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On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part III. Stability criteria for axisymmetric flows

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The general theory developed in Part I of the present series is here applied to axisymmetric solutions of the equations governing the magnetohydrodynamics of ideal incompressible fluids. We first show a helpful analogy between axisymmetric MHD flows and flows of a stratified fluid in the Boussinesq approximation. We then construct a general Casimir as an integral of an arbitrary function of two conserved fields, namely the vector potential of the magnetic field and the scalar field associated with the 'modified vorticity field', the additional frozen-in field introduced in Part I. Using this Casimir, sufficient conditions for linear stability to axisymmetric perturbations are obtained by standard Arnold techniques. We exploit Arnold's method to obtain sufficient conditions for nonlinear (Lyapunov) stability of the MHD flows considered. The appropriate norm is a sum of the magnetic and kinetic energies and the mean square vector potential of the magnetic field.

1. Introduction

In this paper we develop the approach initiated in Part I (Vladimirov and Moffatt 1995) and then continued in Part II (Vladimirov *et al.* 1996). In Part I new variational principles for magnetohydrodynamic (MHD) flows of an ideal incompressible fluid were established. For such flows, the magnetic field $\mathbf{h}(\mathbf{x}, t)$ is frozen in the fluid, but the vorticity field $\boldsymbol{\omega}(\mathbf{x}, t)$ is not frozen, since the Lorentz force is in general rotational. However, in Part I we identified a 'modified vorticity field' $\mathbf{w}(\mathbf{x}, t)$ (see (1.8) below), which is frozen in the fluid, and which reduces to $\boldsymbol{\omega}$ when $\mathbf{h} \equiv 0$. The existence of this additional frozen-in field has consequences for the construction of Casimirs, the integral invariants that play an essential role in the derivation of sufficient conditions for stability (or 'stability criteria') for steady solutions $\{\mathbf{U}(\mathbf{x}), \mathbf{H}(\mathbf{x})\}$ of the governing equations.

We consider here axisymmetric flows (invariant under rotations around a fixed axis), and we demonstrate first a helpful analogy between such flows and flows of a stratified fluid in the Boussinesq approximation (Sec. 2). Analogies between stratified and rotating flows can be traced back to Rayleigh (1916), and have been placed on a firm basis by Vladimirov (1985*a*, *b*). In MHD an analogy of the type to be considered here was already noted by Howard and Gupta (1962), but its precise nature has not been previously revealed. In comparison with Part II, where the analogy between two-dimensional MHD flows and flows of a stratified fluid in the Boussinesq approximation had the precise form of an isomorphism, the analogy considered here is somewhat weaker, but nevertheless it provides a useful guide in some of the calculations in Secs 3–5.

For the axisymmetric situation, the components of the fields $\{\mathbf{U}, \mathbf{H}\}$ are functions of r and z (in cylindrical polar coordinates (r, ϕ, z)). This case is of particular importance in the context of plasma confinement devices (e.g. tokamak and reversed-field pinch). We first show how the presence of two frozen fields, $\mathbf{h}$ and $\mathbf{w}$, leads to determination of an appropriate Casimir as an integral over the fluid domain of a function of two conserved scalar fields (associated with $\mathbf{h}$ and $\mathbf{w}$). Then, following the prescription of Arnold (1965*a*, *b*), we obtain the sufficient conditions for linear stability (to axisymmetric disturbances) of steady axisymmetric MHD flow. The linear criterion obtained in Sec. 3 (Criterion 3.1) is close to the criterion obtained by Almaguer *et al.* (1988). The latter includes compressibility effects, and is obtained from the requirement that a certain 8×8 matrix be positive-definite; as the authors recognize, an improved criterion may in principle be obtained if the fluid domain is bounded. The stability criterion 3.2 that we obtain in Sec. 3 (for the incompressible case) is somewhat simpler, and the improvement of the criterion for the bounded domain is explicitly realized.

A special case that is attracting attention in the context of the MHD of accretion disks (Balbus and Hawley 1991; Ogilvie and Pringle 1996) is that in which the velocity field is purely toroidal (and typically that corresponding to the Keplerian angular velocity distribution). This is a case for which the stability criterion (for axisymmetric disturbances) takes a simplified form (Sec. 3, Criterion 3.4).

In Sec. 4 we consider nonlinear stability of the steady state, i.e. Lyapunov stability with respect to a norm based on the total energy and the mean square vector potential of the perturbed magnetic field. We consider first the case of 'isomagnetic' perturbations under which the magnetic field is a 'frozen-field' perturbation of the steady state, and we obtain conditions (Criterion 4.1) under which the norm of the perturbation remains bounded by a constant multiple of its initial value for all time.

Then, in Sec. 5, we establish nonlinear stability criteria for two particular classes of steady MHD flows, namely for the purely poloidal field and flow (toroidal components of both the velocity and the magnetic field are absent) and for the steady state in which all components of the magnetic field are non-zero while the velocity field has only a toroidal component. For such steady states, we generalize Criterion 4.1 to cover arbitrary axisymmetric perturbations (that are not 'isomagnetic'); the difficulty here centres on the problem of appropriate continuation of functions describing the steady state beyond their initial range of definition; this difficulty is addressed in detail and is successfully overcome.

We conclude this introduction with a statement of the governing equations.

Consider an incompressible, inviscid and perfectly conducting fluid contained in a domain $\mathscr{D}$ with fixed boundary $\partial\mathscr{D}$. Let $\mathbf{u}(\mathbf{x}, t)$ be the velocity field, $\mathbf{h}(\mathbf{x}, t)$

the magnetic field (in Alfvén velocity units), $p(\mathbf{x},t)$ the pressure field (divided by density), and $\mathbf{j} = \boldsymbol{\nabla}\wedge\mathbf{h}$ the current density in the fluid. Then the governing equations are

$$D\mathbf{u} \equiv \left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u} = -\boldsymbol{\nabla}p+\mathbf{j}\wedge\mathbf{h}, \tag{1.1}$$

$$L\mathbf{h} \equiv \frac{\partial \mathbf{h}}{\partial t}-\boldsymbol{\nabla}\wedge(\mathbf{u}\wedge\mathbf{h}) = 0, \tag{1.2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{h} = 0. \tag{1.3}$$

Here D is the material derivative and L is the Lie derivative. Equation (1.2) implies that $\mathbf{h}$ is frozen in the fluid; its flux through any material circuit is conserved. We suppose that the boundary $\partial\mathscr{D}$ is perfectly conducting, and therefore the magnetic field $\mathbf{h}$ does not penetrate through $\partial\mathscr{D}$. The boundary conditions are then

$$\mathbf{n}\cdot\mathbf{u} = 0, \quad \mathbf{n}\cdot\mathbf{h} = 0 \quad \text{on } \partial\mathscr{D}. \tag{1.4}$$

We suppose further that at $t = 0$ the fields $\mathbf{u}$ and $\mathbf{h}$ are smooth and satisfy (1.3), but are otherwise arbitrary.

Taking the curl of (1.1), we obtain

$$L\boldsymbol{\omega} = \boldsymbol{\nabla}\wedge(\mathbf{j}\wedge\mathbf{h}), \tag{1.5}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\wedge\mathbf{u}$ is the vorticity field. Equation (1.5) implies that vortex lines are not frozen in the fluid unless the right-hand side of (1.5) is identically zero (i.e. unless the Lorentz force $\mathbf{j}\wedge\mathbf{h}$ is irrotational). However, as shown in Part I, a 'modified vorticity field' $\mathbf{w}$ may be defined as follows: let $\mathbf{g}(\mathbf{x},t)$ be an arbitrary solenoidal field satisfying

$$\boldsymbol{\nabla}\wedge(\mathbf{g}\wedge\mathbf{h}) = 0, \tag{1.6}$$

and let $\mathbf{m}(\mathbf{x},t)$ be defined by

$$L\mathbf{m} = \mathbf{j}+\mathbf{g}, \qquad \boldsymbol{\nabla}\cdot\mathbf{m} = 0. \tag{1.7}$$

Then the field $\mathbf{w}$ defined by

$$\mathbf{w} = \boldsymbol{\omega}+\boldsymbol{\nabla}\wedge(\mathbf{h}\wedge\mathbf{m}) \tag{1.8}$$

satisfies

$$L\mathbf{w} = 0, \tag{1.9}$$

and provides the appropriate frozen-field generalization of $\boldsymbol{\omega}$ for ideal MHD. Note that, since $\mathbf{h}$ and $\mathbf{w}$ are now two independent frozen-in fields, it follows that $\boldsymbol{\nabla}\wedge(\mathbf{h}\wedge\mathbf{w})$ is also frozen-in (Tur and Yanovsky 1993), and, by iteration, an infinite family of such derived frozen-in fields may be constructed.

The global invariants of the system (1.1)–(1.4) are the total energy

$$\mathscr{E}_{\text{tot}} = \frac{1}{2}\int_{\mathscr{D}} (|\mathbf{u}|^2+|\mathbf{h}|^2)\,d\tau, \tag{1.10}$$

the magnetic helicity

$$\mathscr{H}_M = \int_{\mathscr{D}} (\mathbf{h}\cdot\text{curl}^{-1}\,\mathbf{h})\,d\tau, \tag{1.11}$$

the cross-helicity

$$\mathscr{H}_C = \int_{\mathscr{D}} \mathbf{u}\cdot\mathbf{h}\,d\tau = \int (\mathbf{h}\cdot \mathrm{curl}^{-1}\,\mathbf{w})\,d\tau \tag{1.12}$$

and the 'generalized' helicity

$$\mathscr{H}_W = \int_{\mathscr{D}} \mathbf{w}\cdot \mathrm{curl}^{-1}\,\mathbf{w}\,d\tau. \tag{1.13}$$

By the arguments of Moffatt (1969), the three helicities are all topological in character.

2. Analogy between axisymmetric MHD and stratified flow

Suppose that $\mathbf{u}$ and $\mathbf{h}$ are invariant under rotations around a fixed axis. It is natural to use cylindrical polar coordinates (r, ϕ, z), z being a coordinate along the axis of the symmetry. The fluid domain is supposed to be axisymmetric. Let $\mathscr{D}$ denote the meridian section of this domain, with boundary $\partial\mathscr{D}$ on which $\mathbf{n} = (n_r, 0, n_z)$ is the unit normal. For simplicity, we shall suppose that $\mathscr{D}$ is simply connected.

We now decompose $\mathbf{u}$ and $\mathbf{h}$ into poloidal and toroidal parts:

$$\left.\begin{aligned} \mathbf{h}(r,z,t) &= \mathbf{b} + r\rho_2\,\mathbf{e}_\phi, \qquad \mathbf{b} = \frac{1}{r}\frac{\partial\rho}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial\rho}{\partial z}\mathbf{e}_r,\\ \mathbf{u}(r,z,t) &= \mathbf{v} + \frac{\rho_1}{r}\mathbf{e}_\phi, \qquad \mathbf{v} = \frac{1}{r}\frac{\partial\psi}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial\psi}{\partial z}\mathbf{e}_r. \end{aligned}\right\} \tag{2.1}$$

Then the poloidal parts of (1.1) and (1.2) may be written in the form

$$D\mathbf{v} = -\boldsymbol{\nabla}\tilde{p} - (\rho_1)^2\,\boldsymbol{\nabla}\Phi_1 - (\rho_2)^2\,\boldsymbol{\nabla}\Phi_2 - \rho\boldsymbol{\nabla}\Phi, \tag{2.2}$$

$$D\rho = 0, \tag{2.3}$$

where $D = \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla}$, $\boldsymbol{\nabla} = (\partial/\partial r, \partial/\partial z)$ and

$$\Phi_1 = (2r^2)^{-1}, \qquad \Phi_2 = \tfrac{1}{2}r^2, \qquad \tilde{p} = p + (\rho_2)^2\,\Phi_2 - \rho\Phi, \tag{2.4}$$

$$\Phi = -\hat{K}\rho, \qquad \hat{K} = \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial z^2}. \tag{2.5}$$

The toroidal parts of (1.1) and (1.2) become

$$D\rho_1 = \frac{1}{r}\{\rho, r^2\rho_2\}, \tag{2.6}$$

$$D\rho_2 = \frac{1}{r}\left\{\rho, \frac{\rho_1}{r^2}\right\}, \tag{2.7}$$

where now $\{a, b\} \equiv \partial(a,b)/\partial(r,z)$. The boundary conditions (1.4) become

$$\mathbf{v}\cdot\mathbf{n} = 0, \quad \rho = \text{const} \quad \text{on } \partial\mathscr{D}. \tag{2.8}$$

Equations (2.2)–(2.7) admit the following physical interpretation. The term $-(\rho_1)^2\,\boldsymbol{\nabla}\Phi_1$ in (2.2) represents the centrifugal force associated with the swirl (toroidal) velocity $(\rho_1/r)\,\mathbf{e}_\phi$; $-(\rho_2)^2\,\boldsymbol{\nabla}\Phi_2$ represents the analogous 'hoop stress' of the toroidal magnetic field acting in the inward radial direction; and $-\rho\boldsymbol{\nabla}\Phi$ represents the Lorentz force associated with the ϕ component of $\mathbf{j}$ and the poloidal field $\mathbf{b}$†. The equation $D\rho = 0$ reflects the frozen-in character of the poloidal field $\mathbf{b}$: equation (2.6) for $D\rho_1$ represents generation of swirl by the ϕ component of $\mathbf{j}\wedge\mathbf{h}$; and equation (2.7) for $D\rho_2$ represents generation of toroidal field by differential rotation acting on $\mathbf{b}$.

Equations (2.2)–(2.8) admit an alternative interpretation as the equations of the flow of an analogue Boussinesq fluid endowed with three densities ρ_1^2, ρ_2^2 and ρ influenced by force fields with potentials Φ_1, Φ_2 and Φ respectively. Note, however, that source terms are present in (2.6) and (2.7) for ρ_1 and ρ_2. The analogy may therefore seem somewhat strained, but nevertheless it provides a useful guide in some of manipulations that follow.

