Kirchoff vortices

Consider an elliptical 'patch' of vorticity in a two-dimensional inviscid incompressible fluid. For this calculation we assume that there is no imposed shear flow, as in the shearing sheet: far enough away from the vortex the fluid is still. Within the patch the vorticity is a constant $\zeta = \zeta_0$, while outside it is zero. The vortex has semimajor axis a, semiminor axis b and the flow outside is irrotational with velocity tending to zero as $|\boldsymbol{x}| \to \infty$. At a given instant, we select Cartesian coordinates aligned with the semi-major and minor axes.

Our task is to compute the streamfunction of the overall flow (ψ) from Poisson's equation (see lectures)

$$\nabla^2 \psi = -\zeta \tag{1}$$

In order to facilitate our calculations we introduce elliptical coordinates (ξ, η) such that

$$x = c \cosh \xi \cos \eta, \qquad y = c \sinh \xi \sin \eta, \qquad \xi > 0, \qquad 0 \leqslant \eta < 2\pi,$$

and choose $c > 0$ such that the boundary of the vortex patch at a particular instant corresponds to the curve $\xi = \xi_0 =$ constant. Then the semi-axes are $a = c \cosh \xi_0$ and $b = c \sinh \xi_0$ (and thus $c^2 = a^2 - b^2$). Details of elliptical coordinates can be looked up in standard textbooks, for example, Arfken's *Mathematical Methods for Physicists.* I now list the most important. The scale vectors are

$$\boldsymbol{h}_\xi = \frac{\partial \boldsymbol{x}}{\partial \xi} = c(\sinh \xi \cos \eta, \cosh \xi \sin \eta)$$

$$\boldsymbol{h}_\eta = \frac{\partial \boldsymbol{x}}{\partial \eta} = c(-\cosh \xi \sin \eta, \sinh \xi \cos \eta) = \boldsymbol{e}_z \times \boldsymbol{h}_\xi$$

The system is orthogonal because $\boldsymbol{h}_\xi \cdot \boldsymbol{h}_\eta = 0$. The scale factors are given by

$$\begin{aligned} h_\xi^2 &= |\boldsymbol{h}_\xi|^2 \\ &= c^2(\sinh^2 \xi \cos^2 \eta + \cosh^2 \xi \sin^2 \eta) \\ &= c^2[\sinh^2 \xi (1 - \sin^2 \eta) + (1 + \sinh^2 \xi) \sin^2 \eta] \\ &= c^2(\sinh^2 \xi + \sin^2 \eta) \end{aligned}$$

$$h_\eta^2 = |\boldsymbol{h}_\eta|^2 = |\boldsymbol{h}_\xi|^2 = h_\xi^2$$

Curves of constant ξ are confocal ellipses,

$$\left(\frac{x}{c \cosh \xi}\right)^2 + \left(\frac{y}{c \sinh \xi}\right)^2 = 1$$

and curves of constant η are confocal hyperbolae,

$$\left(\frac{x}{c \cos \eta}\right)^2 - \left(\frac{y}{c \sin \eta}\right)^2 = 1$$

Note importantly that the coordinates are singular where $h_\xi = h_\eta = 0$, i.e. where $\xi = 0$ and $\eta = 0$ or π, i.e. at $\boldsymbol{x} = (\pm c, 0)$, the two foci.

In elliptical coordinates the Laplacian can be obtained from

$$\begin{aligned}\nabla^2\psi &= \frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta}{h_\xi}\frac{\partial\psi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_\xi}{h_\eta}\frac{\partial\psi}{\partial\eta}\right)\right]\\ &= \frac{1}{h_\xi^2}\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}\right)\end{aligned}$$

and so the Poisson equation for our problem may be written as

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2} = \begin{cases}-\zeta_0 c^2(\sinh^2\xi+\sin^2\eta), & 0<\xi<\xi_0\\ 0, & \xi>\xi_0\end{cases}$$

In fact, it will be convenient to rewrite the forcing on the right side as $-\frac{1}{2}\zeta_0 c^2(\cosh 2\xi - \cos 2\eta)$ (for $0<\xi<\xi_0$).

The boundary conditions require ψ to be periodic in η. We also need the velocity to go to zero as ξ goes to infinity, and for the velocity to be finite at the coordinate singularities. The velocity $\mathbf{u}$ associated with ψ may be computed from

$$\mathbf{u} = \nabla\times(\mathbf{e}_z\psi) = \frac{1}{h_\xi}(\mathbf{e}_\xi\partial_\eta\psi - \mathbf{e}_\eta\partial_\xi\psi).$$

We solve Poisson's equation by finding the particular integral and the complementary function separately. Just by looking at the forcing, an inspired guess gets us the particular integral straight away:

$$\psi_\mathrm{p} = \begin{cases}-\frac{1}{8}\zeta_0 c^2(\cosh 2\xi+\cos 2\eta), & 0<\xi<\xi_0\\ 0, & \xi>\xi_0\end{cases},$$

The homogeneous problem, on the other hand, can be solved readily enough by separation of variables, imposing the condition that the solution must be periodic in η. Solutions of the homogeneous equation are $\cos(n\eta)$ or $\sin(n\eta)$ multiplied by $\cosh(n\xi)$ or $\sinh(n\xi)$ [or $\exp(\pm n\xi)$] if n is a positive integer, or 1 or η multiplied by 1 or ξ in the case $n=0$.

