6 Elementary analysis

6.1 Motivation

Analysis is the careful study of infinite processes such as limits, convergence, continuity, differential and integral calculus, and is one of the foundations of mathematics. This section covers some of the basic concepts including the important problem of the convergence of infinite series. It also introduces the remarkable properties of analytic functions of a complex variable.

6.2 Sequences

6.2.1 Limits of sequences

We first consider a *sequence* of real or complex numbers f_n, defined for all integers $n \geqslant n_0$. Possible behaviours as n increases are:

- f_n tends towards a particular value
- f_n does not tend to any value but remains limited in magnitude
- f_n is unlimited in magnitude

Definition 1. *The sequence f_n converges to the limit L as $n \to \infty$ if, for any positive number ϵ, $|f_n - L| < \epsilon$ for sufficiently large n.*

In other words *the members of the sequence are eventually contained within an arbitrarily small disk centred on L*. We write this as

Note that L here is a finite number.

To say that a property holds *for sufficiently large n* means that there exists an integer N such that the property is true for all $n \geqslant N$.

Example:

Proof:

If f_n does not tend to a limit it may nevertheless be *bounded*.

Definition 2. *The sequence f_n is bounded as $n \to \infty$ if there exists a positive number K such that $|f_n| < K$ for sufficiently large n.*

Example:

Proof:

6.2.2 Cauchy's principle of convergence

A necessary and sufficient condition for the sequence f_n to converge is that, for any positive number ϵ, $|f_{n+m} - f_n| < \epsilon$ for all positive integers m, for sufficiently large n. Note that this condition does not require a knowledge of the value of the limit L.

6.3 Convergence of series

6.3.1 Meaning of convergence

What is the meaning of an infinite series such as

$$\sum_{n=n_0}^{\infty} u_n$$

involving the addition of an infinite number of terms?

We define the *partial sum*

The infinite series $\sum u_n$ is said to *converge* if the sequence of partial sums S_N tends to a limit S as $N \to \infty$. The value of the series is then S. Otherwise the series *diverges*.

Note that whether a series converges or diverges does not depend on the value of n_0 (i.e. on when the series begins) but only on the behaviour of the terms for large n.

According to Cauchy's principle of convergence, a necessary and sufficient condition for $\sum u_n$ to converge is that, for any positive number ϵ,

for all positive integers m, for sufficiently large N.

6.3.2 Classic examples

The *geometric series* $\sum z^n$ has the partial sum

$$\sum_{n=0}^{N} z^n = \begin{cases} \frac{1-z^{N+1}}{1-z}, & z \neq 1 \\ N+1, & z = 1 \end{cases}$$

Therefore $\sum z^n$ converges if $|z| < 1$, and the sum is $1/(1-z)$. If $|z| \geqslant 1$ the series diverges.

The *harmonic series* $\sum n^{-1}$ diverges. Consider the partial sum

$$S_N = \sum_{n=1}^{N} \frac{1}{n} > \int_1^{N+1} \frac{\mathrm{d}x}{x} = \ln(N+1)$$

Therefore S_N increases without bound and does not tend to a limit as $N \to \infty$.

6.3.3 Absolute and conditional convergence

If $\sum |u_n|$ converges, then $\sum u_n$ also converges. $\sum u_n$ is said to converge *absolutely*.

If $\sum |u_n|$ diverges, then $\sum u_n$ may or may not converge. If it does, it is said to converge *conditionally*.

[Proof of the first statement above: If $\sum |u_n|$ converges then, for any positive number ϵ,

for all positive integers m, for sufficiently large N. But then

and so $\sum u_n$ also converges.]

6.3.4 Necessary condition for convergence

A necessary condition for $\sum u_n$ to converge is that $u_n \to 0$ as $n \to \infty$. Formally, this can be shown by noting that

If the series converges then

and therefore $\lim u_n = 0$.

This condition is *not* sufficient for convergence, as exemplified by the harmonic series.

6.3.5 The comparison test

This refers to series of non-negative real numbers. We write these as $|u_n|$ because the comparison test is most often applied in assessing the absolute convergence of a series of real or complex numbers.

If $\sum |v_n|$ converges and

then $\sum |u_n|$ also converges. This follows from the fact that

and each partial sum is a non-decreasing sequence, which must either tend to a limit or increase without bound.

More generally, if $\sum |v_n|$ converges and

for sufficiently large n, where K is a constant, then $\sum |u_n|$ also converges.

Conversely, if $\sum |v_n|$ diverges and

for sufficiently large n, where K is a positive constant, then $\sum |u_n|$ also diverges.

In particular, if $\sum |v_n|$ converges (diverges) and

then $\sum |u_n|$ also converges (diverges).

6.3.6 D'Alembert's ratio test

This uses a comparison between a given series $\sum u_n$ of complex terms and a geometric series $\sum v_n = \sum r^n$, where $r > 0$.

