Dynamics of Astrophysical Discs — Mathematical Tripos, Part III
Dr Henrik Latter — Lent Term 2024

Example Sheet 3

1. **Particles in vortices**

Consider a stable steady Kida vortex in the shearing sheet model of a protoplanetary disk. As shown earlier, the flow in the core of the vortex can be described by

$$\mathbf{u} = \frac{S}{(r-1)}\left(\frac{y}{r}\,\mathbf{e}_x - rx\,\mathbf{e}_y\right),$$

where r is the vortex's aspect ratio.

(i) Suppose small test particles are caught up in the vortex. The motion of a test particle obeys

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \epsilon\Omega(\dot{x} - u_x),$$
$$\ddot{y} + 2\Omega\dot{x} = -\epsilon\Omega(\dot{y} - u_y),$$

where $[x(t),\, y(t)]$ is the position of the test particle, u_x and u_y are the x and y components of the gas's velocity, and ϵ is the Stokes number, which quantifies the strength of the gas drag on the particle.

Solutions to these equations are of the form $\propto \mathrm{e}^{\lambda t}$. Show that

$$\lambda^4 + 2\epsilon\Omega\lambda^3 + (\kappa^2 + \epsilon^2\Omega^2)\lambda^2 - 2\Omega^2\zeta_0\epsilon\lambda + \frac{S^2\epsilon^2\Omega^2}{(r-1)^2} = 0. \qquad (1)$$

(ii) Consider the limit $0 < \epsilon \ll 1$, which corresponds to relatively large particles. By appropriately expanding λ in small ϵ, determine the four trajectories described by Equation (1) and show that they all end up in the centre of the vortex if

$$-\frac{\kappa^2}{\Omega} < \zeta_0 < 0.$$

Rewrite this criterion in terms of the vortex ratio r.

2. *Structure formation in Saturn's rings: the 'viscous overstability'*

A dense portion of Saturn's B-ring can be represented by a two-dimensional, compressible, non-self-gravitating, viscous shearing sheet. It has surface density $\Sigma(x, y, t)$ and two-dimensional velocity $\boldsymbol{u}(x, y, t)$, governed by the equation of mass conservation,

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla}\cdot(\Sigma\boldsymbol{u}) = 0,$$

and the equation of motion,

$$\Sigma\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}+2\boldsymbol{\Omega}\times\boldsymbol{u}\right)=-\Sigma\boldsymbol{\nabla}\Phi-\boldsymbol{\nabla}P+\boldsymbol{\nabla}\cdot\mathbf{T}.$$

Here $\boldsymbol{\Omega}=\Omega\,\boldsymbol{e}_z$ is the angular velocity of the rotating frame of reference, $\Phi=-\Omega S x^2$ is the effective potential, P is the vertically integrated pressure and

$$\mathbf{T}=2\bar{\nu}\Sigma\,\mathbf{S}+\bar{\nu}_{\mathrm{b}}\Sigma(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,\mathbf{I}$$

is the vertically integrated viscous stress tensor. Also

$$\mathbf{S}=\frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\right]-\frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,\mathbf{I}$$

is the traceless shear tensor, $\mathbf{I}$ is the unit tensor, $\bar{\nu}$ is the mean kinematic shear viscosity and $\bar{\nu}_{\mathrm{b}}$ is the mean kinematic bulk viscosity. Assume that P, $\bar{\nu}$, $\bar{\nu}_{\mathrm{b}}$ are known functions of Σ, with $\mathrm{d}P/\mathrm{d}\Sigma=v_{\mathrm{s}}^2$ being the square of the sound speed.

The basic state of the sheet corresponds to the homogeneous solution in which $\boldsymbol{u}=-Sx\,\boldsymbol{e}_y$, while Σ, P and $\mathbf{T}$ are uniform, with $T_{xy}=-\bar{\nu}\Sigma S$.

(i) Formulate the linearized equations for perturbations Σ', $\boldsymbol{v}$, etc., on this background. Assume that the perturbations are axisymmetric (i.e. independent of y) and depend on x and t through the factor $\exp(st+\mathrm{i}k_x x)$, where s is a complex growth rate. Hence obtain the equations

$$s\Sigma'=-\Sigma\,\mathrm{i}k_x v_x,$$

$$sv_x=2\Omega v_y-\mathrm{i}k_x v_{\mathrm{s}}^2\frac{\Sigma'}{\Sigma}-(\bar{\nu}_{\mathrm{b}}+\tfrac{4}{3}\bar{\nu})k_x^2 v_x,$$

$$sv_y=-(2\Omega-S)v_x-\bar{\nu}k_x^2 v_y-\mathrm{i}k_x S\frac{\mathrm{d}(\bar{\nu}\Sigma)}{\mathrm{d}\Sigma}\frac{\Sigma'}{\Sigma},$$

where unprimed quantities represent their values in the basic state. (Note that the crucial final term comes from the fact the shear stress T_{xy} is affected by the density perturbation in the wave as well as the perturbed velocity gradient.)

(ii) Deduce that the dispersion relation is

$$s^3+(\bar{\nu}_{\mathrm{b}}+\tfrac{7}{3}\bar{\nu})k_x^2 s^2+[\kappa^2+v_{\mathrm{s}}^2 k_x^2+\bar{\nu}(\bar{\nu}_{\mathrm{b}}+\tfrac{4}{3}\bar{\nu})k_x^4]s+2\Omega S\frac{\mathrm{d}(\bar{\nu}\Sigma)}{\mathrm{d}\Sigma}k_x^2+v_{\mathrm{s}}^2\bar{\nu}k_x^4=0,$$

where $\kappa^2=2\Omega(2\Omega-S)$ is the square of the epicyclic frequency (assumed positive). Verify that this reduces to the expected result in the case of an inviscid disc.