3. Stability of axisymmetric MHD flows

Consider now a steady solution of (2.2), (2.3) and (2.6)–(2.8) in the form

$$\mathbf{v} = \mathbf{V}(r,z) = \frac{1}{r}\frac{\partial\Psi}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial\Psi}{\partial z}\mathbf{e}_r, \qquad \tilde{p} = P(r,z), \tag{3.1a}$$

$$\mathbf{b} = \mathbf{B}(r,z) = \frac{1}{r}\frac{\partial A}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial A}{\partial z}\mathbf{e}_r, \qquad \rho_1 = R_1(r,z), \qquad \rho_2 = R_2(r,z), \tag{3.1b}$$

Capital letters will be used throughout for properties of the steady state whose stability is to be investigated. In this state, (2.3) reduces to

$$\mathbf{V}\cdot\boldsymbol{\nabla}A = \frac{1}{r}\{\Psi, A\} = 0.$$

This in turn implies a functional dependence $\Psi = \Psi(A)$, and therefore

$$\mathbf{V} = \Psi'(A)\,\mathbf{B}, \qquad \Psi'(A) \equiv \frac{d\Psi}{dA}. \tag{3.2}$$

Equations (2.6) and (2.7) simplify to the equations

$$\frac{1}{r}\{A, \Psi'(A)\,R_1 - r^2R_2\} = 0, \qquad \frac{1}{r}\left\{A, \Psi'(A)\,R_2 - \frac{R_1}{r^2}\right\} = 0,$$

so that

$$\Psi'(A)\,R_2 - \frac{R_1}{r^2} = G_1(A), \qquad \Psi'(A)\,R_1 - r^2R_2 = G_2(A) \tag{3.3}$$

for some functions $G_1(A)$ and $G_2(A)$. The toroidal component of the curl of (2.2) gives, after use of (3.2) and (3.3),

$$\frac{1}{r}\{A, \Psi'(A)\,Q - J - G_1'(A)\,R_1 - G_2'(\Psi)\,R_2 + \Psi''(A)\,R_1R_2\} = 0, \tag{3.4}$$

where

$$Q \equiv -\hat{K}\Psi, \qquad J \equiv -\hat{K}A. \tag{3.5}$$

† This term has already been identified for two-dimensional flow in Part II, but here Φ is modified to take account of the coordinate system.

Hence we have the generalized Grad–Shafranov equation

$$-\Psi'(A)\hat{K}\Psi+\hat{K}A-G_1'(A)R_1-G_2'(A)R_2+\Psi''(A)R_1R_2=G(A), \qquad (3.6)$$

for some function $G(A)$.

Consider now (1.6)–(1.9). Again, we decompose the solenoidal fields $\mathbf{m}$ and $\mathbf{g}$ (defined by (1.6) and (1.7)) into poloidal and toroidal parts

$$\mathbf{m}(r,z,t)=\mathbf{m}_p+r\eta\mathbf{e}_\phi, \qquad \mathbf{m}_p=\frac{1}{r}\frac{\partial\chi}{\partial r}\mathbf{e}_z-\frac{1}{r}\frac{\partial\chi}{\partial z}\mathbf{e}_r, \qquad (3.7a)$$

$$\mathbf{g}(r,z,t)=\mathbf{g}_p+r\mu\mathbf{e}_\phi, \qquad \mathbf{g}_p=\frac{1}{r}\frac{\partial\zeta}{\partial r}\mathbf{e}_z-\frac{1}{r}\frac{\partial\zeta}{\partial z}\mathbf{e}_r, \qquad (3.7b)$$

so that $\chi(r,z,t)$ and $\zeta(r,z,t)$ are axisymmetric 'flux functions' for the poloidal components of the fields $\mathbf{m}$ and $\mathbf{g}$.

Equation (1.6) is satisfied by taking

$$\zeta=\zeta(\rho), \qquad \mu=\rho_2\zeta'(\rho)+g(\rho), \qquad (3.8)$$

where $\zeta(\rho)$ and $g(\rho)$ are arbitrary functions. The toroidal part of (1.7) then takes the form

$$D\eta=\frac{1}{r}\left\{\chi,\frac{\rho_1}{r^2}\right\}+\Phi+g(\rho)+\rho_2\zeta'(\rho), \qquad (3.9a)$$

while the poloidal part simplifies to the equation

$$D\chi=r^2\rho_2+\zeta(\rho). \qquad (3.9b)$$

Equations (2.6) and (3.9b) have the consequence that

$$D\left(\rho_1-\frac{1}{r}\{\rho,\chi\}\right)=0. \qquad (3.10)$$

This equation is in fact equivalent to the poloidal part of (1.9) governing the evolution of the 'generalized vorticity' (1.8); the quantity $\rho_1-r^{-1}\{\rho,\chi\}$ is the axisymmetric 'flux function' for poloidal components of the 'modified vorticity' $\mathbf{w}$. Equation (3.10) has an obvious particular solution:

$$\rho_1=\frac{1}{r}\{\rho,\chi\}+C(\rho), \qquad (3.11)$$

with arbitrary $C(\rho)$. According to (3.10), if (3.11) is satisfied at the initial instant $t=0$ then it will remain true for all $t>0$. (Note that (3.11) does not impose any restriction on the class of admissible motions. It only implies a special choice of initial data for the field χ.) Therefore (2.6) is no longer needed. Using (3.11), we can eliminate ρ_1 from (2.2) and (2.7) and consider (3.11) as a replacement for (2.6).

The toroidal component w_ϕ of the 'modified vorticity field' is given by

$$w_\phi=rq+\{\eta,\rho\}+\{\chi,\rho_2\}, \qquad q\equiv-\hat{K}\psi, \qquad (3.12)$$

and satisfies the equation

$$D\left(\frac{w_\phi}{r}\right) = \frac{1}{r}\left\{\rho_1 - \frac{1}{r}\{\rho, \chi\}, \frac{\rho_1}{r^2}\right\}, \tag{3.13}$$

which represents the toroidal part of (1.9). (Note that rq is the toroidal component of the vorticity $\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \wedge \mathbf{u}$.)

Let (3.11) be satisfied. Then, using (3.11) and (3.13), it may be shown that

$$D\lambda = 0, \tag{3.14}$$

where

$$\lambda \equiv \frac{w_\phi}{r} - C'(\rho)\,\rho_2 = q + \frac{1}{r}\{\eta, \rho\} + \frac{1}{r}\{\chi, \rho_2\} - C'(\rho)\,\rho_2. \tag{3.15}$$

(The validity of (3.14) may also be verified by straightforward calculation, without any reference to the 'modified vorticity field'.)

In the steady state (3.1),

$$\chi = X(r,z), \qquad \eta = N(r,z), \tag{3.16}$$

and, from (3.9*b*), $X(r,z)$ satisfies

$$\frac{1}{r}\{\Psi, X\} = r^2 R_2 + \zeta(A). \tag{3.17}$$

In order to ensure the single-valuedness of the function $X(r,z)$ in $\mathscr{D}$, the function $\zeta(A)$ is chosen in the following way. If $\mathscr{D}$ is bounded then the streamlines $\Psi =$ const. are closed curves in $\mathscr{D}$, and we define

$$\zeta(A(\Psi)) \equiv -\oint_{\Psi=\text{const.}} |\mathbf{V}|^{-1}\, r^2 R_2\, dl \bigg/ \oint_{\Psi=\text{const.}} |\mathbf{V}|^{-1}\, dl. \tag{3.18}$$

If $\mathscr{D}$ is unbounded then on any streamlines that are not closed we simply choose ζ so that $\zeta(A)$ is smooth function of A (for example, if no streamlines are closed, we simply take $\zeta \equiv 0$). From (3.9*a*), $N(r,z)$ satisfies

$$\frac{1}{r}\{\Psi, N\} = \frac{1}{r}\left\{X, \frac{R_1}{r^2}\right\} + J + g(A) + R_2\,\zeta'(A). \tag{3.19}$$

Here the function $\zeta(A)$ has already been defined. Now we choose an arbitrary function $g(A)$ similarly to the choice of $\zeta(A)$: we take

$$g(A(\Psi)) \equiv -\oint_{\Psi=\text{const.}} |\mathbf{V}|^{-1}\left[\frac{1}{r}\left\{X, \frac{R_1}{r^2}\right\} + J + R_2\,\zeta'(A)\right] dl \bigg/ \oint_{\Psi=\text{const.}} |\mathbf{V}|^{-1}\, dl \tag{3.20}$$

for closed streamlines and make a smooth continuation of $g(A(\Psi))$ for all streamlines that are not closed. With this choice of $g(A)$, the function $N(r,z)$ is always single-valued in $\mathscr{D}$.

For the steady state, (3.11) reduces to

$$R_1 = \frac{1}{r}\{A, X\} + C(A). \tag{3.21}$$

The function $C(A)$ is related to $G_2(A)$, (3.3b), by the equality

$$G_2(A) = \zeta(A) + \Psi'(A)\, C(A), \tag{3.22}$$

which may be deduced from (3.17) and (3.21).

Equation (3.14) implies that in the steady state, $\lambda = \Lambda(A)$, where

$$\begin{aligned}\Psi'(A)\,\Lambda(A) = \Psi'(A)\,Q - J - G_1'(A)\,R_1 - G_2'(A)\,R_2 \\ + \Psi'''(A)\,R_1\,R_2 + C(A)\,G_1'(A) - g(A)\end{aligned} \tag{3.23}$$

(this last equation is obtained by simple manipulations with (3.15), (3.19) and (3.21)). Comparison of (3.23) with (3.6) shows that

$$\Psi'(A)\,\Lambda(A) - C(A)\,G_1'(A) + g(A) = G(A), \tag{3.24}$$

and this is an axisymmetric counterpart of (3.10) of Part II.

Under general unsteady evolution, the fields ρ and λ are conserved ($D\rho = D\lambda = 0$); therefore the appropriate Casimir invariant is given by

$$\mathscr{C} \equiv \int_{\mathscr{D}} F(\rho, \lambda)\, d\tau, \tag{3.25}$$

where $d\tau = r\,dr\,dz$, and $F(\rho, \lambda)$ is an arbitrary function of ρ and λ.

Now consider the conserved functional

$$\mathscr{R}\{\psi, \rho, \rho_2, \chi, \eta\} \equiv \mathscr{E} + \mathscr{C} + \mathscr{H}_M, \tag{3.26}$$

where

$$\mathscr{E} \equiv \frac{1}{2}\int_{\mathscr{D}} \left[|\mathbf{v}|^2 + \frac{1}{r^2}(\boldsymbol{\nabla}\rho)^2 + \rho_1^2\,\Phi_1 + \rho_2^2\,\Phi_2\right] d\tau \tag{3.27}$$

is the conserved energy of the system and

$$\mathscr{H}_M \equiv \int_{\mathscr{D}} S(\rho)\,\rho_2\, d\tau \tag{3.28}$$

is the conserved 'generalized' magnetic helicity (cf. (1.11)), with an arbitrary function $S(\rho)$. The dependence of $\mathscr{R}$ on χ and η enters through both λ and ρ_1. Let $\delta\psi$, $\delta\rho$, $\delta\rho_2$, $\delta\chi$ and $\delta\eta$ be independent variations, and let

$$\delta\mathbf{v} = \frac{1}{r}\frac{\partial\delta\psi}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial\delta\psi}{\partial z}\mathbf{e}_r, \qquad \delta q = -\hat{K}\delta\psi. \tag{3.29}$$

From (3.11) and (3.15), the first and second variations of ρ_1 and λ are

$$\delta^1\rho_1 = \frac{1}{r}\{A, \delta\chi\} + \frac{1}{r}\{\delta\rho, X\} + C'(A)\,\delta\rho, \tag{3.30a}$$

$$\begin{aligned}\delta^1\lambda = \delta q + \frac{1}{r}\{N, \delta\rho\} + \frac{1}{r}\{\delta\eta, A\} \\ + \frac{1}{r}\{\delta\chi, R_2\} + \frac{1}{r}\{X, \delta\rho_2\} - C''(A)\,R_2\,\delta\rho - C'(A)\,\delta\rho_2,\end{aligned} \tag{3.30b}$$

$$\delta^2\rho_1 = \frac{1}{r}\{\delta\rho, \delta\chi\} + \frac{1}{2}C''(A)(\delta\rho)^2, \tag{3.30c}$$

$$\delta^2\lambda = \frac{1}{r}\{\delta\eta, \delta\rho\} + \frac{1}{r}\{\delta\chi, \delta\rho_2\} - \frac{1}{2}C'''(A)R_2(\delta\rho)^2 - C''(A)\,\delta\rho\delta\rho_2. \tag{3.30d}$$

The first variation of $\mathscr{R}$ calculated at the steady solution (3.1) is

$$\delta\mathscr{R} = \int_{\mathscr{D}}\left[\frac{1}{r^2}\nabla\Psi\cdot\nabla\delta\psi + \frac{1}{r^2}\nabla A\cdot\nabla\delta\rho + 2R_1\Phi_1\delta^1\rho_1 \right.$$
$$\left. + [2R_2\Phi_2 + S(A)]\,\delta\rho_2 + [F_A + S'(A)R_2]\,\delta\rho + F_\Lambda\,\delta^1\lambda\right]d\tau, \tag{3.31}$$

where $F_A = \partial F(A,\Lambda)/\partial A$ and $F_\Lambda = \partial F(A,\Lambda)/\partial\Lambda$. From here on, F and its derivatives are always evaluated at $\rho = A$ and $\lambda = \Lambda$.

Since the streamfunction Ψ is defined (by (3.1)) up to a constant and the domain $\mathscr{D}$ is simply connected, we take

$$\Psi = 0 \quad \text{on } \partial\mathscr{D}. \tag{3.32}$$

For $\delta\rho$, we take the boundary condition

$$\delta\rho = 0 \quad \text{on } \partial\mathscr{D}, \tag{3.33}$$

which is consistent with (2.8).

