi\Big)^2 + \Big(\nabla\delta\rho\Big)^2 + \frac{\partial^2 F}{\partial^2 w}\Big(\delta' w - \frac{dW}{dA}\delta\rho\Big)^2 \\
&+2\frac{\partial\Psi}{\partial A}\delta\rho\nabla^2\delta\psi + \Big(\frac{\nabla A\cdot\nabla(\nabla^2 A)}{(\nabla A)^2} - \frac{\nabla\Psi\cdot\nabla(\nabla^2\Psi)}{(\nabla A)^2}\Big)\Big(\delta\rho\Big)^2\Bigg]d\tau.
\end{aligned}
\tag{A 7}
$$

Also, we have

$$
\begin{aligned}
\int_{\mathscr{D}}\Big[\nabla\Big(\frac{d\Psi}{dA}\delta\rho\Big)\Big]^2 &= \int_{\mathscr{D}}\Big[\Big(\nabla\frac{d\Psi}{dA}\Big)^2(\delta\rho)^2 + \Big(\frac{d\Psi}{dA}\Big)^2(\nabla\delta\rho)^2 + \frac{d\Psi}{dA}\nabla\frac{d\Psi}{dA}\cdot\nabla(\delta\rho)^2\Big]d\tau \\
&= \int_{\mathscr{D}}\Big[\Big(\frac{d\Psi}{dA}\Big)^2(\nabla\delta\rho)^2 - \frac{d\Psi}{dA}\Big(\nabla^2\frac{d\Psi}{dA}\Big)(\delta\rho)^2\Big]d\tau \quad \text{integrating by parts.}
\end{aligned}
\tag{A 8}
$$

Finally, after integrating by parts the third term in (A7) and using (A4) we arrive at equation (3.26).

Appendix B. Extension of the definition of functions $F_1(a)$ and $F_2(a)$

It will be sufficient to construct explicitly a continuation of $F_1(a)$ and $F_2(a)$ to all $a > A^+$. Continuation to all $a < A^-$ can be achieved in a similar way.

Suppose that (4.32*a*) is true. Then three different situations are possible: (i) $F_1''(A^+) > 0$, (ii) $F_1''(A^+) < 0$, or (iii) $F_1''(A^+) = 0$.

(i) If $F_1''(A^+) > 0$, we define $F_1(a)$ for $a > A^+$ such that

$$F_1'(a) = F_1'(A^+) + F_1''(A^+)\frac{z}{1+\psi_1 z} \tag{B 1}$$

where

$$z \equiv a - A^+ , \quad \psi_1 \equiv F_1''(A^+)/(1 - |\sigma_0| - F_1'(A^+)) \; > \; 0 . \tag{B 2}$$

It is easy to see that, with this definition, $F_1(a)$ is twice continuously differentiable for all $a \geqslant A^+$ and satisfies (4.33*c*). Note that

$$F_1''(a) = F_1''(A^+)\frac{1}{(1+\psi_1 z)^2} \; \leqslant \; F_1''(A^+) \tag{B 3}$$

for all $a \geqslant A^+$. Hence, (4.33*b*) is also satisfied.

With $F_1(a)$ given by (B 1), we choose the function $F_2(a)$ for $a > A^+$ in such a way that (4.33*a*) is satisfied. Before doing this, let us introduce functions $\hat{\alpha}^-(a)$ and $\hat{\alpha}^+(a)$ such that

$$\hat{\alpha}^-(a) \equiv \min_{x\in\mathscr{D}} \alpha(\boldsymbol{x}, a) , \quad \hat{\alpha}^+(a) \equiv \max_{x\in\mathscr{D}} \alpha(\boldsymbol{x}, a) \tag{B 4}$$

where $\alpha(\boldsymbol{x}, a)$ is still given by (4.5*a*). From (4.30), (B 4), it is obvious that

$$\alpha_0^- \leqslant \hat{\alpha}^-(a) \leqslant \alpha(\boldsymbol{x}, a) \leqslant \hat{\alpha}^+(a) \leqslant \alpha_0^+ . \tag{B 5}$$

Also, from the definitions of $\hat{\alpha}^-(a)$ and $\hat{\alpha}^+(a)$ it follows that

$$\hat{\alpha}^-(a) = F_2''(a) + \begin{cases} F_1''(a)\min Q(\boldsymbol{x}), & \text{if } F_1''(a) > 0 \\ 0, & \text{if } F_1''(a) = 0 \\ F_1''(a)\max Q(\boldsymbol{x}), & \text{if } F_1''(a) < 0; \end{cases} \tag{B 6}$$

and

$$\hat{\alpha}^+(a) = F_2''(a) + \begin{cases} F_1''(a)\max Q(\boldsymbol{x}), & \text{if } F_1''(a) > 0 \\ 0, & \text{if } F_1''(a) = 0 \\ F_1''(a)\min Q(\boldsymbol{x}), & \text{if } F_1''(a) < 0. \end{cases} \tag{B 7}$$