The *absolute ratio of successive terms* is

Suppose that r_n tends to a limit r as $n \to \infty$. Then

- if $r < 1$,

- if $r > 1$,

- if $r = 1$,

Even if r_n does not tend to a limit, if $r_n \leqslant r$ for sufficiently large n, where $r < 1$ is a constant, then $\sum u_n$ converges absolutely.

Example: for the harmonic series $\sum n^{-1}$,

A different test is required, such as the integral comparison test used above.

The ratio test is useless for series in which some of the terms are zero. However, it can easily be adapted by relabelling the series to remove the vanishing terms.

6.3.7 Cauchy's root test

The same conclusions as for the ratio test apply when instead

This result also follows from a comparison with a geometric series. It is more powerful than the ratio test but usually harder to apply.

6.4 Functions of a continuous variable

6.4.1 Limits and continuity

We now consider how a real or complex function $f(z)$ of a real or complex variable z behaves near a point z_0.

Definition 3. *The function $f(z)$ tends to the limit L as $z \to z_0$ if, for any positive number ϵ, there exists a positive number δ, depending on ϵ, such that $|f(z) - L| < \epsilon$ for all z such that $|z - z_0| < \delta$.*

We write this as

The value of L would normally be $f(z_0)$. However, cases such as

must be expressed as limits because $\sin 0/0 = 0/0$ is indeterminate.

Definition 4. *The function $f(z)$ is continuous at the point $z = z_0$ if $f(z) \to f(z_0)$ as $z \to z_0$.*

Definition 5. *The function $f(z)$ is bounded as $z \to z_0$ if there exist positive numbers K and δ such that $|f(z)| < K$ for all z with $|z - z_0| < \delta$.*

Definition 6. *The function $f(z)$ tends to the limit L as $z \to \infty$ if, for any positive number ϵ, there exists a positive number R, depending on ϵ, such that $|f(z) - L| < \epsilon$ for all z with $|z| > R$.*

We write this as

Definition 7. *The function $f(z)$ is bounded as $z \to \infty$ if there exist positive numbers K and R such that $|f(z)| < K$ for all z with $|z| > R$.*

There are different ways in which z can approach z_0 or ∞, especially in the complex plane. Sometimes the limit or bound applies only if approached in a particular way.

For example, consider $\tanh(z)$ as $|z| \to \infty$

$$\lim_{z\to+\infty} \tanh z = 1, \qquad \lim_{z\to-\infty} \tanh z = -1$$

This notation implies that z is approaching positive or negative real infinity along the real axis. But if z approaches infinity along the imaginary axis, i.e. $z \to \pm\mathrm{i}\infty$, the limit of $\tanh$ is not even defined.

In the context of real variables $x \to \infty$ usually means specifically $x \to +\infty$. A related notation for one-sided limits is exemplified by

$$\lim_{x\to0^+} \frac{x(1+x)}{|x|} = 1, \qquad \lim_{x\to0^-} \frac{x(1+x)}{|x|} = -1$$

6.4.2 The O notation

The useful symbols O, o and $\sim$ are used to compare the rates of growth or decay of different functions.

- $f(z) = O(g(z))$ as $z \to z_0$ means that

- $f(z) = o(g(z))$ as $z \to z_0$ means that

- $f(z) \sim g(z)$ as $z \to z_0$ means that

If $f(z) \sim g(z)$ we say that f is *asymptotically equal* to g. This should *not* be written as $f(z) \to g(z)$.

Notes:

- these definitions also apply when $z_0 = \infty$
- $f(z) = O(1)$ means that $f(z)$ is bounded
- either $f(z) = o(g(z))$ or $f(z) \sim g(z)$ implies $f(z) = O(g(z))$
- only $f(z) \sim g(z)$ is a symmetric relation

Examples:

6.5 Taylor's theorem for functions of a real variable

Let $f(x)$ be a (real or complex) function of a real variable x, which is differentiable at least n times in the interval $x_0 \leqslant x \leqslant x_0 + h$. Then

where

$$R_n = \int_{x_0}^{x_0+h} \frac{(x_0 + h - x)^{n-1}}{(n-1)!} f^{(n)}(x)\,\mathrm{d}x$$

is the remainder after n terms of the Taylor series.

[Proof: integrate R_n by parts n times.]

The remainder term can be expressed in various ways. Lagrange's expression for the remainder is

$$R_n = \frac{h^n}{n!} f^{(n)}(\xi)$$

where ξ is an unknown number in the interval $x_0 < \xi < x_0 + h$. So

If $f(x)$ is infinitely differentiable in $x_0 \leqslant x \leqslant x_0 + h$ (it is a *smooth* function) we can write an infinite Taylor series

which converges for sufficiently small h (as discussed below).