After simple manipulations using (3.30), (3.32) and (3.33) and the identity (A 1) of Appendix A, (3.31) becomes

$$\delta\mathscr{R} = \int_{\mathscr{D}}\left[(F_\Lambda + \Psi)\,\delta q + \left(r^2R_2 + S + \frac{1}{r}\{F_\Lambda, X\} - F_\Lambda C'\right)\delta\rho_2\right.$$
$$+\left(\frac{1}{r}\{R_2, F_\Lambda\} - \frac{1}{r}\left\{A, \frac{R_1}{r^2}\right\}\right)\delta\chi + \delta\eta\,\frac{1}{r}\{A, F_\Lambda\}$$
$$\left.+\left(F_A + J + S'R_2 + \frac{1}{r}\{F_\Lambda, N\} + \frac{1}{r}\left\{X, \frac{R_1}{r^2}\right\} - C''F_\Lambda R_2 + \frac{R_1}{r^2}C'\right)\delta\rho\right]d\tau$$
$$+\oint_{\partial\mathscr{D}} F_\Lambda(\delta\chi\,\nabla R_2 - \delta\rho_2\,\nabla X)\cdot d\mathbf{l}. \tag{3.34}$$

From (3.34), the sufficient conditions for $\delta\mathscr{R} = 0$ are

$$\left.\begin{aligned} &F_\Lambda = -\Psi, \quad S(A) = -r^2R_2 + F_\lambda C' - \frac{1}{r}\{F_\Lambda, X\},\\ &F_A + J + S'R_2 + \frac{1}{r}\{F_\Lambda, N\} + \frac{1}{r}\left\{X, \frac{R_1}{r^2}\right\} - C''F_\Lambda R_2 + \frac{R_1}{r^2}C' = 0,\\ &\frac{1}{r}\{A, F_\Lambda\} = 0, \quad \frac{1}{r}\{R_2, F_\Lambda\} - \frac{1}{r}\left\{A, \frac{R_1}{r^2}\right\} = 0 \end{aligned}\right\} \quad \text{in } \mathscr{D},$$

$F_\Lambda = 0 \quad \text{on } \partial\mathscr{D}.$

It may be shown that all these conditions are satisfied if we choose arbitrary functions $F(A, \Lambda)$ and $S(A)$ such that

$$F_\Lambda(A, \Lambda) = -\Psi(A), \tag{3.35a}$$

$$F_A(A, \Lambda) = g(A) + C'(A)\, G_1(A), \tag{3.35b}$$

$$S(A) = \zeta(A) - \Psi(A)\, C'(A). \tag{3.35c}$$

These equations do not determine $F(\rho, \lambda)$ uniquely, since in the steady state (3.1) the functions $A(r, z)$ and $\Lambda(r, z)$ are dependent. The remaining freedom in $F(\rho, \lambda)$ will be used later. Note that, by differentiating (3.35a, b) with respect to A, we obtain

$$F_{\Lambda\Lambda}(A, \Lambda)\, \Lambda'(A) + F_{\Lambda A}(A, \Lambda) = -\Psi'(A), \tag{3.36a}$$

$$F_{AA}(A, \Lambda) + F_{\Lambda A}(A, \Lambda)\, \Lambda'(A) = g'(A) + [G_1(A)\, C'(A)]'. \tag{3.36b}$$

Now consider the second variation $\delta^2\mathscr{R}$ evaluated at the steady solution (3.1). It is given by

$$\begin{aligned}\delta^2\mathscr{R} = \frac{1}{2}\int_{\mathscr{D}} \bigg[& \frac{1}{r^2}(\nabla\delta\psi)^2 + \frac{1}{r^2}(\nabla\delta\rho)^2 + \frac{1}{r^2}(\delta^1\rho_1)^2 + r^2(\delta\rho_2)^2 \\ & + (F_{AA} + S''R_2)(\delta\rho)^2 + 2F_{\Lambda A}\,\delta\rho\,\delta^1\lambda + F_{\Lambda\Lambda}(\delta^1\lambda)^2 \\ & + 2S'\,\delta\rho\,\delta\rho_2 + \frac{2}{r^2}R_1\,\delta^2\rho_1 + 2F_\Lambda\,\delta^2\lambda \bigg] d\tau. \end{aligned} \tag{3.37}$$

After standard transformations (see Appendix A), this may be expressed in the form

$$\begin{aligned}\delta^2\mathscr{R} = \frac{1}{2}\int_{\mathscr{D}} \bigg[& \frac{1}{r^2}[\nabla(\delta\psi - \sigma\,\delta\rho)]^2 + \frac{1}{r^2}(1-\sigma^2)(\nabla\delta\rho)^2 \\ & + \frac{1}{r^2}(\delta^1\rho_1)^2 + r^2(\delta\rho_2)^2 - 2\sigma\delta^1\rho_1\,\delta\rho_2 + 2f_1\,\delta\rho\,\delta^1\rho_1 + 2f_2\,\delta\rho\,\delta\rho_2 \\ & + f_3(\delta\rho)^2 + F_{\Lambda\Lambda}[\delta^1\lambda - \Lambda'(A)\,\delta\rho]^2 \bigg] d\tau, \end{aligned} \tag{3.38}$$

where

$$\sigma = \Psi'(A), \tag{3.38a}$$

$$f_1 \equiv \frac{\sigma\nabla A\cdot\nabla R_2 - \nabla A\cdot\nabla(R_1/r^2)}{(\nabla A)^2}, \tag{3.38b}$$

$$f_2 \equiv \frac{\sigma\nabla A\cdot\nabla R_1 - \nabla A\cdot\nabla(r^2 R_2)}{(\nabla A)^2}, \tag{3.38c}$$

$$f_3 \equiv \frac{\nabla\Psi\cdot\nabla Q - \nabla A\cdot\nabla J}{(\nabla A)^2} + \sigma\hat{K}\sigma - \frac{f_1\,\nabla A\cdot\nabla R_1 + f_2\,\nabla A\cdot\nabla R_2}{(\nabla A)^2}. \tag{3.38d}$$

The quantities f_1, f_2 and f_3 defined by (3.38b–d) are functions of r and z in $\mathscr{D}$.

Note that, by using (3.3) and (3.6), the functions f_1, f_2 and f_3 may be expressed in terms of $G(A)$, $G_1(A)$ and $G_2(A)$:

$$\left.\begin{aligned} f_1 &= G_1'(A) - R_2\,\Psi''(A), \qquad f_2 = G_2'(A) - R_1\,\Psi''(A), \\ f_3 &= \sigma\hat{K}\sigma - \Psi''(A)\,Q + G'(A) - \Psi'''(A)\,R_1\,R_2 + R_1\,G_1''(A) + R_2\,G_2''(A). \end{aligned}\right\} \quad (3.39)$$

According to the general theory of Arnold (1965*a*, *b*), if the quadratic form $\delta^2\mathscr{R}$ is definite in sign for all admissible variations $(\delta\psi, \delta\rho, \delta\rho_2, \delta\chi, \delta\eta)$ then the system considered is linearly stable. Here $\delta^2\mathscr{R}$ is positive-definite provided that

$$F_{\Lambda\Lambda} \geqslant 0, \qquad \sigma^2 \leqslant 1, \qquad f_3 - \left(\frac{f_2}{r}\right)^2 - \frac{(rf_1 + \sigma f_2/r)^2}{1-\sigma^2} \geqslant 0 \qquad (3.40)$$

throughout $\mathscr{D}$. We satisfy the first of these by using the remaining freedom in the choice of $F(A, \Lambda)$. Let

$$F(A, \Lambda) = F_0(\Lambda) + \Lambda F_1(A) + F_2(A), \qquad (3.41)$$

where F_0, F_1 and F_2 are smooth functions such that

$$F_0''(\Lambda) \geqslant 0, \qquad F_1(A) = -\Psi - F_0'(\Lambda), \qquad F_2'(A) = g(A) + C'(A)\,G_2(A) - \Lambda F_1'(A). \qquad (3.42)$$

Then (3.35*a*, *b*) are satisfied. We thus obtain the following stability criterion.

Criterion 3.1. *The steady state (3.1) is linearly stable provided that*

$$[\Psi'(A)]^2 \leqslant 1 \qquad (\textit{or, equivalently}, |\mathbf{V}| \leqslant |\mathbf{B}|), \qquad (3.43)$$

$$I_0 + I_1 \geqslant 0, \qquad (3.44)$$

throughout $\mathscr{D}$, *where*

$$I_0 \equiv \frac{\nabla\Psi\cdot\nabla Q - \nabla A\cdot\nabla J}{(\nabla A)^2} + \frac{d\Psi}{dA}\hat{K}\frac{d\Psi}{dA}, \qquad (3.45a)$$

$$I_1 \equiv -\frac{f_1\,\nabla A\cdot\nabla R_1 + f_2\,\nabla A\cdot\nabla R_2}{(\nabla A)^2} - \frac{f_2^2}{r^2} - \frac{(rf_1 + \sigma f_2/r)^2}{1-\sigma^2}, \qquad (3.45b)$$

the functions f_1 *and* f_2 *being given by (3.38b, c).*

The condition (3.43) means that the poloidal flow must be *sub-Alfvénic*; (3.44) involves both poloidal and toroidal components of the velocity and the magnetic field, and its physical interpretation will be discussed later.

Previously the stability criterion for axisymmetric MHD flow was obtained by Almaguer *et al.* (1988). Their criterion includes compressibility effects, and is obtained from the requirement that a certain 8×8 matrix be positive-definite; as they recognize, an improved criterion may in principle be obtained if the fluid domain is bounded. We shall explicitly realize this improvement for the incompressible case considered here.

Suppose that the domain $\mathscr{D}$ is bounded. Then an improvement of Criterion 3.1 may be achieved by using the Poincaré-type inequality

$$\int_{\mathscr{D}} \frac{1}{r^2}(\boldsymbol{\nabla}\delta\rho)^2\,d\tau \geqslant C\int_{\mathscr{D}} \frac{1}{r^2}(\delta\rho)^2\,d\tau, \tag{3.46}$$

where C is some constant depending on the domain $\mathscr{D}$. (Actually, similar inequalities may also be obtained for some classes of unbounded domains, see e.g. Appendix B.) Indeed, consider the eigenvalue problem

$$r\frac{\partial}{\partial r}\left[\frac{1}{r}(1-\Psi'^2)\frac{\partial\phi}{\partial r}\right]+\frac{\partial}{\partial z}\left[(1-\Psi'^2)\frac{\partial\phi}{\partial z}\right]+\lambda^2\phi = 0 \quad \text{in } \mathscr{D}, \tag{3.47a}$$

$$\phi = 0 \quad \text{on } \partial\mathscr{D} \tag{3.47b}$$

and let λ_*^2 be the least eigenvalue. Then it may be shown that

$$\int_{\mathscr{D}} (1-\Psi'^2)\frac{1}{r^2}(\boldsymbol{\nabla}\delta\rho)^2\,d\tau \geqslant \lambda_*^2\int_{\mathscr{D}} \frac{1}{r^2}(\delta\rho)^2\,d\tau. \tag{3.48}$$

Hence $\delta^2\mathscr{R}$ is positive-definite provided that (3.43) and the inequality

$$\lambda_*^2/r^2+I_0+I_1 \geqslant 0 \tag{3.49}$$

are satisfied. Thus we obtain the following stability criterion.

Criterion 3.2. *The steady state (3.1) is linearly stable provided that*

$$[\Psi'(A)]^2 \leqslant 1 \qquad (or,\ equivalently,\ |\mathbf{V}| \leqslant |\mathbf{B}|), \tag{3.50}$$

$$\lambda_*^2/r^2+I_0+I_1 \geqslant 0 \tag{3.51}$$

throughout $\mathscr{D}$, *where* I_0 *and* I_1 *are given by (3.45) and* λ_*^2 *is the least eigenvalue of the problem (3.47).*

We conclude this section by formulating the stability criteria for two particular classes of axisymmetric MHD flows that may help to clarify the conditions (3.43) and (3.44).

3.1. Purely poloidal field and flow

Suppose that in the steady state (3.1),

$$\mathbf{V} = \Psi'(A)\,\mathbf{B}, \qquad R_1 = R_2 = 0. \tag{3.52}$$

Then I_1 entering the inequality (3.44) vanishes and Criterion 3.1 simplifies to the following.

Criterion 3.3. *The steady state (3.52) is linearly stable provided that*

$$[\Psi'(A)]^2 \leqslant 1 \qquad (or,\ equivalently,\ |\mathbf{V}| \leqslant |\mathbf{B}|), \tag{3.53}$$

$$\frac{\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}Q-\boldsymbol{\nabla}A\cdot\boldsymbol{\nabla}J}{(\boldsymbol{\nabla}A)^2}+\frac{d\Psi}{dA}\hat{K}\frac{d\Psi}{dA} \geqslant 0 \tag{3.54}$$

throughout $\mathscr{D}$.

The conditions (3.53) and (3.54) represent the axisymmetric counterpart of Criterion 3.1 of Part II, and the second is much simpler than (3.44); it places a constraint on the degree of misalignment of the fields (Ψ, Q) and (A, J). Note that if the unperturbed state is magnetostatic (i.e. $\Psi = 0$) then (3.54) reduces to

$$\frac{dJ}{dA} \leqslant 0.$$

Returning to interpretation of the condition (3.44), we conclude that I_0 appearing in the inequality (3.44) is related to the stability of the purely poloidal unperturbed state, while the term I_1 imposes an additional constraint due to the presence of the toroidal components of the velocity and the magnetic field.

Example 3.1: parallel flow and field inside a circular cylinder of radius a. Suppose that

$$\mathbf{B} = B(r)\,\mathbf{e}_z, \quad \mathbf{V} = V(r)\,\mathbf{e}_z, \quad \text{for } r \leqslant a, \ -\infty < z < \infty.$$

Then the conditions (3.53) and (3.54) reduce to

$$B^2 \geqslant V^2,$$

$$\frac{1}{rB}\frac{d}{dr}\left[\left(1-\frac{V^2}{B^2}\right)\frac{1}{r}\frac{dB}{dr}\right] \geqslant 0$$

for $0 \leqslant r \leqslant a$. These conditions may be considerably strengthened by noting that, under a Gallilean transformation $z' = z - V_0 t$, $t' = t$, V transforms to $V - V_0$ while B is invariant. If the state $(V(r), B(r))$ is stable in any reference frame then it is stable in all frames related by such Gallilean transformations. Hence the state is stable if there exists a value of V_0 such that the inequalities

$$\left.\begin{aligned} &\min_{0\leqslant y\leqslant L} [B^2 - (V-V_0)^2] \geqslant 0, \\ &\min_{0\leqslant y\leqslant L} \left\{\frac{1}{rB}\frac{d}{dr}\left[\left(1-\frac{(V-V_0)^2}{B^2}\right)\frac{1}{r}\frac{dB}{dr}\right]\right\} \geqslant 0 \end{aligned}\right\} \tag{3.55}$$

are both satisfied.

If the magnetic field is homogeneous, i.e. $B = \text{const}$, then the second of these inequalities is satisfied identically, and the stability condition reduces to the first only. Thus any parallel flow inside a circular cylinder is stable if it is sub-Alfvénic in some inertial reference frame, a result first obtained by Howard and Gupta (1962).