(iii) Consider the limit of long wavelengths, $k_x\to 0$. Show that one root of the dispersion relation behaves as

$$s=-\frac{2\Omega S}{\kappa^2}\frac{\mathrm{d}(\bar{\nu}\Sigma)}{\mathrm{d}\Sigma}k_x^2+O(k_x^4)$$

in this limit. This root yields exponential growth (*viscous instability*) when

$$\frac{\mathrm{d}(\bar{\nu}\Sigma)}{\mathrm{d}\Sigma}<0.$$

(iv) Show that the other two roots behave in this limit as

$$s = \pm \mathrm{i}\kappa \left(1 + \frac{v_\mathrm{s}^2 k_x^2}{2\kappa^2}\right) + \frac{1}{2}\left[-(\bar{\nu}_\mathrm{b} + \tfrac{7}{3}\bar{\nu}) + \frac{2\Omega S}{\kappa^2}\frac{\mathrm{d}(\bar{\nu}\Sigma)}{\mathrm{d}\Sigma}\right] k_x^2 + O(k_x^4).$$

These roots yield exponentially growing oscillations (*viscous overstability*) when

$$\frac{\mathrm{d}(\bar{\nu}\Sigma)}{\mathrm{d}\Sigma} > (\bar{\nu}_\mathrm{b} + \tfrac{7}{3}\bar{\nu})\frac{\kappa^2}{2\Omega S}.$$

Show that this condition is satisfied even when $\bar{\nu}$ is independent of Σ, $\bar{\nu}_\mathrm{b} = 0$ and the disc is Keplerian.

3. *Horseshoe orbits*

Consider a disc of non-interacting particles in the local approximation:

$$\ddot{x} - 2\Omega\dot{y} = 3\Omega^2 x - \frac{\partial \Psi}{\partial x},$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial \Psi}{\partial y},$$

with $\Psi = -GM_\mathrm{s}\left(x^2 + y^2\right)^{-1/2}$ the gravitational potential of an embedded satellite.

(i) Demonstrate that $\epsilon = (\dot{x}^2 + \dot{y}^2)/2 - \frac{3}{2}\Omega^2 x^2 + \Psi$ is an integral of the motion.

(ii) Show that for $y = 0$ there are three equilibrium points where accelerations and velocities are zero, and calculate their locations.

(iii) Particle streamlines can be classified into three types: (a) particles with large $|x|$ are only marginally perturbed and stream past the satellite; (b) particles close to the satellite form bound orbits that encircle it; (c) particles that start with small x but large y undergo horseshoe turns once they come near the satellite. Separating these three classes are special orbits (sepatrices) that join two of the equilibrium points and also extend from them to $\pm\infty$. Sketch the equilibrium points, the separatrices, and some of the streamlines.

(iv) By following the flow from an equilibrium point to infinity, calculate the width w of the horseshoe region.

(v) Consider a stream of particles in the horseshoe region at distance $x = x_0 > 0$ far away from the satellite ($y > 0$). Show that a stream of particles of width δx, surface density $\Sigma(x_0)$, induces a mass flow towards the satellite $\delta\dot{M} = \Sigma(x_0)v_{y0}(x_0)\delta x$. Argue that the resulting torque onto the satellite is $\delta\Gamma = \Omega x_0 \delta\dot{M}$. Show that the torque due to a stream originating at $x = -x_0$ is $\delta\Gamma = -\Omega x_0 \Sigma(-x_0)v_{y0}(x_0)\delta x$.

Now assume that the disk exhibits asymmetric background density structure described by $\Sigma = \Sigma_0(1 + kx)$. Integrate over all streams making a horseshoe turn for $x_0 > 0$ to get the total torque from one side. Then show that the total torque is given by $\Gamma = (3/4)k\Sigma_0\Omega^2 w^4$.

4. **Mechanical analogue of the MRI**

In the local approximation, the dynamics of two particles of mass m connected by a spring of spring constant $k = \beta m$ is described by the equations

$$\begin{aligned} \ddot{x}_1 - 2\Omega\dot{y}_1 - 2\Omega S x_1 &= \beta(x_2 - x_1) \\ \ddot{y}_1 + 2\Omega\dot{x}_1 &= \beta(y_2 - y_1) \\ \ddot{z}_1 + \Omega_z^2 z_1 &= \beta(z_2 - z_1), \end{aligned}$$

together with similar equations in which the suffixes 1 and 2 are interchanged.

(i) Give a physical interpretation of the equations, explaining the meaning of the symbols Ω, S and Ω_z.

(ii) Assume that the quantities β, Ω, S, $\kappa^2 = 2\Omega(2\Omega - S)$ and Ω_z^2 are all positive. Show that relative motions of the two particles in the (x, y) plane proportional to $\exp(\lambda t)$ are possible, where

$$\lambda^4 + (\kappa^2 + 4\beta)\lambda^2 + 4\beta(\beta - \Omega S) = 0.$$

(iii) Determine the range of β for which instability occurs. For fixed Ω and S, find the maximum growth rate of the instability and the value of β for which this occurs. Write down the explicit form of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ for this optimal solution.

(iv) Discuss the relation of this problem to the MRI in astrophysical discs. In the magnetohydrodynamic case, what quantity would correspond to β in the above analysis?

Please send any comments and corrections to `hl278@cam.ac.uk`