6.6 Analytic functions of a complex variable

6.6.1 Complex differentiability

Definition 8. *The derivative of the function $f(z)$ at the point $z = z_0$ is*

If this exists, the function $f(z)$ is differentiable at $z = z_0$.

Another way to write this is

Requiring a function of a *complex* variable to be differentiable is a surprisingly strong constraint. The limit must be the same when $\delta z \to 0$ in any direction in the complex plane.

6.6.2 The Cauchy–Riemann equations

Separate $f = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$ into their real and imaginary parts:

If $f'(z)$ exists we can calculate it by assuming that $\delta z = \delta x + \mathrm{i}\,\delta y$ approaches 0 along the real axis, so $\delta y = 0$:

$$
\begin{aligned}
f'(z) &= \\
&= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + \mathrm{i}v(x+\delta x, y) - u(x,y) - \mathrm{i}v(x,y)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + \mathrm{i} \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x} \\
&=
\end{aligned}
$$

The derivative should have the same value if δz approaches 0 along the imaginary axis, so $\delta x = 0$:

$$
\begin{aligned}
f'(z) &= \\
&= \lim_{\delta y \to 0} \frac{u(x, y+\delta y) + \mathrm{i}v(x, y+\delta y) - u(x,y) - \mathrm{i}v(x,y)}{\mathrm{i}\,\delta y} \\
&= -\mathrm{i} \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y} \\
&=
\end{aligned}
$$

Comparing the real and imaginary parts of these expressions, we deduce the *Cauchy–Riemann equations*

These are necessary conditions for $f(z)$ to have a complex derivative. They are also sufficient conditions, provided that the partial derivatives are also continuous.

6.6.3 Analytic functions

If a function $f(z)$ has a complex derivative at every point z in a region R of the complex plane, it is said to be *analytic* in R. To be analytic at a point $z = z_0$, $f(z)$ must be differentiable throughout some neighbourhood $|z - z_0| < \epsilon$ of that point.

Examples of functions that are analytic in the whole complex plane (known as *entire functions*):

- $f(z) = c$, a complex constant
- $f(z) = z$, for which $u = x$ and $v = y$, and we easily verify the CR equations
- $f(z) = z^n$, where n is a positive integer

- $f(z) = P(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0$, a general polynomial function with complex coefficients

- $f(z) = \exp(z)$

In the case of the exponential function we have

$$f(z) = \mathrm{e}^z = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y} =$$

The CR equations are satisfied for all x and y:

$$\frac{\partial u}{\partial x} =$$

$$\frac{\partial v}{\partial x} =$$

The derivative of the exponential function is

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\,\frac{\partial v}{\partial x} =$$

as expected.

Sums, products and compositions of analytic functions are also analytic, e.g.

The usual product, quotient and chain rules apply to complex derivatives of analytic functions. Familiar relations such as

$$\frac{\mathrm{d}}{\mathrm{d}z} z^n = nz^{n-1}, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z$$

apply as usual.

Many complex functions are analytic everywhere in the complex plane except at isolated points, which are called the *singular points* or *singularities* of the function.

Examples:

- $f(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials. This is called a *rational function* and is analytic except at points where $Q(z) = 0$.

- $f(z) = z^c$, where c is a complex constant, is analytic except at $z = 0$ (unless c is a non-negative integer)

- $f(z) = \ln z$ is also analytic except at $z = 0$

The last two examples are in fact *multiple-valued functions*, which require special treatment (see next term).

Examples of non-analytic functions:

- $f(z) = \mathrm{Re}(z)$, for which $u = x$ and $v = 0$, so the CR equations are not satisfied anywhere

- $f(z) = z^*$, for which $u = x$ and $v = -y$

- $f(z) = |z|$, for which $u = (x^2 + y^2)^{1/2}$ and $v = 0$

- $f(z) = |z|^2$, for which $u = x^2 + y^2$ and $v = 0$

In the last case the CR equations are satisfied only at $x = y = 0$ and we can say that $f'(0) = 0$. However, $f(z)$ is not analytic even at $z = 0$ because it is not differentiable throughout any neighbourhood $|z| < \epsilon$ of 0.

6.6.4 Consequences of the Cauchy–Riemann equations

If we know the real part of an analytic function in some region, we can find its imaginary part (or vice versa) up to an additive constant by integrating the CR equations.

Example:

$$u(x, y) = x^2 - y^2$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \quad \Rightarrow \quad v = 2xy + g(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \Rightarrow \quad 2y + g'(x) = 2y \quad \Rightarrow \quad g'(x) = 0$$

Therefore $v(x, y) = 2xy + c$, where c is a real constant, and we recognize

The real and imaginary parts of an analytic function satisfy Laplace's equation (they are *harmonic functions*):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =$$

$$=$$

$$=$$

The proof that $\nabla^2 v = 0$ is similar. This provides a useful method for solving Laplace's equation in two dimensions. Furthermore,

$$\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v =$$

$$=$$

$$=$$

and so the curves of constant u and those of constant v intersect at right angles. u and v are said to be *conjugate harmonic functions*.