3.2. Arbitrary magnetic field and purely toroidal flow

A special case of the steady axisymmetric state with no poloidal components of the velocity is attracting attention in the context of the MHD of accretion disks (Balbus and Hawley 1991; Ogilvie and Pringle 1996). In this state,

$$\Psi \equiv \text{const}, \qquad \rho_1 = R_1(r,z), \qquad \rho_2 = R_2(r,z), \qquad \rho = A(r,z), \tag{3.56}$$

and (3.3) and (3.6) become

$$G_1(A) = -\frac{R_1}{r^2}, \quad G_2(A) = -r^2 R_2, \quad G(A) = -J - G_1'(A)\,R_1 - G_2'(A)\,R_2. \tag{3.57}$$

Criterion 3.2 reduces to the single inequality

$$\lambda_*^2 + r^2 \frac{\nabla A \cdot \nabla(\hat{K} A)}{(\nabla A)^2} + r^3 \frac{\partial A}{\partial r} \frac{\nabla A \cdot \nabla(R_1^2/r^4) - r^{-4} \nabla A \cdot \nabla(r^4 R_2^2)}{(\nabla A)^4} \geqslant 0. \tag{3.58}$$

Thus we obtain the following stability criterion.

Criterion 3.4. *The steady state (3.56) is linearly stable provided that the inequality (3.58) is satisfied throughout $\mathscr{D}$.*

The inequality (3.58) may be expressed in terms of the velocity and the magnetic field in the undisturbed state. Let $\Omega \equiv R_1/r^2$ (i.e. Ω is the angular velocity of the toroidal flow) and let H_ϕ denote the toroidal component of the magnetic field (i.e. $H_\phi \equiv r R_2$). Then (3.58) takes the form

$$\lambda_*^2 + r^2 \frac{\hat{K}(|\mathbf{B}|^2)}{2|\mathbf{B}|^2} - r \frac{\nabla \cdot [r^{-1}(\mathbf{B} \cdot \nabla)\,\mathbf{B}]}{|\mathbf{B}|^2} - \frac{J_\phi^2}{|\mathbf{B}|^2} + r \frac{B_z}{|\mathbf{B}|^4} \left[D_B(\Omega^2) - \frac{1}{r^4} D_B(r^2 H_\phi^2) \right] \geqslant 0, \tag{3.59}$$

where $J_\phi \equiv \partial B_r/\partial z - \partial B_z/\partial r$ is the toroidal current and

$$D_B \equiv B_z \frac{\partial}{\partial r} - B_r \frac{\partial}{\partial z}.$$

Example 3.2: stability of steady MHD flow inside a circular cylinder of radius a. Suppose that

$$\mathbf{B} = B(r)\,\mathbf{e}_z, \quad H_\phi = H_\phi(r), \quad \Omega = \Omega(r) \quad \text{for } r \leqslant a, \ -\infty < z < \infty. \tag{3.60}$$

In this example the domain $\mathscr{D}$ extends to infinity in the z direction; therefore we have to restrict the class of admissible variations $\delta\mathbf{v}$, $\delta\rho_1$, $\delta\rho_2$ and $\delta\rho$ to such variations that are either vanishing at infinity or periodic in z, in the latter case all integrals over $\mathscr{D}$ being replaced by integrals over the period. In both cases it is possible to establish inequalities similar to (3.46).

Consider the situation where $\delta\mathbf{v}$, $\delta\rho_1$, $\delta\rho_2$ and $\delta\rho$ are vanishing sufficiently rapidly as $|z| \to \infty$. Then it may be shown (see Appendix B) that

$$\int_{\mathscr{D}} \frac{1}{r^2} (\nabla \delta\rho)^2 \, d\tau \geqslant \frac{\lambda_0^2}{a^2} \int_{\mathscr{D}} \frac{1}{r^2} (\delta\rho)^2 \, d\tau, \tag{3.61}$$

where λ_0 is the first zero of the Bessel function of the first order $J_1(x)$: $J_1(\lambda_0) = 0$. Hence the sufficient condition for stability (3.59) becomes

$$\frac{\lambda_0^2}{r^2 a^2} + \frac{1}{2rB^2} \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (B^2) \right] - \frac{1}{r^2 B^2} \left(\frac{dB}{dr} \right)^2 + \frac{1}{rB^2} \left[\frac{d}{dr} (\Omega^2) - \frac{1}{r^4} \frac{d}{dr} (r^2 H_\phi^2) \right] \geqslant 0. \tag{3.62}$$

If the magnetic field along the z axis is homogeneous ($B = \text{const}$) and the toroidal magnetic field is absent ($H_\phi = 0$) then the stability condition (3.62) reduces to

$$\frac{\lambda_0^2 B^2}{r^2 a^2} \geqslant -\frac{1}{r}\frac{d}{dr}(\Omega^2), \tag{3.63}$$

which is similar to Chandrasekhar's stability criterion (see Chandrasekhar 1961, p. 388). From (3.63), we conclude that instability may possibly occur only if the angular velocity $\Omega(r)$ decreases with r and the magnetic field is sufficiently small that the condition (3.63) is not satisfied. Note that, according to (3.62), the presence of the toroidal magnetic field may either make the situation 'more unstable' (if $d(rH_\phi)^2/dr > 0$) or stabilize it (if $d(rH_\phi)^2/dr < 0$).

4. Nonlinear stability criterion

Now consider a *finite-amplitude* perturbation of the steady solution (3.1), given by

$$\left.\begin{aligned} \psi(r,z,t) &= \Psi(r,z)+\tilde{\psi}(r,z,t), \quad \mathbf{v}(r,z,t) = \mathbf{V}(r,z)+\tilde{\mathbf{v}}(r,z,t),\\ \rho(r,z,t) &= A(r,z)+\tilde{\rho}(r,z,t), \quad \mathbf{b}(r,z,t) = \mathbf{B}(r,z)+\tilde{\mathbf{b}}(r,z,t), \end{aligned}\right\} \quad (r,z)\in\mathscr{D}, \tag{4.1}$$

with

$$\tilde{\mathbf{v}}\cdot\mathbf{n} = 0, \quad \tilde{\rho} = 0 \quad \text{on} \quad \partial\mathscr{D}, \tag{4.2}$$

so that the constant value of ρ on $\partial\mathscr{D}$ is unchanged by the perturbation. Let $A^- \equiv \min_{\mathscr{D}} A(r,z)$ and $A^+ \equiv \max_{\mathscr{D}} A(r,z)$, and let $\mathscr{A}$ be the closed interval $[A^-, A^+]$. Let us introduce the notation

$$\boldsymbol{\nu} = (\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7) \equiv \left(\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r}, \frac{1}{r}\frac{\partial\tilde{\psi}}{\partial z}, \frac{1}{r}\frac{\partial\tilde{\rho}}{\partial r}, \frac{1}{r}\frac{\partial\tilde{\rho}}{\partial z}, \tilde{\rho}, \frac{\tilde{\rho}_1}{r}, r\tilde{\rho}_2\right). \tag{4.3}$$

To measure the deviation of the perturbed solution (4.1) from the unperturbed one (3.1), we shall exploit the norm (or, more accurately, the seminorm) given by

$$\|\boldsymbol{\nu}\|^2 \equiv \int_{\mathscr{D}} \nu_i\nu_i\,d\tau = \int_{\mathscr{D}}\left[\frac{1}{r^2}(\boldsymbol{\nabla}\tilde{\psi})^2+\frac{1}{r^2}(\boldsymbol{\nabla}\tilde{\rho})^2+\tilde{\rho}^2+\frac{1}{r^2}\tilde{\rho}_1^2+r^2\tilde{\rho}_2^2\right]d\tau. \tag{4.4}$$

We adopt the standard Lyapunov definition of stability: the steady state (3.1) is stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\boldsymbol{\nu}(0)\| < \delta \Rightarrow \|\boldsymbol{\nu}(t)\| < \epsilon$.

For the subsequent analysis, it is convenient to define the following functions:

$$\sigma(a) \equiv \Psi'(a), \quad \sigma'(a) \equiv \Psi''(a), \quad \sigma''(a) \equiv \Psi'''(a), \quad a\in\mathscr{A}, \tag{4.5a}$$

$$\left.\begin{aligned} \alpha(r,z;a) &\equiv -Q(r,z)\,\sigma'(a)+G'(a)+R_1(r,z)\,G_1''(a)\\ &\qquad +R_2(r,z)\,G_2''(a)-R_1(r,z)\,R_2(r,z)\,\sigma''(a),\\ \beta(r,z;a) &\equiv -R_2(r,z)\,\sigma'(a)+G_1'(a),\\ \gamma(r,z;a) &\equiv -R_1(r,z)\,\sigma'(a)+G_2'(a), \end{aligned}\right\}\quad (r,z)\in\mathscr{D}, a\in\mathscr{A}, \qquad \begin{aligned}&(4.5b)\\&(4.5c)\\&(4.5d)\end{aligned}$$

$$\mu_1(\xi,\sigma) \equiv \frac{1-\xi}{(1-\xi)^2-\sigma^2}, \quad \mu_2(\xi,\sigma) \equiv \frac{\sigma}{(1-\xi)^2-\sigma^2}. \tag{4.5e}$$

with $G_1(A)$, $G_2(A)$ and $G(A)$ given by (3.3) and (3.6). (Note that comparison of α, β and γ with the functions f_1, f_2 and f_3 given by (3.39) shows that $\beta(r,z;A(r,z)) = f_1(r,z)$, $\gamma(r,z;A(r,z)) = f_2(r,z)$ and $\alpha(r,z;A(r,z)) = f_3(r,z) - \Psi'\hat{K}\Psi'$.) It is also useful to introduce the notation, related to these functions,

$$\left.\begin{aligned} &\alpha^-(r,z) \equiv \min_{a\in\mathscr{A}}\{\alpha(r,z;a)\}, \quad \alpha^+(r,z) \equiv \max_{a\in\mathscr{A}}\{\alpha(r,z;a)\}, \\ &\beta_0(r,z) \equiv \max_{a\in\mathscr{A}}|\beta(r,z;a)|, \quad \gamma_0(r,z) \equiv \max_{a\in\mathscr{A}}|\gamma(r,z;a)| \end{aligned}\right\} \quad (r,z)\in\mathscr{D}, \qquad (4.6a,b)$$

$$\sigma_0 \equiv \max_{a\in\mathscr{A}}|\Psi'(a)|, \qquad \sigma_0' \equiv \max_{a\in\mathscr{A}}|\Psi''(a)|. \qquad (4.6c)$$

Isomagnetic perturbations. Consider a particular class of finite-amplitude perturbations with general initial data for the streamfunction $\tilde{\psi}(r,z,0)$ and with initial data for the flux function (density) $\tilde{\rho}(r,z,0)$ such that

$$A^- \leqslant \rho(r,z,0) = A(r,z) + \tilde{\rho}(r,z,0) \leqslant A^+. \qquad (4.7)$$

Note that if (4.7) is satisfied then, according to (2.3),

$$A^- \leqslant \rho(r,z,t) = A(r,z) + \tilde{\rho}(r,z,t) \leqslant A^+$$

for all $t > 0$. Such perturbations may be imagined as obtained at the initial instant $t = 0$ by displacement of fluid particles from their position in the unperturbed state (3.1), the value of the flux function ρ for each fluid particle being unchanged. For this class of perturbations, we shall obtain the following nonlinear stability criterion.

Criterion 4.1. *Suppose that*

(i) the function $\sigma(A) \equiv d\Psi/dA$ is a twice continuously differentiable function for all $A\in\mathscr{A}$;

(ii) the functions $G_1(A)$, $G_2(A)$ and $G(A)$ defined by (3.3) and (3.6) are such that $G(A)$ is continuously differentiable, and $G_1(A)$ and $G_2(A)$ are twice continuously differentiable for all $A\in\mathscr{A}$;

(iii) there exist constants ϵ^- and ϵ^+ such that, for $a\in\mathscr{A}$ and $(r,z)\in\mathscr{D}$,

$$0 < \epsilon^- < 1, \qquad \epsilon^+ > 2-\epsilon^-, \qquad |\Psi'(a)| < 1-\epsilon^-, \qquad (4.8a)$$

$$\alpha^- > \epsilon^- + \mu_1(\epsilon^-,\sigma_0)\left(\sigma_0'^2\,\mathbf{B}^2 + r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right) + 2\mu_2(\epsilon^-,\sigma_0)\,\beta_0\,\gamma_0, \qquad (4.8b)$$

$$\alpha^+ < \epsilon^+ + \mu_1(\epsilon^+,\sigma_0)\left(\sigma_0'^2\,\mathbf{B}^2 + r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right) - 2\mu_2(\epsilon^+,\sigma_0)\,\beta_0\,\gamma_0, \qquad (4.8c)$$

where α^-, α^+, β_0 and γ_0 are functions of r and z defined by (4.6a, b).

Then the steady state (3.1) is nonlinearly stable to perturbations with initial data satisfying (4.7). Moreover, the following a priori estimate holds:

$$\epsilon^-\|\mathbf{v}(t)\| \leqslant \epsilon^+\|\mathbf{v}(0)\|. \qquad (4.9)$$

Proof. We shall prove this theorem following Arnold's (1965*a*) technique.

Consider the conserved functional $\mathscr{F} = \mathscr{E} + \tilde{\mathscr{C}} + \mathscr{H}_M + \mathscr{H}_C + \mathscr{L}$, where $\mathscr{E}$ and $\mathscr{H}_M$ are given by (3.27) and (3.28), $\mathscr{H}_C$ is the conserved 'generalized' cross-helicity (cf. (1.12))

$$\mathscr{H}_C = \int_{\mathscr{D}} N_1(\rho)\,(\mathbf{v}\cdot\mathbf{b} + \rho_1\,\rho_2)\,d\tau, \tag{4.10}$$

$\mathscr{L}$ is the conserved 'generalized' angular momentum

$$\mathscr{L} = \int_{\mathscr{D}} L(\rho)\,\rho_1\,d\tau \tag{4.11}$$

and $\tilde{\mathscr{C}}$ is the 'reduced' version of the functional $\mathscr{C}$ (cf. (3.25)),

$$\tilde{\mathscr{C}} = \int_{\mathscr{D}} N_2(\rho)\,d\tau. \tag{4.12}$$

The functions $N_1(\rho)$, $N_2(\rho)$ and $L(\rho)$ are arbitrary. In comparison with the functional $\mathscr{R}$ introduced in Sec. 3, $\mathscr{F}$† involves two new integral invariants $\mathscr{H}_C$ and $\mathscr{L}$, and the Casimir invariant $\mathscr{C}$ (given by (3.25)) is replaced by its 'reduced' version $\tilde{\mathscr{C}}$. It may be shown, however, that the sum $\mathscr{H}_C + \tilde{\mathscr{C}}$ is in fact equivalent to $\mathscr{C}$ with some special choice of function $F(\rho, \lambda)$. (The precise connection between the functionals $\mathscr{F}$ and $\mathscr{R}$ is discussed in Appendix C.) The reasons for using $\mathscr{F}$ (rather than $\mathscr{R}$) here are as follows:

(i) the subsequent calculations become simpler;

(ii) the auxiliary fields η and χ (which are somewhat artificial and have no obvious physical meaning) do not appear in $\mathscr{F}$, and this allows us to eliminate η and χ from consideration.