We now can start thinking about imposing the other boundary conditions. If one goes through all the boring details one finds that only $n=0$ and $n=2$ are present in the solution and only even functions of η are required. For $\xi>\xi_0$ we must select the decaying solution $\exp(-2\xi)$ in the case $n=2$. (The scale factors $h\propto\exp\xi$ for large ξ, so $\psi\propto\exp(2\xi)$ would give velocity $\propto\exp\xi$.) At the singular points $\xi=0$, $\eta=0$ or π we require the velocity to be finite, so $\partial\psi/\partial\xi$ and $\partial\psi/\partial\eta$ should vanish. This means the solutions ξ and $\sinh 2\xi\cos 2\eta$ can be rejected for $\xi<\xi_0$.

Putting all that together, the solution must have the form

$$\psi = \begin{cases}-\frac{1}{8}\zeta_0 c^2(\cosh 2\xi+\cos 2\eta)+C_1+C_2\cosh 2\xi\cos 2\eta, & 0<\xi<\xi_0\\ C_3+C_4\xi+C_5\exp(-2\xi)\cos 2\eta, & \xi>\xi_0\end{cases}$$

where the C_i are constants. We determine four of these constants by requiring continuity of the velocity at the vortex boundary, i.e. ψ and $\partial\psi/\partial\xi$ must be continuous at $\xi = \xi_0$. (An overall additive constant remains undetermined.) The boundary conditions are

$$-\tfrac{1}{8}\zeta_0 c^2 \cosh 2\xi_0 + C_1 = C_3 + C_4\xi_0$$

$$-\tfrac{1}{8}\zeta_0 c^2 + C_2 \cosh 2\xi_0 = C_5 \exp(-2\xi_0)$$

$$-\tfrac{1}{4}\zeta_0 c^2 \sinh 2\xi_0 = C_4$$

$$2C_2 \sinh 2\xi_0 = -2C_5 \exp(-2\xi_0)$$

Thus

$$C_4 = -\tfrac{1}{4}\zeta_0 c^2 \sinh 2\xi_0$$

$$C_1 - C_3 = \tfrac{1}{8}\zeta_0 c^2(\cosh 2\xi_0 + 2\xi_0 \sinh 2\xi_0)$$

$$C_2 = \tfrac{1}{8}\zeta_0 c^2 \exp(-2\xi_0)$$

$$C_5 = -\tfrac{1}{8}\zeta_0 c^2 \sinh 2\xi_0$$

From this complicated expression we can calculate the actual flow and the associated pressure distribution. It is a bit nasty though. There is a simpler approximate form of the solution in the vortex core (near the origin), which we will use in example sheet 3.

The vortex will rotate if there is some normal velocity on its boundary. Let us determine the rotation rate associated with this velocity. To find the normal velocity on the boundary, evaluate ψ at $\xi = \xi_0$:

$$\begin{aligned}\psi_0(\eta) &= C_3 + C_4\xi_0 + C_5 \exp(-2\xi_0)\cos 2\eta \\ &= \text{constant} - \tfrac{1}{8}\zeta_0 c^2 \sinh 2\xi_0 \exp(-2\xi_0)\cos 2\eta\end{aligned}$$

Next compare this with the variation around the boundary of the streamfunction of a uniformly rotating flow, $\psi_\omega = -\frac{1}{2}\omega(x^2 + y^2)$, corresponding to $\boldsymbol{u} = (-\omega y, \omega x)$:

$$\begin{aligned}\psi_\omega(\eta) &= -\tfrac{1}{2}\omega c^2(\cosh^2\xi_0 \cos^2\eta + \sinh^2\xi_0 \sin^2\eta) \\ &= \text{constant} - \tfrac{1}{4}\omega c^2(\cosh^2\xi_0 - \sinh^2\xi_0)\cos 2\eta\end{aligned}$$

The two streamfunctions ψ_0 and ψ_ω are of the same form and we can identify

$$\omega = \tfrac{1}{2}\zeta_0 \sinh 2\xi_0 \exp(-2\xi_0)$$

Since $a + b = c\cosh\xi_0 + c\sinh\xi_0 = c\exp\xi_0$, we get a nice expression for the elliptical vortex's rotation rate:

$$\omega = \frac{ab\,\zeta_0}{(a+b)^2}$$