6.7 Taylor series for analytic functions

If a function of a complex variable is analytic in a region R of the complex plane, not only is it differentiable everywhere in R, it is also differentiable any number of times. If $f(z)$ is analytic at $z = z_0$, it has an infinite Taylor series

which converges within some neighbourhood of z_0 (as discussed below). In fact this can be taken as a definition of analyticity.

6.8 Zeros, poles and essential singularities

6.8.1 Zeros of complex functions

The zeros of $f(z)$ are the points $z = z_0$ in the complex plane where $f(z_0) = 0$. A zero is of order N if

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(N-1)}(z_0) = 0 \qquad \text{but} \quad f^{(N)}(z_0) \neq 0$$

The first non-zero term in the Taylor series of $f(z)$ about $z = z_0$ is then proportional to $(z - z_0)^N$. Indeed

A *simple zero* is a zero of order 1. A *double zero* is one of order 2, etc.

Examples:

- $f(z) = z$ has a simple zero at $z = 0$
- $f(z) = (z - \mathrm{i})^2$ has a double zero at $z = \mathrm{i}$
- $f(z) = z^2 - 1 = (z - 1)(z + 1)$ has simple zeros at $z = \pm 1$

Example . . . .

▷ *Find and classify the zeros of $f(z) = \sinh z$.*

. . . .

6.8.2 Poles

Suppose $g(z)$ is analytic and non-zero at $z = z_0$. Consider the function

Obviously, $f(z)$ is not analytic at $z = z_0$. We say that $f(z)$ has a *pole of order N*. Note that

$$f(z) \sim g(z_0)(z - z_0)^{-N} \qquad \text{as} \quad z \to z_0\,.$$

A *simple pole* is a pole of order 1. A *double pole* is one of order 2, etc. The behaviour of $f(z)$ near $z = z_0$ can be explored through an expansion of the following type. Because $g(z)$ is analytic

Then

$$f(z) = (z - z_0)^{-N} g(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n$$

with $a_n = b_{n+N}$ and so $a_{-N} \neq 0$. This is not a Taylor series because it includes negative powers of $z - z_0$, and $f(z)$ is not analytic at $z = z_0$.

Notes:

- if $f(z)$ has a zero of order N at $z = z_0$, then $1/f(z)$ has a pole of order N there, and vice versa
- if $f(z)$ is analytic and non-zero at $z = z_0$ and $g(z)$ has a zero of order N there, then $f(z)/g(z)$ has a pole of order N there

Example:

$$f(z) = \frac{2z}{(z+1)(z-\mathrm{i})^2}$$

has a simple pole at $z = -1$ and a double pole at $z = \mathrm{i}$ (as well as a simple zero at $z = 0$). The expansion about the double pole can be carried out by letting $z = \mathrm{i} + w$ and expanding in w:

$$\begin{aligned}
f(z) &= \frac{2(\mathrm{i} + w)}{(\mathrm{i} + w + 1)w^2} \\
&= \frac{2\mathrm{i}(1 - \mathrm{i}w)}{(\mathrm{i}+1)\left[1 + \frac{1}{2}(1-\mathrm{i})w\right] w^2} \\
&= \frac{2\mathrm{i}}{(\mathrm{i}+1)w^2}(1 - \mathrm{i}w)\left[1 - \frac{1}{2}(1-\mathrm{i})w + O(w^2)\right] \\
&= (1+\mathrm{i})w^{-2}\left[1 - \frac{1}{2}(1+\mathrm{i})w + O(w^2)\right] \\
&=
\end{aligned}$$

6.8.3 Laurent series

It can be shown that any function that is analytic (and single-valued) throughout an annulus $a < |z - z_0| < b$ centred on a point $z = z_0$ has a unique *Laurent series*

which converges for all values of z within the annulus.

If $a = 0$, then $f(z)$ is analytic throughout the disk $|z - z_0| < b$ except possibly at $z = z_0$ itself, and the Laurent series determines the behaviour of $f(z)$ near $z = z_0$. There are three possibilities:

- if the first non-zero term in the Laurent series has $n \geqslant 0$, then $f(z)$ is analytic at $z = z_0$ and the series is just a Taylor series
- if the first non-zero term in the Laurent series has $n = -N < 0$, then $f(z)$ has a pole of order N at $z = z_0$
- otherwise, if the Laurent series involves an infinite number of terms with $n < 0$, then $f(z)$ has an *essential singularity* at $z = z_0$

A classic example of an essential singularity is $f(z) = \mathrm{e}^{1/z}$ at $z = 0$. Here we can generate the Laurent series from a Taylor series in $1/z$:

The behaviour of a function near an essential singularity is remarkably complicated. *Picard's theorem* states that, in any neighbourhood of an essential singularity, the function takes all possible complex values (possibly with one exception) at infinitely many points. (In the case of $f(z) = \mathrm{e}^{1/z}$, the exceptional value 0 is never attained.)