Following the prescription of Arnold (1965*a*), we decompose $\mathscr{F}$ in the form

$$\mathscr{F} = \mathscr{F}_0 + \mathscr{F}_1 + \mathscr{F}_2, \tag{4.13}$$

with

$$\begin{aligned}\mathscr{F}_0 = \int_{\mathscr{D}} \bigg[& \frac{(\boldsymbol{\nabla}\Psi)^2}{2r^2} + \frac{(\boldsymbol{\nabla}A)^2}{2r^2} + \frac{R_1^2}{2r^2} + r^2\frac{R_2^2}{2} \\ & + N_1(A)\left(\frac{1}{r^2}\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}A + R_1\,R_2\right) + N_2(A) + L(A)\,R_1 + S(A)\,R_2\bigg]\,d\tau,\end{aligned} \tag{4.14}$$

$$\begin{aligned}\mathscr{F}_1 = \int_{\mathscr{D}} \bigg[& \frac{1}{r^2}\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}\tilde{\psi} + \frac{1}{r^2}\boldsymbol{\nabla}A\cdot\boldsymbol{\nabla}\tilde{\rho} + \frac{R_1}{r^2}\tilde{\rho}_1 + r^2 R_1\,\tilde{\rho}_2 \\ & + N_1'(A)\left(\frac{1}{r^2}\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}A + R_1\,R_2\right)\tilde{\rho} \\ & + N_1(A)\left(\frac{1}{r^2}\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}\tilde{\rho} + \frac{1}{r^2}\boldsymbol{\nabla}A\cdot\boldsymbol{\nabla}\tilde{\psi} + R_1\,\tilde{\rho}_2 + R_2\,\tilde{\rho}_1\right) \\ & + [N_2'(A) + L'(A)\,R_1 + S'(A)\,R_2]\,\tilde{\rho} + L(A)\,\tilde{\rho}_1 + S(A)\,\tilde{\rho}_2\bigg]\,d\tau,\end{aligned} \tag{4.15}$$

† A functional similar to $\mathscr{F}$ was used by Almaguer *et al.* (1988) to obtain sufficient conditions for linear stability of compressible MHD flows.

$$\mathscr{F}_2 = \int_{\mathscr{D}} \bigg[\frac{(\nabla\tilde{\psi})^2}{2r^2} + \frac{(\nabla\tilde{\rho})^2}{2r^2} + \frac{\tilde{\rho}_1^2}{2r^2} + r^2\frac{\tilde{\rho}_2^2}{2} + \left(\frac{\nabla\Psi\cdot\nabla A}{r^2} + R_1 R_2 \right)\Delta_2 N_1$$
$$+ \left(\frac{\nabla\Psi\cdot\nabla\tilde{\rho}}{r^2} + \frac{\nabla A\cdot\nabla\tilde{\psi}}{r^2} + R_1\tilde{\rho}_2 + R_2\tilde{\rho}_1 \right)\Delta_1 N_1 + R_1\Delta_2 L + R_2\Delta_2 S$$
$$+ \Delta_2 N_2 + \left(\frac{\nabla\tilde{\psi}\cdot\nabla\tilde{\rho}}{r^2} + \tilde{\rho}_1\tilde{\rho}_2 \right) N_1(A+\tilde{\rho}) + \tilde{\rho}_1\Delta_1 L + \tilde{\rho}_2\Delta_1 S \bigg], \tag{4.16}$$

where we have used the notation

$$\Delta_1 f \equiv f(A+\tilde{\rho}) - f(A), \qquad \Delta_2 f \equiv f(A+\tilde{\rho}) - f(A) - f'(A)\,\tilde{\rho}. \tag{4.17}$$

The functions $N_1(\rho)$, $N_2(\rho)$, $S(\rho)$ and $L(\rho)$ are still arbitrary. Now we choose them in such a way that $\mathscr{F}_1 = 0$. It may be shown that an appropriate choice is given by

$$\left.\begin{aligned} &N_1(A) = -\Psi'(A), \quad S(A) = \Psi'(A)\,R_1 - r^2 R_2, \quad L(A) = \Psi'(A)\,R_2 - \frac{R_1}{r^2}, \\ &N_2'(A) = \Psi'(A)\,Q - J + \Psi''(A)\,R_1 R_2 - S'(A)\,R_2 - L'(A)\,R_1. \end{aligned}\right\} \tag{4.18}$$

Comparing these equations with (3.3) and (3.6), we find that

$$S(A) = G_2(A), \qquad L(A) = G_1(A), \qquad N_2'(A) = G(A). \tag{4.19}$$

Since $\mathscr{F}_0$ does not depend on time, $\mathscr{F}_1 = 0$ and $\mathscr{F}$ is the conserved functional, we conclude that $\mathscr{F}_2$ is an invariant of the exact nonlinear problem (2.2), (2.3), (2.6)–(2.8).

Let us now transform $\mathscr{F}_2$ to a form that is convenient for the subsequent stability analysis. It may be shown that

$$I = \int_{\mathscr{D}} \left(\frac{\nabla\Psi\cdot\nabla A}{r^2}\Delta_2 N_1 + \frac{\nabla\Psi\cdot\nabla\tilde{\rho}}{r^2}\Delta_1 N_1 \right) d\tau$$
$$= \int_{\mathscr{D}} \frac{1}{r^2}\nabla\Psi\cdot\nabla[F_1(A+\tilde{\rho}) - F_1(A) - F_1'(A)\,\tilde{\rho}]\,d\tau = \int_{\mathscr{D}} \frac{1}{r^2}\nabla\Psi\cdot\nabla(\Delta_2 F_1)\,dt, \tag{4.20}$$

where the function $F_1(a)$ is such that $F_1'(a) = N_1(a)$. (Note that, according to (4.18), $F_1(a) = -\Psi(a)$ for all $a \in \mathscr{A}$.) Integrating (4.20) by parts and using (4.2b), we find that

$$I = -\int_{\mathscr{D}} \hat{K}\Psi\,\Delta_2 F_1\,d\tau = \int_{\mathscr{D}} Q\,\Delta_2 F_1\,d\tau.$$

Substitution of this into (4.16) results in

$$\mathscr{F}_2 = \int_{\mathscr{D}} \bigg[\frac{(\nabla\tilde{\psi})^2}{2r^2} + \frac{(\nabla\tilde{\rho})^2}{2r^2} + \frac{\tilde{\rho}_1^2}{2r^2} + r^2\frac{\tilde{\rho}_2^2}{2}$$
$$+ (Q\,\Delta_2 F_1 + \Delta_2 N_2 + R_1 R_2\,\Delta_2 N_1 + R_1\Delta_2 L + R_2\Delta_2 S)$$
$$+ \left(\frac{\nabla\tilde{\psi}\cdot\nabla\tilde{\rho}}{r^2} + \tilde{\rho}_1\tilde{\rho}_2 \right) N_1(A+\tilde{\rho}) + \frac{\nabla A\cdot\nabla\tilde{\psi}}{r^2}\Delta_1 N_1$$
$$+ \tilde{\rho}_1(R_2\Delta_1 N_1 + \Delta_1 L) + \tilde{\rho}_2(R_1\Delta_1 N_1 + \Delta_1 S) \bigg] d\tau. \tag{4.21}$$

Using Taylor's formula with remainder in Lagrange's form, we obtain

$$\left.\begin{aligned} &N_1(A+\tilde{\rho}) = -\sigma(a_0), \quad \Delta_1 N_1 = -\sigma'(a_1)\,\tilde{\rho}, \\ &Q\,\Delta_2 F_1 + \Delta_2 N_2 + R_1 R_2 \Delta_2 N_1 + R_1 \Delta_2 L + R_2 \Delta_2 S = \tfrac{1}{2}\alpha(r,z;a_2)\,\tilde{\rho}^2, \\ &R_2 \Delta_1 N_1 + \Delta_1 L = \beta(r,z;a_3)\,\tilde{\rho}, \quad R_1 \Delta_1 N_1 + \Delta_1 S = \gamma(r,z;a_4)\,\tilde{\rho}, \end{aligned}\right\} \tag{4.22}$$

where

$$a_0 = A+\tilde{\rho}, \quad a_i = A+\theta_i\,\tilde{\rho} \quad (i = 1,2,3,4), \tag{4.23}$$

and we use the definitions (4.5*a–d*) and the fact that, according to (4.18) and (4.19),

$$\left.\begin{aligned} &N_1'(a) = -\sigma'(a), \quad N_1''(a) = -\sigma''(a), \quad N_2''(a) = G'(a), \\ &S'(a) = G_2'(a), \quad S''(a) = G_2''(a), \quad L'(a) = G_1'(a), \quad L''(a) = G_1''(a), \end{aligned}\right\} \tag{4.24}$$

for all $a \in \mathscr{A}$. In (4.23), θ_i $(i = 1,2,3,4)$ are functions of A and $\tilde{\rho}$ such that

$$0 \leqslant \theta_i \leqslant 1.$$

Note that for perturbations with initial data in the range (4.7), $a_0, a_i \in \mathscr{A}$. By using (4.22) and the notation (4.3), (4.21) may be written in the form

$$2\mathscr{F}_2 = \int_{\mathscr{D}} F_{ik}\,\nu_i\,\nu_k\,d\tau, \tag{4.25a}$$

where F_{ik} are elements of the symmetric 7×7 matrix given by

$$\hat{F} \equiv \begin{pmatrix} 1 & 0 & -\sigma & 0 & -\sigma' A_r/r & 0 & 0 \\ 0 & 1 & 0 & -\sigma & -\sigma' A_z/r & 0 & 0 \\ -\sigma & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\sigma & 0 & 1 & 0 & 0 & 0 \\ -\sigma' A_r/r & -\sigma' A_z/r & 0 & 0 & \alpha & r\beta & \gamma/r \\ 0 & 0 & 0 & 0 & r\beta & 1 & -\sigma \\ 0 & 0 & 0 & 0 & \gamma/r & -\sigma & 1 \end{pmatrix}. \tag{4.25b}$$

If the positive constants ϵ^- and ϵ^+ in the conditions (4.8) are such that

$$\epsilon^- \int_{\mathscr{D}} \nu_i\,\nu_i\,d\tau \leqslant 2\mathscr{F}_2 \leqslant \epsilon^+ \int_{\mathscr{D}} \nu_i\,\nu_i\,d\tau \tag{4.26}$$

then the a priori estimate (4.9) and hence the nonlinear stability of (3.1) follow immediately from the fact that $\mathscr{F}_2$ is an invariant of the exact nonlinear problem (2.2), (2.3), (2.6)–(2.8). The stability analysis therefore reduces to obtaining upper and lower bounds on $\mathscr{F}_2$. Obviously, the inequalities (4.26) are satisfied provided that the two quadratic forms

$$(F_{ik} - \epsilon^- \delta_{ik})\,\nu_i\,\nu_k \quad \text{and} \quad (\epsilon^+ \delta_{ik} - F_{ik})\,\nu_i\,\nu_k$$

are positive-definite. The necessary and sufficient conditions for this are

$$\epsilon^- < 1, \quad \sigma^2 < (1-\epsilon^-)^2, \tag{4.27a}$$

$$\alpha > \epsilon^- + \mu_1(\epsilon^-,\sigma)\left(\sigma'^2 \mathbf{B}^2 + r^2\beta^2 + \frac{\gamma^2}{r^2}\right) + 2\mu_2(\epsilon^-,\sigma)\,\beta\gamma \tag{4.27b}$$

and

$$\epsilon^+ > 1, \qquad \sigma^2 < (1-\epsilon^+)^2, \tag{4.28a}$$

$$\alpha < \epsilon^+ + \mu_1(\epsilon^+, \sigma)\left(\sigma'^2|\mathbf{B}|^2 + r^2\beta^2 + \frac{\gamma^2}{r^2}\right) + 2\mu_2(\epsilon^+, \sigma)\,\beta\gamma \tag{4.28b}$$

for all $(r,z)\in\mathscr{D}$ and $a_0, a_1, a_2, a_3, a_4 \in \mathscr{A}$. (Recall that, according to (4.22), the functions $\sigma(a)$, $\sigma'(a)$, $\alpha(r,z;a)$, $\beta(r,z;a)$ and $\gamma(r,z;a)$ appearing in the inequalities (4.27) and (4.28) are taken at $a=a_0$, $a=a_1$, $a=a_2$, $a=a_3$ and $a=a_4$ respectively.)

We now show that (4.27) and (4.28) are satisfied under the conditions (4.8). First, we take ϵ^+ and ϵ^- such that

$$0 < \epsilon^- < 1-\sigma_0, \qquad 2-\epsilon^- < \epsilon^+ < \infty. \tag{4.29}$$

Then the inequalities (4.27a) and (4.28a) are satisfied. Secondly, according to the definitions (4.5e) and (4.6), we have

$$\left.\begin{aligned} &\mu_1(\epsilon^-, \sigma)\left(\sigma'^2|\mathbf{B}|^2 + r^2\beta^2 + \frac{\gamma^2}{r^2}\right) \leqslant \mu_1(\epsilon^-, \sigma_0)\left(\sigma_0'^2|\mathbf{B}|^2 + r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right),\\ &\mu_2(\epsilon^-, \sigma)\,\beta\gamma \leqslant \mu_2(\epsilon^-, \sigma_0)\,\beta_0\,\gamma_0, \end{aligned}\right\} \tag{4.30a}$$

$$\left.\begin{aligned} &\mu_1(\epsilon^+, \sigma)\left(\sigma'^2|\mathbf{B}|^2 + r^2\beta^2 + \frac{\gamma^2}{r^2}\right) \geqslant \mu_1(\epsilon^+, \sigma_0)\left(\sigma_0'^2|\mathbf{B}|^2 + r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right),\\ &\mu_2(\epsilon^+, \sigma)\,\beta\gamma \geqslant -\mu_2(\epsilon^+, \sigma_0)\,\beta_0\,\gamma_0. \end{aligned}\right\} \tag{4.30b}$$

Thirdly, we suppose that there exist ϵ^- and ϵ^+ satisfying (4.29) such that

$$\epsilon^- + \mu_1(\epsilon^-, \sigma_0)\left(\sigma_0'^2|\mathbf{B}|^2 + r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right) + 2\mu_2(\epsilon^-, \sigma_0)\,\beta_0\,\gamma_0 < \alpha^-, \tag{4.31a}$$

$$\epsilon^+ + \mu_1(\epsilon^+, \sigma_0)\left(\sigma_0'^2|\mathbf{B}|^2 + r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right) - 2\mu_2(\epsilon^+, \sigma_0)\,\beta_0\,\gamma_0 > \alpha^+, \tag{4.31b}$$

where α^-, α^+, β_0 and γ_0 are functions of r and z defined by (4.6a, b). If (4.31a, b) are satisfied then, in view of the inequalities (4.30a, b), the conditions (4.27b) and (4.28b) are satisfied too. Hence we have shown that the conditions (4.29) and (4.31) are in fact sufficient for all six inequalities (4.27) and (4.28) to be satisfied.

