## Example Sheet 0

*This is a revision sheet. If you did NST Mathematics A or B last year you should be able to do the questions already (let me know if I have made an incorrect assumption here, especially if you did Course A). If you did the Mathematical Tripos last year you will have to read up on Fourier Series for question 4. Some of the material will be touched on in the first couple of lectures, so you might prefer to wait until then.*

1. Let $f(x)$ be a function of one variable. Working to $O((\delta x)^3)$, write down the Taylor expansion of $f(x+\delta x)$. Let $g(x,y,z)$ be a function of three variables. Working to $O(\delta x, \delta y, \delta z)$, write down the Taylor expansion of $g(x+\delta x, y+\delta y, z+\delta z)$. Express the latter using vector notation.

2. (a) Let $h$ be a function of one variable. Working from first principles, differentiate with respect to $x$ the function

$$I(x) = \int_a^x h(y)\,\mathrm{d}y\,,$$

where $a$ is a constant.

(b) Let $f(x,y)$ be a function of two variables. Working from first principles, differentiate with respect to $x$ the function

$$J(x) = \int_a^x f(x,y)\,\mathrm{d}y\,.$$

3. Find functions $y(x)$ which satisfy

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; 3x\frac{\mathrm{d}y}{\mathrm{d}x} \;-\; 15y \;=\; 0\,.$$

4. (a) The function $h(x) = x(\pi - x)$ is defined on the interval $0 \le x < \pi$. Sketch the *odd* $2\pi$-periodic continuation of $h(x)$ over the whole real line. Compute the Fourier (sine) series for $h$ and hence demonstrate that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}\,.$$

(b) Let $g(x)$ be an even periodic function of $x$ with period $4L$, i.e. $g(-x) = g(x)$ and $g(x+4L) = g(x)$. Given that $g(x) = (x-L)/L$ for $0 \leqslant x \leqslant 2L$, sketch $g(x)$ for $-4L \leqslant x \leqslant 4L$ and find the Fourier series for $g$.

5. Show the following:

   (a) For vector components $a_i$ $(i = 1, 2, 3)$,

   $$\sum_{j=1}^{3} a_j \delta_{ij} = a_i\,;$$

   where $\delta_{ij}$ is the Kronecker delta,

   (b) For independent variables $q_i$ $(i = 1, 2, 3)$,

   $$\frac{\partial q_i}{\partial q_j} = \delta_{ij}\,.$$

6. Show that, if $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$ $(i, j = 1, 2, 3)$ and $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$, then

   $$\boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1 \quad \text{and} \quad \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2\,.$$

7. Using a coordinate system of your choice, write down an example function possessing each of the following kinds of symmetry:

   (a) A function of 2 variables which has circular symmetry

   (b) A function of 3 variables which is spherically symmetric

   (c) A function of 3 variables which has axial symmetry

8. Evaluate $\boldsymbol{\nabla} \cdot \boldsymbol{r}$ and $\boldsymbol{\nabla} \times \boldsymbol{r}$, where $\boldsymbol{r}$ is the position vector.

9. Show that, if $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, where $\boldsymbol{\omega}$ is a constant vector and $\boldsymbol{r}$ is the position vector, then $\boldsymbol{\nabla} \times \boldsymbol{v} = 2\boldsymbol{\omega}$. (Feel free to use suffix notation.) Comment.

10. Let $\psi(\boldsymbol{r})$ and $\chi(\boldsymbol{r})$ be scalar fields, and $\boldsymbol{u}(\boldsymbol{r})$ and $\boldsymbol{v}(\boldsymbol{r})$ vector fields. Show the following (feel free to use suffix notation):

    (a)

    $$\boldsymbol{\nabla} \times \psi \boldsymbol{v} \;=\; \boldsymbol{\nabla}\psi \times \boldsymbol{v} + \psi \boldsymbol{\nabla} \times \boldsymbol{v}\,,$$

    (b)

    $$\boldsymbol{\nabla} \cdot \boldsymbol{u} \times \boldsymbol{v} \;=\; \boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \times \boldsymbol{v}\,,$$

    and hence that

    $$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\chi) \;=\; 0\,.$$

*This example sheet is available at* `http://www.damtp.cam.ac.uk/user/hl278/NST`
*Hints and answers will be posted on the Cambridge University Moodle.*
*Supervisors may request these early.*

Comments/corrections to `hl278@cam.ac.uk`.