6.8.4 Behaviour at infinity

We can examine the behaviour of a function $f(z)$ as $z \to \infty$ by defining a new variable $\zeta = 1/z$ and a new function $g(\zeta) = f(z)$. Then $z = \infty$ maps to a single point $\zeta = 0$, the *point at infinity*.

If $g(\zeta)$ has a zero, pole or essential singularity at $\zeta = 0$, then we can say that $f(z)$ has the corresponding property at $z = \infty$.

Examples:

$$f_1(z) = e^z =$$

has an essential singularity at $z = \infty$.

$$f_2(z) = z^2 =$$

has a double pole at $z = \infty$.

$$f_3(z) = e^{1/z} =$$

is analytic at $z = \infty$.

6.9 Convergence of power series

6.9.1 Circle of convergence

If the power series

converges for $z = z_1$, then the series converges absolutely for all z such that $|z - z_0| < |z_1 - z_0|$.

[Proof: The necessary condition for convergence,

$$\lim_{n\to\infty} a_n(z_1 - z_0)^n = 0$$

implies that

$$|a_n(z_1 - z_0)^n| < \epsilon$$

for sufficiently large n, for any $\epsilon > 0$. Therefore

$$|a_n(z - z_0)^n| < \epsilon\, r^n$$

for sufficiently large n, with

$$r = |(z - z_0)/(z_1 - z_0)| < 1$$

By comparison with the geometric series $\sum r^n$, $\sum |a_n(z - z_0)^n|$ converges.]

It follows that, if the power series diverges for $z = z_2$, then it diverges for all z such that $|z - z_0| > |z_2 - z_0|$.

Therefore there must exist a real, non-negative number R such that the series converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. R is called the *radius of convergence* and may be zero (exceptionally), positive or infinite.

$|z - z_0| = R$ is the *circle of convergence*. The series converges inside it and diverges outside. On the circle, it may either converge or diverge.

6.9.2 Determination of the radius of convergence

The absolute ratio of successive terms in a power series is

Suppose that $|a_{n+1}/a_n| \to L$ as $n \to \infty$. Then $r_n \to r = L|z - z_0|$. According to the ratio test, the series converges for $L|z - z_0| < 1$ and diverges for $L|z - z_0| > 1$. The radius of convergence is $R = 1/L$.

The same result, $R = 1/L$, follows from Cauchy's root test if instead $|a_n|^{1/n} \to L$ as $n \to \infty$.

The radius of convergence of the Taylor series of a function $f(z)$ about the point $z = z_0$ is equal to the distance of the nearest singular point of the function $f(z)$ from z_0. Since a convergent power series defines an analytic function, no singularity can lie inside the circle of convergence.

6.9.3 Examples

The following examples are generated from familiar Taylor series.

Here $|a_{n+1}/a_n| = n/(n+1) \to 1$ as $n \to \infty$, so $R = 1$. The series converges for $|z| < 1$ and diverges for $|z| > 1$. (In fact, on the circle $|z| = 1$, the series converges except at the point $z = 1$.) The function has a singularity at $z = 1$ that limits the radius of convergence.

Thought of as a power series in $(-z^2)$, this has $|a_{n+1}/a_n| = (2n+1)/(2n+3) \to 1$ as $n \to \infty$. Therefore $R = 1$ in terms of $(-z^2)$. But since $|-z^2| = 1$ is equivalent to $|z| = 1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$.

Here $|a_{n+1}/a_n| = 1/(n+1) \to 0$ as $n \to \infty$, so $R = \infty$. The series converges for all z; this is an entire function.

7 Ordinary differential equations

7.1 Motivation

Very many scientific problems are described by differential equations. Even if these are partial differential equations, they can often be reduced to ordinary differential equations (ODEs), e.g. by the method of separation of variables.

The ODEs encountered most frequently are linear and of either first or second order. In particular, second-order equations describe oscillatory phenomena.

Part IA dealt with first-order ODEs and also with linear second-order ODEs with constant coefficients. Here we deal with general linear second-order ODEs.

The general linear inhomogeneous first-order ODE

can be solved using the integrating factor

to obtain the general solution

Provided that the integrals can be evaluated, the problem is completely solved. An equivalent method does not exist for second-order ODEs, but an extensive theory can still be developed.

7.2 Classification

The general second-order ODE is an equation of the form for an unknown function $y(x)$, where $y' = \mathrm{d}y/\mathrm{d}x$ and $y'' = \mathrm{d}^2y/\mathrm{d}x^2$.

The general linear second-order ODE has the form

$$Ly = f$$

where L is a linear operator such that

where a, b, c, f are functions of x.

The equation is *homogeneous* (unforced) if $f = 0$, otherwise it is *inhomogeneous* (forced).