Finally, comparing (4.29) and (4.31) with (4.8), we see that they coincide. Criterion 4.1 is thus established. □

Remark on general perturbations. The inequalities (4.8) give sufficient conditions for nonlinear stability of (3.1) only with respect to perturbations with initial data satisfying (4.7). For arbitrary perturbations, however, the quantities a_0, a_1, a_2, a_3 and a_4, defined by (4.23), may be outside $\mathscr{A}$ In this situation, the stability analysis involves an appropriate extension of the definition of the functions $N_1(a)$, $N_2(a)$, $S(a)$ and $L(a)$ (defined only for $a\in\mathscr{A}$ by (4.18)) to all real

a, and becomes very complicated. We shall not consider the general situation here. Instead, in the next section we shall obtain the sufficient conditions for nonlinear stability to *arbitrary* axisymmetric perturbations for two simpler particular problems.

5. Nonlinear stability criteria for two particular problems

5.1. Purely poloidal field and flow

Consider a particular case of the steady state (3.1), when the toroidal components of both the magnetic field and the velocity are zero:

$$\mathbf{V} = \Psi'(A)\,\mathbf{B}, \qquad R_1 = R_2 = 0. \tag{5.1}$$

The functions $G_1(A)$, $G_2(A)$ and $G(A)$ (given by (3.3) and (3.6)) are then

$$G(A) = \Psi'(A)\,Q - J, \qquad G_1(A) = G_2(A) = 0. \tag{5.2}$$

The conditions (4.8) reduce to

$$0 < \epsilon^- < 1, \qquad \epsilon^+ > 2 - \epsilon^-, \qquad |\Psi'(a)| < 1 - \epsilon^-, \tag{5.3a}$$

$$\alpha^-(r,z) > \epsilon^- + \mu_1(\epsilon^-, \sigma_0)\,\sigma_0'^2\,|\mathbf{B}|^2\,(r,z), \tag{5.3b}$$

$$\alpha^+(r,z) < \epsilon^+ + \mu_1(\epsilon^+, \sigma_0)\,\sigma_0'^2\,|\mathbf{B}|^2\,(r,z), \tag{5.3c}$$

where the functions $\alpha^-(r,z)$ and $\alpha^+(r,z)$ are now given by

$$\left.\begin{aligned} \alpha^-(r,z) &= \min_{a\in\mathscr{A}}[\alpha(r,z;a)] = \min_{a\in\mathscr{A}}[-Q(r,z)\,\Psi''(a) + G'(A)], \\ \alpha^+(r,z) &= \max_{a\in\mathscr{A}}[\alpha(r,z;a)] = \max_{a\in\mathscr{A}}[-Q(r,z)\,\Psi''(a) + G'(A)]. \end{aligned}\right\} \tag{5.4}$$

The inequalities (5.3) give the sufficient conditions for nonlinear stability of (5.1) to perturbations with initial data satisfying (4.7).

Now consider a general situation when the perturbations are quite arbitrary (without restriction (4.7) on the initial data) Let α_0^- and α_0^+ be the minimum and maximum values of the function $\alpha(r,z;a)$ for all $(r,z)\in\mathscr{D}$ and $a\in\mathscr{A}$, i.e.

$$\left.\begin{aligned} \alpha_0^- &\equiv \min_{(r,z)\in\mathscr{D},\,a\in\mathscr{A}} \alpha(r,z;a) = \min_{(r,z)\in\mathscr{D}} \alpha^-(r,z), \\ \alpha_0^+ &\equiv \max_{(r,z)\in\mathscr{D},\,a\in\mathscr{A}} \alpha(r,z;a) = \max_{(r,z)\in\mathscr{D}} \alpha^+(r,z). \end{aligned}\right\} \tag{5.5}$$

We shall obtain the following nonlinear stability criterion.

Criterion 5.1. *Suppose that*

(i) the function $\Psi(A)$ defined by (5.1) is a twice continuously differentiable function for all $A\in\mathscr{A}$;

(ii) the function $G(A)$ defined by (5.2) is continuously differentiable for all $A\in\mathscr{A}$;

(iii) there exist constants ϵ^-, ϵ^+ and ϵ^* such that, for $a\in\mathscr{A}$ and $(r,z)\in\mathscr{D}$,

$$0 < \epsilon^- < \epsilon^* < 1, \quad \epsilon^+ > 2 - \epsilon^-, \quad |\Psi'(a)| < 1 - \epsilon^*, \tag{5.6a}$$

$$\alpha_0^- > \epsilon^- + \mu_1(\epsilon^-, 1-\epsilon^*)\,\sigma_0'^2\,|\mathbf{B}|^2\,(r,z), \tag{5.6b}$$

$$\alpha_0^+ < \epsilon^+ + \mu_1(\epsilon^+, 1-\epsilon^*)\,\sigma_0'^2\,|\mathbf{B}|^2\,(r,z). \tag{5.6c}$$

Then the a priori estimate (4.9) holds, and the steady state (5.1) is stable to arbitrary finite-amplitude perturbations.

Proof. To prove this proposition, it is sufficient to show that the inequalities (4.27) and (4.28) hold provided that the conditions (5.6) are satisfied. For arbitrary perturbations, however, the quantities a_0, a_1, a_2, a_3 and a_4 defined by (4.23) may be outside $\mathscr{A}$. The inequalities (4.27) and (4.28) must therefore be satisfied for *all* real a_0, a_1, a_2, a_3 and a_4.

For the steady state (5.1), the functions β and γ in (4.27) and (4.28) are identically zero; the functions $\alpha(r,z;a)$ and $\sigma(a)$ are now considered as defined by

$$\sigma(a) = -N_1(a), \quad \alpha(r,z;a) = N_1'(a)\,Q(r,z) + N_2''(a), \tag{5.7}$$

so that, according to (4.18) and (4.19), for $a \in \mathscr{A}$, the new definitions coincide with the old ones (4.5a, b). Initially, $N_1(a)$ and $N_2(a)$ were arbitrary, and then were defined only for $a \in \mathscr{A}$ by (4.18). Hence $N_1(a)$ and $N_2(a)$ are still arbitrary for $a \notin \mathscr{A}$. We can therefore extend the definitions of $N_1(a)$ and $N_2(a)$ to all real a in any way we need, and then extend the definitions of $\alpha(r,z;a)$ and $\sigma(a)$ using (5.7).

We shall continue $N_1(a)$ and $N_2(a)$ to all $a \notin \mathscr{A}$ in such a way that, first, $N_1(a)$ remains continuously differentiable and $N_2(a)$ twice continuously differentiable, and, secondly, the inequalities

$$\alpha_0^- \leqslant \alpha(r,z,a) = Q(r,z)\,N_1''(a) + N_2''(a) \leqslant \alpha_0^+, \tag{5.8a}$$

$$|\sigma'(a)| = |N_1'(a)| \leqslant \max_{\eta \in \mathscr{A}} |\Psi''(\eta)| = \sigma_0', \tag{5.8b}$$

$$|\sigma(a)| = |N_1(a)| \leqslant 1 - \epsilon^* \tag{5.8c}$$

hold for all $a \notin \mathscr{A}$. If such a continuation is possible then the proof of Criterion 5.1 reduces effectively to the proof of Criterion 4.1.

Extension of the definition of the functions $N_1(a)$ and $N_2(a)$. It will be sufficient to construct explicitly a continuation of $N_1(a)$ and $N_2(a)$ to all $a > A^+$. Continuation to all $a < A^-$ can be achieved in a similar way.

Three different situations are possible: (i) $N_1'(A^+) > 0$, (ii) $N_1'(A^+) < 0$ or (iii) $N_1'(A^+) = 0$.

(i) If $N_1'(A^+) > 0$ then we define $N_1(a)$ for $a > A^+$ such that

$$N_1(a) = N_1(A^+) + N_1'(A^+)\frac{z}{1+\psi_1 z}, \tag{5.9a}$$

where

$$z \equiv a - A^+, \quad \psi_1 \equiv \frac{N_1'(A^+)}{1-\epsilon^* - N_1(A^+)} > 0. \tag{5.9b}$$

It is easy to see that, with this definition, $N_1(a)$ is continuously differentiable for all $a \geqslant A^+$ and satisfies (5.8c). Note that

$$N_1'(a) = N_1'(A^+)\frac{1}{(1+\psi_1 z)^2} \leqslant N_1'(A^+) \leqslant \sigma_0' \tag{5.10}$$

for all $a \geqslant A^+$. Hence (5.8b) is also satisfied.

With $N_1(a)$ given by (5.9a), we choose the function $N_2(a)$ for $a > A^+$ in such a way that (5.8a) is satisfied. Before doing this, let us introduce functions $\hat{\alpha}^-(a)$ and $\hat{\alpha}^+(a)$ such that

$$\hat{\alpha}^-(a) \equiv \min_{(r,z)\in\mathscr{D}} \alpha(r,z;a), \qquad \hat{\alpha}^+(a) \equiv \max_{(r,z)\in\mathscr{D}} \alpha(r,z;a), \tag{5.11}$$

where $\alpha(r,z;a)$ is given by (5.7). It is obvious from (5.5) and (5.11) that

$$\alpha_0^- \leqslant \hat{\alpha}^-(a) \leqslant \alpha(r,z;a) \leqslant \hat{\alpha}^+(a) \leqslant \alpha_0^+. \tag{5.12}$$

Also, it follows from the definitions of $\hat{\alpha}^-(a)$ and $\hat{\alpha}^+(a)$ that

$$\hat{\alpha}^-(a) = N_2''(a) + \begin{cases} N_1'(a)\min Q(r,z) & \text{if } N_1'(a) > 0, \\ 0 & \text{if } N_1'(a) = 0, \\ N_1'(a)\max Q(r,z) & \text{if } N_1'(a) < 0, \end{cases} \tag{5.13a}$$

and

$$\hat{\alpha}^+(a) = N_2''(a) + \begin{cases} N_1'(a)\max Q(r,z) & \text{if } N_1'(a) > 0, \\ 0 & \text{if } N_1'(a) = 0, \\ N_1'(a)\min Q(r,z) & \text{if } N_1'(a) < 0. \end{cases} \tag{5.13b}$$

Since in our case $N_1'(a) > 0$ for $a \geqslant A^+$, we have

$$\hat{\alpha}^-(a) = N_2''(a) + N_1'(a) \min_{(r,z)\in\mathscr{D}} Q(r,z), \qquad \hat{\alpha}^+(a) = N_2''(a) + N_1'(a) \max_{(r,z)\in\mathscr{D}} Q(r,z). \tag{5.14}$$

Now, for all $a > A^+$, we take $N_2(a)$ such that

$$N_2''(a) = \hat{\alpha}^-(A^+) - N_1'(a) \min_{(r,z)\in\mathscr{D}} Q(r,z), \tag{5.15}$$

where $N_1(a)$ is given by (5.9a). With this choice, $N_2(a)$ is twice continuously differentiable for all $a \geqslant A^+$, and, from (5.14),

$$\hat{\alpha}^-(a) = \hat{\alpha}^-(A^+) \geqslant \alpha_0^-, \qquad a \geqslant A^+. \tag{5.16}$$

Also, from (5.14), we obtain

$$\hat{\alpha}^+(a) = \hat{\alpha}^-(A^+) + N_1'(a)\,[\max_{(r,z)\in\mathscr{D}} Q(r,z) - \min_{(r,z)\in\mathscr{D}} Q(r,z)]. \tag{5.17}$$

From (5.17), we have

$$\hat{\alpha}^+(A^+) - \hat{\alpha}^-(A^+) = N_1'(A^+)\,[\max_{(r,z)\in\mathscr{D}} Q(r,z) - \min_{(r,z)\in\mathscr{D}} Q(r,z)]. \tag{5.18}$$

Eliminating $\hat{\alpha}^-(A^+)$ from (5.17) and (5.18), we find

$$\hat{\alpha}^+(a) = \hat{\alpha}^+(A^+) - [N_1'(A^+) - N_1'(a)]\,[\max_{(r,z)\in\mathscr{D}} Q(r,z) - \min_{(r,z)\in\mathscr{D}} Q(r,z)]. \tag{5.19}$$

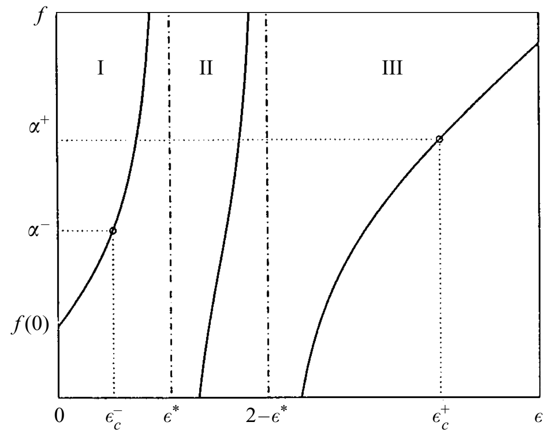

Figure 1

Finally, since $N_1'(A^+) \geqslant N_1'(a)$ for all $a \geqslant A^+$, we obtain

$$\hat{\alpha}^+(a) \leqslant \hat{\alpha}^+(A^+) \leqslant \alpha_0^+. \tag{5.20}$$

The inequalities (5.16) and (5.20) are valid for all $a \geqslant A^+$ and coincide with (5.8a), so that our continuation satisfies all the conditions (5.8).