Since in our case $F_1''(a) > 0$ for $a \geqslant A^+$, we have

$$\hat{\alpha}^-(a) = F_2''(a) + F_1''(a)\min_{x\in\mathscr{D}} Q(\boldsymbol{x}) , \quad \hat{\alpha}^+(a) = F_2''(a) + F_1''(a)\max_{x\in\mathscr{D}} Q(\boldsymbol{x}) . \tag{B 8}$$

Now for all $a > A^+$ we take $F_2(a)$ such that

$$F_2''(a) = \hat{\alpha}^-(A^+) - F_1''(a)\min_{x\in\mathscr{D}} Q(\boldsymbol{x}) \tag{B 9}$$

where $F_1(a)$ is given by (B 1). With this choice, $F_2(a)$ is twice continuously differentiable for all $a \geqslant A^+$ and from (B 8)

$$\hat{\alpha}^-(a) = \hat{\alpha}^-(A^+) \geqslant \alpha_0^- , \quad a \geqslant A^+ . \tag{B 10}$$

Also from (B 8) we obtain

$$\hat{\alpha}^+(a) = \hat{\alpha}^-(A^+) + F_1''(a)\Big(\max_{x\in\mathscr{D}} Q(\boldsymbol{x}) - \min_{x\in\mathscr{D}} Q(\boldsymbol{x})\Big) . \tag{B 11}$$

From (B 11), we have

$$\hat{\alpha}^+(A^+) - \hat{\alpha}^-(A^+) = F_1''(A^+)\Big(\max_{x\in\mathscr{D}} Q(\boldsymbol{x}) - \min_{x\in\mathscr{D}} Q(\boldsymbol{x})\Big) . \tag{B 12}$$

Eliminating $\hat{\alpha}$ we find

$$\hat{\alpha}^+(a) = \hat{\alpha}^+(A^+) - \Big(F_1''(A^+) - F_1''(a)\Big)\Big(\max_{x\in\mathscr{D}} Q(\boldsymbol{x}) - \min_{x\in\mathscr{D}} Q(\boldsymbol{x})\Big) . \tag{B 13}$$

Finally, since $F_1''(A^+) \geqslant F_1''(a)$ for all $a \geqslant A^+$ we obtain

$$\hat{\alpha}^+(a) \leqslant \hat{\alpha}^+(A^+) \leqslant \alpha_0^- . \tag{B 14}$$

The inequalities (B 10), (B 14) are valid for all $a \geqslant A^+$ and coincide with (4.33*a*), so that our continuation satisfies all the conditions (4.33).

(ii) If $F_1''(A^+) < 0$, then for $a > A^+$ we define $F_1(a)$ such that

$$F_1'(a) = F_1'(A^+) + F_1''(A^+)\frac{z}{1+\psi_2 z} , \tag{B 15}$$

where

$$\psi_2 \equiv -F_1''(A^+)/(1-|\sigma_0| + F_1'(A^+)) \; > \; 0 . \tag{B 16}$$

It is easy to verify that with this choice $F_1(a)$ is twice continuously differentiable and satisfies (4.33*b,c*). In the case under consideration, $F_1''(A^+) < 0$, and hence from (B 6), (B 7) we have

$$\hat{\alpha}^-(a) = F_2''(a) + F_1''(a)\max_{x\in\mathscr{D}} Q(\boldsymbol{x}) , \quad \hat{\alpha}^+(a) = F_2''(a) + F_1''(a)\min_{x\in\mathscr{D}} Q(\boldsymbol{x}) . \tag{B 17}$$

We choose the function $F_2(a)$ for $a > A^+$ such that

$$F_2''(a) = \hat{\alpha}^-(A^+) - F_1''(a)\max_{x\in\mathscr{D}} Q(\boldsymbol{x}) \tag{B 18}$$

where $F_1(a)$ and $\hat{\alpha}^-(a)$ are given by (B 15), (B 17) respectively. It may be shown that $F_2(a)$, defined by (B 18), is twice continuously differentiable for all $a \geqslant A^+$ and that the inequalities (4.33*a*) are satisfied.

(iii) If $F_1''(A^+) = 0$, then we take $F_1(a)$ for $a > A^+$ such that $F_1'(a) = F_1'(A^+)$ and $F_2(a)$ such that $F_2''(a) = F_2''(A^+)$, so that equations (4.33) are satisfied.

Suppose now that (4.32*b*) is true; then, as in case (iii), we take $F_1'(a) = F_1'(A^+)$ and $F_2''(a) = F_2''(A^+)$, and the inequalities (4.33) are satisfied.

Thus, we have shown that a smooth continuation of the functions $F_1(a)$ and $F_2(a)$ to all a such that the conditions (4.33) remain satisfied can be constructed.

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