The principle of superposition applies to linear ODEs as to all linear equations.

Although the solution may be of interest only for real x, it is often informative to analyse the solution in the complex domain.

7.3 Homogeneous linear second-order ODEs

7.3.1 Linearly independent solutions

Divide through by the coefficient of y'' to obtain a standard form

Suppose that $y_1(x)$ and $y_2(x)$ are two solutions of this equation. They are *linearly independent* if

$$Ay_1(x) + By_2(x) = 0 \quad \text{(for all } x\text{)} \qquad \text{implies} \quad A = B = 0$$

i.e. if one is not simply a constant multiple of the other.

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions, then the *general solution* of the ODE is where A and B are arbitrary constants. There are two arbitrary constants because the equation is of second order.

7.3.2 The Wronskian

The *Wronskian* $W(x)$ of two solutions $y_1(x)$ and $y_2(x)$ of a second-order ODE is the determinant of the Wronskian matrix:

Suppose that $Ay_1(x) + By_2(x) = 0$ in some interval of x. Then also $Ay_1'(x) + By_2'(x) = 0$, and so

If this is satisfied for non-trivial A, B then $W = 0$ (in that interval of x). Therefore the solutions are linearly independent if $W \neq 0$.

7.3.3 Calculation of the Wronskian

Consider

$$\begin{aligned} W' &= y_1 y_2'' - y_2 y_1'' \\ &= y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1) \\ &= -pW \end{aligned}$$

since both y_1 and y_2 solve the ODE. This first-order ODE for W has the solution

This expression involves an indefinite integral and could be written as

Notes:

- the indefinite integral involves an arbitrary additive constant, so W involves an arbitrary multiplicative constant

- apart from that, W is the same for any two solutions y_1 and y_2

- W is therefore an intrinsic property of the ODE

- if $W \neq 0$ for one value of x (and p is integrable) then $W \neq 0$ for all x, so linear independence need be checked at only one value of x

7.3.4 Finding a second solution

Suppose that one solution $y_1(x)$ is known. Then a second solution $y_2(x)$ can be found as follows.

First find W as described above. The definition of W then provides a first-order linear ODE for the unknown y_2:

$$y_1 y_2' - y_2 y_1' = W$$

$$\frac{y_2'}{y_1} - \frac{y_2 y_1'}{y_1^2} = \frac{W}{y_1^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_2}{y_1}\right) = \frac{W}{y_1^2}$$

Again, this result involves an indefinite integral and could be written as

Notes:

- the indefinite integral involves an arbitrary additive constant, since any amount of y_1 can be added to y_2
- W involves an arbitrary multiplicative constant, since y_2 can be multiplied by any constant
- this expression for y_2 therefore provides the general solution of the ODE

Example . . . .

▷ *Given that* $y = x^n$ *is a solution of* $x^2y'' - (2n-1)xy' + n^2y = 0$, *find the general solution.* Standard form

Wronskian

Second solution

. . . .

The same result can be obtained by writing $y_2(x) = y_1(x)u(x)$ and obtaining a first-order linear ODE for u'. This method applies to higher-order linear ODEs and is reminiscent of the factorization of polynomial equations.

7.4 Series solutions

7.4.1 Ordinary and singular points

We consider a homogeneous linear second-order ODE in standard form:

A point $x = x_0$ is an *ordinary point* of the ODE if:

$p(x)$ and $q(x)$ are both analytic at $x = x_0$

Otherwise it is a *singular point*.

A singular point $x = x_0$ is *regular* if:

$(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both analytic at $x = x_0$

Example . . . .

▷ *Identify the singular points of Legendre's equation*

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0$$

where ℓ is a constant, and determine their nature. Divide through by $(1 - x^2)$ to obtain the standard form with

Both $p(x)$ and $q(x)$ are analytic for all x except $x = \pm 1$. These are the singular points. They are both regular:

are both analytic at $x = 1$, and similarly for $x = -1$. . . . .

7.4.2 Series solutions about an ordinary point

Wherever $p(x)$ and $q(x)$ are analytic, the ODE has two linearly independent solutions that are also analytic. If $x = x_0$ is an ordinary point, the ODE has two linearly independent solutions of the form

The coefficients a_n can be determined by substituting the series into the ODE and comparing powers of $(x - x_0)$. The radius of convergence of the series solutions is the distance to the nearest singular point of the ODE in the complex plane.