(ii) If $N_1'(A^+) < 0$ then we define $N_1(a)$ for $a > A^+$ such that

$$N_1(a) = N_1(A^+) + N_1'(A^+)\frac{z}{1+\psi_2 z}, \tag{5.21a}$$

where

$$\psi_2 \equiv -\frac{N'(A^+)}{1-\epsilon^* + N_1(A^+)} > 0. \tag{5.21b}$$

It is easy to verify that, with this choice, $N_1(a)$ is continuously differentiable and satisfies (5.8b, c). In the case under consideration $N_1'(A^+) < 0$, and hence, from (5.13), we have

$$\hat{\alpha}^-(a) = N_2''(a) + N_1'(a) \max_{(r,z)\in\mathscr{D}} Q(r,z), \qquad \hat{\alpha}^+(a) = N_2''(a) + N_1'(a) \min_{(r,z)\in\mathscr{D}} Q(r,z). \tag{5.22}$$

We choose the function $N_2(a)$ for $a > A^+$ such that

$$N_2''(a) = \hat{\alpha}^-(A^+) - N_1'(a) \max_{(r,z)\in\mathscr{D}} Q(r,z), \tag{5.23}$$

where $N_1(a)$ and $\hat{\alpha}^-(a)$ are given by (5.21a) and (5.22) respectively. It may be shown that $N_2(a)$ defined by (5.23) is twice continuously differentiable for all $a \geqslant A^+$ and that the inequalities (5.8a) are satisfied.

(iii) If $N_1'(A^+) = 0$ then we take $N_1(a)$ for $a > A^+$ such that $N_1(a) = N_1(A^+)$ and $N_2(a)$ such that $N_2''(a) = N_2''(A^+)$, so that (5.8) are satisfied.

Thus we have shown that a smooth continuation of the functions $N_1(a)$ and $N_2(a)$ to all $a \in \mathbb{R}$ such that the conditions (5.8) remain satisfied can be constructed. The rest of the proof is the same as for Criterion 4.1. □

Let us now discuss the existence of constants ϵ^-, ϵ^+ and ϵ^* satisfying the conditions (5.6) of Criterion 5.1. It is convenient to rewrite the inequalities (5.6b, c) in the form

$$\alpha_0^- > f(\epsilon^-), \quad \alpha_0^+ < f(\epsilon^+), \tag{5.24}$$

where
$$f(\epsilon) \equiv \epsilon + \frac{1-\epsilon}{(1-\epsilon)^2-(1-\epsilon^*)^2}\,\sigma_0'^2\,|\mathbf{B}|^2. \tag{5.25}$$

The graph of $f(\epsilon)$ (see Fig. 1) consists of three branches, denoted by I, II and III. It follows from (5.6a) that

$$0 < \epsilon^- < \epsilon^*, \qquad \epsilon^+ > 2-\epsilon^*, \tag{5.26}$$

so that ϵ^- lies on I and ϵ^+ on III. The condition (5.24a) is satisfied for any $\epsilon < \epsilon_c^-$, and (5.24b) is valid for $\epsilon > \epsilon_c^+$, where ϵ_c^- and ϵ_c^+ are the points on the first and third branches of the graph corresponding to $f(\epsilon_c^-) = \alpha^-$ and $f(\epsilon_c^+) = \alpha^+$ respectively. Figure 1 also shows that a constant ϵ^- satisfying the conditions (5.6) does exist provided that

$$\alpha_0^- > f(0) = \frac{\sigma_0'^2}{1-(1-\epsilon^*)^2}\,|\mathbf{B}|^2\,(r,z) \quad \text{for all } (r,z)\in\mathscr{D}, \tag{5.27}$$

while a constant ϵ^+ always exists for any given ϵ^- and α^+. Let $\delta \equiv 1-\epsilon^*$. Then we can formulate the following corollary.

Corollary. *The steady state (5.1) is nonlinearly stable to arbitrary axisymmetric perturbations provided that there exists a constant δ such that*

$$0 < \delta < 1, \qquad \frac{1}{\delta^2}|\mathbf{V}|^2\,(r,z) < |\mathbf{B}|^2\,(r,z) < \frac{1-\delta^2}{\sigma_0'^2}\,\alpha_0^-. \tag{5.28}$$

The criterion in this form admits comparison with the linear stability criterion 3.3 involving the inequalities (3.53) and (3.54).

5.2. Arbitrary field and purely toroidal flow

Consider another particular situation when in the steady state (3.1) the poloidal velocity is identically zero. The steady state whose stability is to be investigated is given by (3.56). The functions α, β and γ defined by (4.5b–d) reduce to

$$\alpha(r,z;a) = G'(a) + R_1(r,z)\,G_1''(a) + R_2(r,z)\,G_2''(a), \tag{5.29}$$

$$\beta(a) = G_1'(a), \qquad \gamma(a) = G_2'(a), \tag{5.30}$$

with $G_1(A)$, $G_2(A)$ and $G(A)$ given by (3.57). Note that the functions β and γ depend only on a (in contrast with (4.5c, d), where β and γ are functions of r, z and a). The conditions of Criterion 4.1 take a simplified form:

$$0 < \epsilon^- < 1, \qquad \epsilon^+ > 1, \tag{5.31a}$$

$$\alpha^-(r,z) > \epsilon^- + \frac{1}{1-\epsilon^-}\left(r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right), \tag{5.31b}$$

$$\alpha^+(r,z) > \epsilon^+ + \frac{1}{1-\epsilon^+}\left(r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right), \tag{5.31c}$$

where $\alpha^{\pm}$, β_0 and γ_0 are given by (4.6a, b), with $\alpha(r,z;a)$, $\beta(a)$ and $\gamma(a)$ defined by (5.29) and (5.30).

The inequalities (5.31) give *the sufficient conditions for nonlinear stability of* (3.56) *to perturbations with initial data satisfying* (4.7).

As in Sec. 5.1, this result may be generalized to the case of *arbitrary* perturbations (without the restriction (4.7) on the initial data). The following stability criterion may be obtained.

Criterion 5.2. *Suppose that*

(i) the functions $G_1(A)$, $G_2(A)$ and $G(A)$ defined by (3.57) are such that $G(A)$ is continuously differentiable, and $G_1(A)$ and $G_2(A)$ are twice continuously differentiable for all $A \in \mathscr{A}$;

(ii) there exist constants ϵ^-, ϵ^+ and ϵ^* such that, for $(r,z) \in \mathscr{D}$,

$$0 < \epsilon^- < 1, \qquad \epsilon^+ > 1, \tag{5.32a}$$

$$\alpha_0^- > \epsilon^- + \frac{(1+\epsilon^*)^2}{1-\epsilon^-}\left(r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right), \tag{5.32b}$$

$$\alpha_0^+ < \epsilon^+ + \frac{(1+\epsilon^*)^2}{1-\epsilon^+}\left(r^2\beta_0^2 + \frac{\gamma_0^2}{r^2}\right). \tag{5.32c}$$

Then the a priori estimate (4.9) holds, and the steady state (3.56) is stable to arbitrary axisymmetric finite-amplitude perturbations.

Proof. As in Sec. 5.1, to prove this proposition, it is sufficient to show that the inequalities (4.27) and (4.28), which simplify here to

$$0 < \epsilon^- < 1, \qquad \epsilon^+ > 1, \tag{5.33a}$$

$$\alpha > \epsilon^- + \frac{1}{1-\epsilon^-}\left(r^2\beta^2 + \frac{\gamma^2}{r^2}\right), \tag{5.33b}$$

$$\alpha > \epsilon^+ + \frac{1}{1-\epsilon^+}\left(r^2\beta^2 + \frac{\gamma^2}{r^2}\right), \tag{5.33c}$$

are satisfied for all a_2, a_3, $a_4 \in \mathbb{R}$. (According to (4.22), (5.29) and (5.30), in the inequalities (5.33) $\alpha = \alpha(r,z;a_2)$, $\beta = \beta(a_3)$, $\gamma = \gamma(a_4)$.) We therefore need to continue $N_1(a)$, $L(a)$ and $S(a)$ to all real a in such a way that these functions remain smooth enough and the inequalities

$$\begin{gathered} \alpha_0^- \leqslant \alpha(r,z,a) \leqslant \alpha_0^+, \\ |L'(a)| < \beta_0(1+\epsilon^*), \qquad |S'(a)| < \gamma_0(1+\epsilon^*) \end{gathered} \tag{5.34}$$

with some $\epsilon^* > 0$ hold true for all $a \in \mathbb{R}$.† Such a continuation can be made similarly to that for Criterion 5.1. The rest of the proof is the same as for Criterion 4.1. □

† In general, it is impossible to continue the functions $L(a)$ and $S(a)$ so that (i) they remain sufficiently smooth, (ii) $|L'(a)| \leqslant \beta_0$ and (iii) $|S'(a)| \leqslant \gamma_0$ (to see this, it is sufficient to consider the situation when $\beta_0 = |L'(A^+)|$ and $L''(A^+) \neq 0$). However, it is always possible to continue these functions, keeping (5.34b, c) satisfied for some (arbitrarily small) ϵ^*.

Similarly to Sec. 5.1, the analysis of the existence of positive constants ϵ^+, ϵ^- and ϵ^* satisfying the conditions (5.32) shows that ϵ^+ and ϵ^- always exist for a given ϵ^* provided that

$$\alpha_0^- > (1+\epsilon^*)^2 \left(r^2 \beta_0^2 + \frac{\gamma_0^2}{r^2} \right) \tag{5.35}$$

throughout $\mathscr{D}$. We may thus conclude that *the steady state* (3.56) *is stable to arbitrary finite-amplitude perturbations if there exists a constant* ϵ^* *such that the inequality* (5.35) *is satisfied*.

6. Conclusions

In this paper we have used the general theory developed in Part I (Vladimirov and Moffatt 1995) to obtain linear and nonlinear stability criteria for steady axisymmetric MHD flows of an ideal incompressible fluid (with respect to axisymmetric perturbations). In the unperturbed state both velocity and magnetic field are non-zero and have in general both poloidal and toroidal components. Stability properties of such a steady state is of particular importance in the context of plasma confinement devices (e.g. tokamak or reversed-field pinch).

We have shown that the frozen-in 'modified vorticity field' introduced in Part I was useful in constructing of an appropriate Casimir. This has been used in Sec. 3 to obtain linear stability criteria. We note that in the particular case of circular geometry Howard and Gupta (1962) succeeded (through the use of normal-mode techniques) in obtaining stability criteria for the situation in which at least one of $U_\phi(r)$, $U_z(r)$, $H_\phi(r)$ and $H_z(r)$ is identically zero. However, they were unable to deal with the situation in which all four field components are non-zero. The energy–Casimir technique adopted in this paper is in this respect more powerful, but the stability criteria obtained may possibly be weaker (in the sense that the energy–Casimir method gives only sufficient (but not necessary) conditions for stability) than those obtained in special cases by normal-mode techniques.

In Sec. 4 we have used Arnold's (1965*a*, *b*) theory to obtain nonlinear stability criteria. We have considered first 'isomagnetic' perturbations, i.e. perturbations from the steady state under which the magnetic field is frozen, and the vector potential of a material fluid particle is therefore conserved. This resulted in Criterion 4.1, which gives sufficient conditions for nonlinear stability to 'isomagnetic' perturbations. We then considered two particular situations: (i) when in the steady state only the poloidal parts of both velocity and magnetic field are non-zero (Sec. 5.1) and (ii) when in the steady state all components of the magnetic field are non-zero, while the velocity field has only a toroidal component (Sec. 5.2). For both situations, we succeeded in obtaining sufficient conditions for nonlinear stability with respect to arbitrary initial perturbations (unconstrained by the isomagnetic condition).

The a priori estimate (4.9) gives a global upper bound for perturbations of arbitrary amplitude, provided that the conditions of the stability criteria 5.1 (or 5.2) are satisfied. This is a stronger result than merely Lyapunov stability. On the other hand, the linear stability criteria 3.1–3.4, which ensure stability to infinitesimal perturbations, evidently cover a much wider class of steady states than their nonlinear counterparts. The important question that remains open is whether the linear stability criteria obtained in Sec. 3 give sufficient conditions for stability to perturbations of small but finite amplitude.

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Appendix A. Derivation of (3.38)

Using (3.35a), the last two terms in (3.37) may be written as

$$\mathscr{I} \equiv 2\int_{\mathscr{D}} \left(\frac{R_1}{r^2}\,\delta^2\rho_1 - \Psi(A)\,\delta^2\lambda\right) d\tau.$$

Substituting the expressions (3.30c) and (3.30d) for $\delta^2\rho_1$ and $\delta^2\lambda$ into this formula and then using the identity

$$\int_{\mathscr{D}} f\frac{1}{r}\{g,h\}\,d\tau = \int_{\mathscr{D}} g\frac{1}{r}\{h,f\}\,d\tau + \int_{\partial\mathscr{D}} fg\boldsymbol{\nabla}h\cdot d\mathbf{l}, \qquad \text{(A 1)}$$

we find

$$\begin{aligned}\mathscr{I} = \int_{\mathscr{D}} \Bigg[&\frac{R_1}{r^2}\left(C''(\delta\rho)^2 + \frac{2}{r}\{\delta\rho,\delta\chi\}\right)\\ &-\Psi\left(\frac{2}{r}\{\delta\eta,\delta\rho\} + \frac{2}{r}\{\delta\chi,\delta\rho_2\} - C'''R_2(\delta\rho)^2 - 2C''\delta\rho\,\delta\rho_2\right)\Bigg]d\tau\\ = \int_{\mathscr{D}} \Bigg[&\frac{R_1}{r^2}C''(\delta\rho)^2 + \delta\rho\frac{2}{r}\left\{\delta\chi,\frac{R_1}{r^2}\right\} + \Psi'\,\delta\rho\frac{2}{r}\{\delta\eta,A\}\\ &+\Psi'\,\delta\rho_2\frac{2}{r}\{\delta\chi,A\} + \Psi C'''R_2(\delta\rho)^2 + 2\Psi C''\,\delta\rho\,\delta\rho_2\Bigg]d\tau. \qquad \text{(A 2)}\end{aligned}$$

Here we have used the boundary conditions (3.32) and (3.33). From (3.30b), we have

$$\frac{1}{r}\{A,\delta\eta\} = -\delta^1\lambda + \delta q + \frac{1}{r}\{N,\delta\rho\} + \frac{1}{r}\{\delta\chi,R_2\} + \frac{1}{r}\{X,\delta\rho_2\} - C''R_2\,\delta\rho - C'\,\delta\rho_2.$$