Since $p(x)$ and $q(x)$ are analytic at $x = x_0$,

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

inside some circle centred on $x = x_0$. Let

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y' = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} =$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} =$$

Note the following rule for multiplying power series:

$$AB = \sum_{\ell=0}^{\infty} A_\ell (x - x_0)^\ell \sum_{m=0}^{\infty} B_m (x - x_0)^m$$

$$=$$

Thus

$$py' = \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} p_{n-r} (r+1) a_{r+1} \right] (x - x_0)^n$$

$$qy = \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} q_{n-r} a_r \right] (x - x_0)^n$$

The coefficient of $(x - x_0)^n$ in the ODE $y'' + py' + qy = 0$ is therefore

This is a *recurrence relation* that determines a_{n+2} (for $n \geqslant 0$) in terms of the preceding coefficients $a_0, a_1, \ldots, a_{n+1}$. The first two coefficients a_0 and a_1 are not determined: they are the two arbitrary constants in the general solution.

The above procedure is rarely followed in practice!!

If p and q are *rational functions* (i.e. ratios of polynomials) it is a much better idea to multiply the ODE through by a suitable factor to clear denominators *before* substituting in the power series for y, y' and y''.

Example . . . .

▷ *Find series solutions about $x = 0$ of Legendre's equation*

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0$$

$x = 0$ is an ordinary point, so let

Substitute these expressions into the ODE to obtain

Rewriting

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

we can see that the coefficient of x^n (for $n \geqslant 0$) is

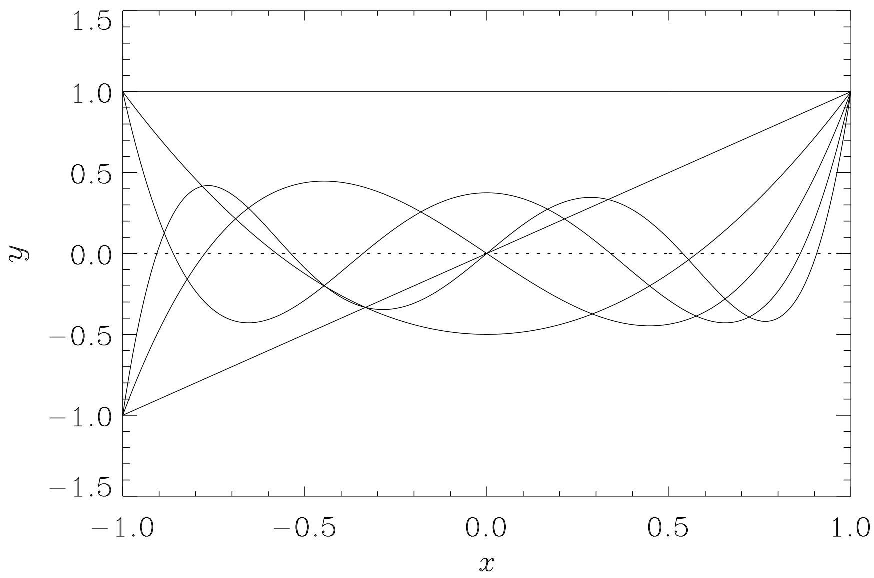
The recurrence relation is therefore

a_0 and a_1 are the arbitrary constants. The other coefficients follow from the recurrence relation. . . . .

Notes:

- an even solution is obtained by choosing $a_0 = 1$ and $a_1 = 0$
- an odd solution is obtained by choosing $a_0 = 0$ and $a_1 = 1$
- these solutions are obviously linearly independent since one is not a constant multiple of the other
- since $|a_{n+2}/a_n| \to 1$ as $n \to \infty$, the even and odd series converge for $|x^2| < 1$, i.e. for $|x| < 1$
- the radius of convergence is the distance to the singular points of Legendre's equation at $x = \pm 1$
- if $\ell \geqslant 0$ is an even integer, then $a_{\ell+2}$ and all subsequent even coefficients vanish, so the even solution is a *polynomial* (terminating power series) of degree ℓ
- if $\ell \geqslant 1$ is an odd integer, then $a_{\ell+2}$ and all subsequent odd coefficients vanish, so the odd solution is a polynomial of degree ℓ

The polynomial solutions are called *Legendre polynomials*, $P_\ell(x)$. They are conventionally normalized (i.e. a_0 or a_1 is chosen) such that $P_\ell(1) = 1$, e.g.


1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−1.0
−0.5
0.0
0.5
1.0
x

7.4.3 Series solutions about a regular singular point

If $x = x_0$ is a regular singular point, *Fuchs's theorem* guarantees that the ODE has at least one solution of the form

i.e. a Taylor series multiplied by a power $(x - x_0)^\sigma$, where the index σ is a (generally complex) number to be determined.

Notes:

- this is a Taylor series only if σ is a non-negative integer
- there may be one or two solutions of this form (see below)
- the condition $a_0 \neq 0$ is required to define σ uniquely

To understand in simple terms why the solutions behave in this way, recall that

are analytic at the regular singular point $x = x_0$. Near $x = x_0$ the ODE can be approximated using the leading approximations to p and q:

$$y'' + \frac{P_0 y'}{x - x_0} + \frac{Q_0 y}{(x - x_0)^2} \approx 0$$

The exact solutions of this approximate equation are of the form $y = (x - x_0)^\sigma$, where σ satisfies the *indicial equation*

with two (generally complex) roots.