After substitution of this expression into (A 2) and some algebra, we obtain

$$\begin{aligned}\mathscr{I} = \int_{\mathscr{D}}\Bigg[&2\Psi'\,\delta\rho\,\delta^1\lambda - 2\Psi'\,\delta\rho\,\delta q + c_1(\delta\rho)^2 + 2(\Psi''C + \Psi'C' + \Psi C'' - \Psi''R_1)\,\delta\rho\,\delta\rho_2\\ &+2\Psi'\,\delta\rho_2\frac{1}{r}\{X,\delta\rho\} - 2\Psi'\,\delta\rho_2\frac{1}{r}\{A,\delta\chi\} + 2(G_1' - \Psi''R_2)\,\delta\rho\frac{1}{r}\{A,\delta\chi\}\Bigg]d\tau, \qquad \text{(A 3)}\end{aligned}$$

where

$$c_1 \equiv \frac{C''R_1}{r^2} + (\Psi C''' + 2\Psi' C'')\, R_2 - \frac{1}{r}\{\Psi, N\}\frac{\Psi''}{\Psi'}. \tag{A 4}$$

From ($3.30a$), we have

$$\frac{1}{r}\{A, \delta\chi\} = \delta^1\rho_1 - \frac{1}{r}\{\delta\rho, X\} - C'\,\delta\rho.$$

By substituting this into (A 3), $\mathscr{I}$ can be transformed into

$$\mathscr{I} = \int_{\mathscr{D}} [2\Psi'\,\delta\rho\,\delta^1\lambda - 2\Psi'\,\delta\rho\,\delta q + c_2(\delta\rho)^2 \\ + 2d_1\,\delta\rho\,\delta\rho_2 + 2(G_1' - \Psi'' R_2)\,\delta\rho\,\delta^1\rho - 2\Psi'\,\delta^1\rho_1\,\delta\rho_2]\,d\tau, \tag{A 5}$$

where

$$\left.\begin{aligned} c_2 &\equiv c_1 - 2(G_1' - \Psi'' R_2)\, C' - (G_1'' - \Psi''' R_2)\frac{1}{r}\{X, A\} + \Psi''\frac{1}{r}\{X, R_2\}, \\ d_1 &= \Psi'' C + 2\Psi' C' + \Psi C'' - \Psi'' R_1. \end{aligned}\right\} \tag{A 6}$$

Substitution of (A 5) into (3.37) results in

$$\delta^2\mathscr{R} = \frac{1}{2}\int_{\mathscr{D}} \left[\frac{1}{r^2}(\boldsymbol{\nabla}\delta\psi)^2 + \frac{1}{r^2}(\boldsymbol{\nabla}\delta\rho)^2 - 2\Psi'\,\delta\rho\,\delta q + \frac{1}{r^2}(\delta^1\rho_1)^2 + r^2(\delta\rho_2)^2 \right. \\ - 2\Psi'\,\delta^1\rho_1\,\delta\rho_2 + (c_2 + F_{AA} + S'' R_2)(\delta\rho)^2 + 2(S' + d_1)\,\delta\rho\,\delta\rho_2 \\ \left. + 2(G_1' - \Psi'' R_2)\,\delta\rho\,\delta^1\rho_1 + F_{\Lambda\Lambda}(\delta^1\lambda)^2 + 2(F_{\Lambda A} + \Psi')\,\delta\rho\,\delta^1\lambda\right] d\tau. \tag{A 7}$$

From ($3.36a$), we have

$$F_{\Lambda A} + \Psi' = -\Lambda'(A)\,F_{\Lambda\Lambda}.$$

Hence

$$\mathscr{I}_1 \equiv \int_{\mathscr{D}} [F_{\Lambda\Lambda}(\delta^1\lambda)^2 + 2(F_{\Lambda A} + \Psi')\,\delta\rho\,\delta^1\lambda]\,d\tau \\ = \int_{\mathscr{D}} [F_{\Lambda\Lambda}[\delta^1\lambda - \Lambda'(A)\,\delta\rho]^2 - F_{\Lambda\Lambda}[\Lambda'(A)]^2\,(\delta\rho)^2]\,d\tau. \tag{A 8}$$

Let

$$\mathscr{I}_2 \equiv \int_{\mathscr{D}} \left[\frac{1}{r^2}(\boldsymbol{\nabla}\delta\psi)^2 + \frac{1}{r^2}(\boldsymbol{\nabla}\delta\rho)^2 - 2\Psi'\,\delta\rho\,\delta q\right] d\tau.$$

After integration by parts of the last term in this integral (using the fact that, according to (3.29), $\delta q = -\hat{K}\,\delta\psi$) and simple algebra, we find

$$\mathscr{I}_2 = \int_{\mathscr{D}} \left[\frac{1}{r^2}[\boldsymbol{\nabla}(\delta\psi - \Psi'\,\delta\rho)]^2 + \frac{1}{r^2}(1 - \Psi'^2)(\boldsymbol{\nabla}\delta\rho)^2 + \Psi'(\hat{K}\Psi')(\delta\rho)^2\right] d\tau. \tag{A 9}$$

Substituting (A 8) and (A 9) into (A 7), we obtain

$$\delta^2\mathscr{R} = \frac{1}{2}\int_{\mathscr{D}}\left[\frac{1}{r^2}[\boldsymbol{\nabla}(\delta\psi - \Psi'\,\delta\rho)]^2 + \frac{1}{r^2}(1-\Psi'^2)(\boldsymbol{\nabla}\delta\rho)^2 + \frac{1}{r^2}(\delta^1\rho_1)^2 + r^2(\delta\rho_2)^2 - 2\Psi'\,\delta^1\rho_1\,\delta\rho_2 + F_{\Lambda\Lambda}(\delta^1\lambda - \Lambda'\,\delta\rho)^2 + 2f_1\,\delta\rho\,\delta^1\rho_1 + 2f_2\,\delta\rho\,\delta\rho_2 + f_3(\delta\rho)^2\right]d\tau, \tag{A 10}$$

where

$$f_1 = G_1' - \Psi''R_2 = \Psi'\frac{dR_2}{dA} - \frac{d}{dA}\left(\frac{R_1}{r^2}\right), \tag{A 11}$$

$$f_2 = S' + d_1 = S' + \Psi''C + 2\Psi'C' + \Psi C'' - \Psi''R_1, \tag{A 12}$$

$$f_3 = F_{AA} - F_{\Lambda\Lambda}(\Lambda'(A))^2 + \Psi'(\hat{K}\,\Psi') + c_2 + S''R_2. \tag{A 13}$$

It is easy to see that (A 11) coincides with (3.38b).

Using (3.35c), (3.3a) and (3.22), it may be shown that (A 12) reduces to (3.38c).

From (3.36a, b),

$$F_{AA} - F_{\Lambda\Lambda}\,\Lambda'^2 = \Lambda'(A)\,\Psi'(A) + g'(A) + [G_1(A)\,C'(A)]'.$$

After substitution of this into (A 13) and some manipulations using (3.3), (3.23), (3.35c) and (A 6), we find that (A 13) coincides with (3.38d).

Finally, comparison of (A 10) with (3.38) shows that they coincide.

Appendix B. Derivation of the inequality (3.61)

Let $\hat{\rho}(r,k)$ be the Fourier transform of $\delta\rho$:

$$\hat{\rho}(r,k) = \int_{-\infty}^{\infty} e^{-ikz}\,\delta\rho(r,z)\,dz.$$

Then we have

$$\int_{\mathscr{D}}\frac{1}{r^2}(\boldsymbol{\nabla}\delta\rho)^2\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^a\left(\frac{1}{r^2}\left|\frac{\partial\hat{\rho}}{\partial r}\right|^2 + \frac{k^2}{r^2}|\hat{\rho}|^2\right)r\,dr\,dk \geqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^a\frac{1}{r^2}\left|\frac{\partial\hat{\rho}}{\partial r}\right|^2 r\,dr\,dk. \tag{B 1}$$

On the other hand, it may be shown that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^a\frac{1}{r^2}\left|\frac{\partial\hat{\rho}}{\partial r}\right|^2 r\,dr\,dk \geqslant \frac{\lambda_0^2}{2\pi a^2}\int_{-\infty}^{\infty}\int_0^a\frac{1}{r^2}|\hat{\rho}|^2\,r\,dr\,dk = \frac{\lambda_0^2}{a^2}\int_{\mathscr{D}}\frac{1}{r^2}(\delta\rho)^2\,d\tau, \tag{B 2}$$

where λ_0 is the first zero of the Bessel function of the first order $J_1(x): J_1(\lambda_0) = 0$. The inequality (3.61) follows directly from (B 1) and (B 2).

Appendix C. Relation between the conserved functionals $\mathscr{R}$ and $\mathscr{F}$

We shall show here that the functional $\mathscr{R}$, (3.26), after a special choice of the function $F(\rho, \lambda)$ (appearing in $\mathscr{R}$ through $\mathscr{C}$), may be transformed so that the relation between $\mathscr{R}$ and $\mathscr{F}$ becomes

$$\mathscr{F} = \mathscr{R} + \mathscr{L} - \mu\Gamma, \tag{C 1}$$

where

$$\Gamma \equiv \oint_{\partial\mathscr{D}} \mathbf{v}_g \cdot d\mathbf{l} = \oint_{\partial\mathscr{D}} (\mathbf{v} + \rho\nabla\eta + \rho_2\nabla\chi)\cdot d\mathbf{l} \tag{C 2}$$

is the conserved circulation of the poloidal part $\mathbf{v}_g$ of the 'generalized velocity field' $\mathbf{u}_g$ and μ is a constant. (The field $\mathbf{u}_g$ is related to the 'modified vorticity field' $\mathbf{w}$ by the equation $\mathbf{w} = \nabla \wedge \mathbf{u}_g$. Conservation of the circulation of $\mathbf{u}_g$ round any closed material curve is a consequence of the fact that the field $\mathbf{w}$ is frozen in the fluid.)

First, we choose the function $F(\rho, \lambda)$ such that

$$F(\rho, \lambda) = \lambda F_1(\rho) + F_2(\rho), \tag{C 3}$$

so that, in contrast with (3.41), we take $F_0(\lambda) \equiv 0$ here. With this choice and using the definition (3.15) of λ, we have

$$\mathscr{C} = \int_{\mathscr{D}} \left[qF_1(\rho) + F_2(\rho) + F_1(\rho)\frac{1}{r}\{\eta, \rho\} + F_1(\rho)\frac{1}{r}\{\chi, \rho_2\} - F_1(\rho)\, C'(\rho)\,\rho_2 \right] d\tau. \tag{C 4}$$

Using the identity (A 1) of Appendix A and the boundary condition (2.8), we find that

$$\int_{\mathscr{D}} F_1(\rho)\frac{1}{r}\{\eta, \rho\}\, d\tau = \int_{\mathscr{D}} \eta\frac{1}{r}\{\rho, F_1(\rho)\}\, d\tau + F_1(\rho)\,|_{\partial\mathscr{D}} \oint_{\partial\mathscr{D}} \eta\nabla\rho\cdot d\mathbf{l} = 0, \tag{C 5}$$

$$\int_{\mathscr{D}} F_1(\rho)\frac{1}{r}\{\chi, \rho_2\}\, d\tau = \int_{\mathscr{D}} F'_1(\rho)\,\rho_2\frac{1}{r}\{\rho, \chi\}\, d\tau - F_1(\rho)\,|_{\partial\mathscr{D}} \oint_{\partial\mathscr{D}} \rho_2\nabla\chi\cdot d\mathbf{l}. \tag{C 6}$$

After eliminating $\{\rho, \chi\}$ from (C 6) using (3.11), we obtain

$$\int_{\mathscr{D}} F_1(\rho)\frac{1}{r}\{\chi, \rho_2\}\, d\tau = \int_{\mathscr{D}} [F'_1(\rho)\,\rho_1\,\rho_2 - F'_1(\rho)\, C(\rho)\,\rho_2]\, d\tau - F_1(\rho)\,|_{\partial\mathscr{D}} \oint_{\partial\mathscr{D}} \rho_2\nabla\chi\cdot d\mathbf{l}. \tag{C 7}$$

Also, integrating the first term in (C 4) by parts, we find

$$\int_{\mathscr{D}} qF_1(\rho)\, d\tau = \int_{\mathscr{D}} F'_1(\rho)\,\mathbf{v}\cdot\mathbf{b}\, d\tau - F_1(\rho)\,|_{\partial\mathscr{D}} \oint_{\partial\mathscr{D}} \mathbf{v}\cdot d\mathbf{l}. \tag{C 8}$$

It follows from (C 5), (C 7) and (C 8) that the functional $\mathscr{C}$ may be written in the form

$$\begin{aligned}\mathscr{C} = &\int_{\mathscr{D}} [F'_1(\rho)\,(\mathbf{v}\cdot\mathbf{b} + \rho_1\,\rho_2) + F_2(\rho) - [F_1(\rho)\, C(\rho)]'\,\rho_2]\, d\tau \\ &- F_1(\rho)\,|_{\partial\mathscr{D}} \oint_{\partial\mathscr{D}} (\mathbf{v} + \rho_2\nabla\chi)\cdot d\mathbf{l}.\end{aligned} \tag{C 9}$$

Finally, substitution of (C 9) into $\mathscr{R}$ results in

$$\mathscr{R} = \int_{\mathscr{D}} \left[\frac{|\mathbf{v}|^2}{2} + \frac{(\boldsymbol{\nabla}\rho)^2}{2r^2} + \frac{\rho_1^2}{2r^2} + r^2 \frac{\rho_2^2}{2} + F_1'(\rho)\,(\mathbf{v}\cdot\mathbf{b} + \rho_1\rho_2) + F_2(\rho) + \tilde{S}(\rho)\,\rho_2 \right] d\tau - \mu\Gamma, \tag{C 10}$$

where

$$\tilde{S}(\rho) = S(\rho) - [F_1(\rho)\,C(\rho)]', \qquad \mu = F_1(\rho)\,|_{\partial\mathscr{D}}. \tag{C 11}$$

Here the functions $F_1(\rho)$, $F_2(\rho)$ and $S(\rho)$ (and hence $\tilde{S}(\rho)$ too) are arbitrary, as well as the functions $N_1(\rho)$, $N_2(\rho)$ and $S(\rho)$ appearing in the functional $\mathscr{F}$. In view of (3.28), (4.10) and (4.12), it follows that, up to the change of notation,

$$\mathscr{R} = \mathscr{E} + \mathscr{H}_M + \mathscr{H}_C + \tilde{\mathscr{C}} - \mu\Gamma.$$

Finally, comparing this with $\mathscr{F}$ ($\mathscr{F} = \mathscr{E} + \mathscr{H}_M + \mathscr{H}_C + \tilde{\mathscr{C}} + \mathscr{L}$), we arrive at (C 1). If we now take $\Psi = 0$ and $\tilde{\rho} = 0$ on $\partial\mathscr{D}$ (as we did in Sec. 4) then the constant μ vanishes, and the difference between $\mathscr{F}$ and $\mathscr{R}$ reduces to the functional $\mathscr{L}$ only.

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