[If the roots are equal, the solutions are $(x-x_0)^\sigma$ and $(x-x_0)^\sigma \ln(x-x_0)$.] It is reasonable that the solutions of the full ODE should resemble the solutions of the approximate ODE near the singular point.

Frobenius's method is used to find the series solutions about a regular singular point. This is best demonstrated by example.

Example . . . .

▷ *Find series solutions about $x = 0$ of Bessel's equation*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

where ν is a constant. $x = 0$ is a regular singular point because $p = 1/x$ and $q = 1 - \nu^2/x^2$ are singular there but $xp = 1$ and $x^2q = x^2 - \nu^2$ are both analytic there.

Seek a solution of the form

Then

Bessel's equation requires

Now compare powers of $x^{n+\sigma}$:

$$n = 0: \qquad \left[\sigma^2 - \nu^2\right] a_0 = 0 \tag{1}$$

$$n = 1: \qquad \left[(1+\sigma)^2 - \nu^2\right] a_1 = 0 \tag{2}$$

$$n \geqslant 2: \qquad \left[(n+\sigma)^2 - \nu^2\right] a_n + a_{n-2} = 0 \tag{3}$$

Since $a_0 \neq 0$ by assumption, equation (1) provides the *indicial equation*

with roots $\sigma = \pm\nu$. Equation (2) then requires that $a_1 = 0$ (except possibly in the case $\nu = \pm\frac{1}{2}$, but even then a_1 can be *chosen* to be 0). Equation (3) provides the *recurrence relation*

Since $a_1 = 0$, all the odd coefficients vanish. Since $|a_n/a_{n-2}| \to 0$ as $n \to \infty$, the radius of convergence of the series is infinite.

For most values of ν we therefore obtain two linearly independent solutions (choosing $a_0 = 1$):

$$y_1 = x^{\nu}\left[1 - \frac{x^2}{4(1+\nu)} + \frac{x^4}{32(1+\nu)(2+\nu)} + \cdots\right]$$

$$y_2 = x^{-\nu}\left[1 - \frac{x^2}{4(1-\nu)} + \frac{x^4}{32(1-\nu)(2-\nu)} + \cdots\right]$$

However, if $\nu = 0$ there is clearly only one solution of this form. Furthermore, if ν is a non-zero integer one of the recurrence relations will fail at some point and the corresponding series is invalid. In these cases the second solution is of a different form (see below). . . . .

A general analysis shows that:

- *if the roots of the indicial equation are equal*, there is only one solution of the form $\sum a_n(x-x_0)^{n+\sigma}$
- *if the roots differ by an integer*, there is generally only one solution of this form because the recurrence relation for the smaller value of $\mathrm{Re}(\sigma)$ will usually (but not always) fail
- otherwise, there are two solutions of the form $\sum a_n(x-x_0)^{n+\sigma}$

- the radius of convergence of the series is the distance from the point of expansion to the nearest singular point of the ODE

If the roots σ_1, σ_2 of the indicial equation are equal or differ by an integer, one solution is of the form

and the other is of the form

The coefficients b_n and c can be determined (with some difficulty) by substituting this form into the ODE and comparing coefficients of $(x-x_0)^n$ and $(x-x_0)^n \ln(x-x_0)$. In exceptional cases c may vanish.

Alternatively, y_2 can be found (also with some difficulty) using the Wronskian method (section 7.3.4).

Example: Bessel's equation of order $\nu = 0$:

$$y_1 = 1 - \frac{x^2}{4} + \frac{x^4}{64} + \cdots$$

$$y_2 = y_1 \ln x + \frac{x^2}{4} - \frac{3x^4}{128} + \cdots$$

Example: Bessel's equation of order $\nu = 1$:

$$y_1 = x - \frac{x^3}{8} + \frac{x^5}{192} + \cdots$$

$$y_2 = y_1 \ln x - \frac{2}{x} + \frac{3x^3}{32} + \cdots$$

These examples illustrate a feature that is commonly encountered in scientific applications: one solution is regular (i.e. analytic) and the other is singular. Usually only the regular solution is an acceptable solution of the scientific problem.

7.4.4 Irregular singular points

If either $(x-x_0)p(x)$ or $(x-x_0)^2q(x)$ is not analytic at the point $x=x_0$, it is an *irregular* singular point of the ODE. The solutions can have worse kinds of singular behaviour there.

Example: the equation $x^4y''+2x^2y'-y=0$ has an irregular singular point at $x=0$. Its solutions are $\exp(\pm x^{-1})$, both of which have an essential singularity at $x=0$.

In fact this example is just the familiar equation $\mathrm{d}^2y/\mathrm{d}z^2=y$ with the substitution $x=1/z$. Even this simple ODE has an irregular singular point at $z=$