Mathematical Methods I
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**Example Sheet 1**

1. In the following, the indices $i$, $j$, $k$, $\ell, m$ take the values 1, 2, 3, and the summation convention applies. In particular let $\boldsymbol{n}$ be a unit vector, i.e. $n_i n_i = 1$.

   (a) Simplify the following expressions

   $$\delta_{ij}a_j\,, \quad \delta_{ij}\delta_{jk}\,, \quad \delta_{ij}\delta_{ji}\,, \quad \delta_{ij}n_i n_j\,, \quad \epsilon_{ijk}\delta_{jk}\,, \quad \epsilon_{ijk}\epsilon_{ij\ell}\,, \quad \epsilon_{ijk}\epsilon_{ikj}\,, \quad \epsilon_{ijk}(\boldsymbol{a}\times\boldsymbol{b})_k\,.$$

   (b) For each of the following equations, either give the equivalent in vector or matrix notation or explain why the equation is invalid.

   $$x_i = a_i b_k c_k + d_i\,, \quad x_i = a_j b_i + c_k d_i e_k f_j\,, \quad u = \epsilon_{jk\ell} v_k w_\ell x_j\,,$$

   $$\epsilon_{ijk} x_j y_k \epsilon_{i\ell m} x_\ell y_m = \mu\,, \qquad A_{ik}B_{k\ell} = T_{ik}\delta_{k\ell}\,, \qquad x_k = A_{ki}B_{ji}y_j\,.$$

   (c) Write the following equations in suffix notation using the summation convention.

   $$(\boldsymbol{x}+\mu\boldsymbol{y})\cdot(\boldsymbol{x}-\mu\boldsymbol{y}) = \kappa\,, \qquad \boldsymbol{x} = |\boldsymbol{a}|^2\boldsymbol{b} - |\boldsymbol{b}|^2\boldsymbol{a}\,, \qquad (2\boldsymbol{x}\times\boldsymbol{y})\cdot(\boldsymbol{a}+\boldsymbol{b}) = \lambda\,.$$

   (d) Given that $A_{ij} = \epsilon_{ijk}a_k$ (for all $i, j$), show that $2a_k = \epsilon_{kij}A_{ij}$ (for all $k$).

   (e) Show that $\epsilon_{ijk}s_{ij} = 0$ (for all $k$) if and only if $s_{ij} = s_{ji}$ (for all $i, j$).

   (f) Given that $N_{ij} = \delta_{ij} - \epsilon_{ijk}n_k + n_i n_j$ and $M_{ij} = \delta_{ij} + \epsilon_{ijk}n_k$, show that $N_{ij}M_{jk} = 2\delta_{ik}$.

2. A fluid flow has the constant velocity vector (in Cartesian coordinates)

   $$\boldsymbol{v}(\boldsymbol{r}) = (0, 0, W)\,.$$

   Explicitly calculate the volume flux of fluid,

   $$Q = \int \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S}\,,$$

   flowing across (a) the open hemispherical surface $r = a$, $z \geqslant 0$, and (b) the disc $r \leqslant a$, $z = 0$. Verify that the divergence theorem holds.

3. For a surface $S$ enclosing a volume $V$, apply the divergence theorem to a vector field $\boldsymbol{F} = \boldsymbol{a}p$, where $\boldsymbol{a}$ is an arbitrary constant vector and $p(\boldsymbol{r})$ is a scalar field. Deduce that

   $$\int_V \boldsymbol{\nabla} p\, \mathrm{d}V = \oint_S p\, \mathrm{d}\boldsymbol{S}\,.$$

4. A time-independent magnetic field $\boldsymbol{B}(\boldsymbol{r})$ is given by

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi}\frac{\boldsymbol{e}_z \times \boldsymbol{r}}{x^2+y^2}\,,$$

where $\mu_0$ is the magnetic permeability and $I$ is a constant. Using Cartesian coordinates, calculate the electric current density $\boldsymbol{J}$ given by the steady Maxwell equation $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J}$. Also evaluate $\oint_C \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{r}$, where $C$ is a circle of radius $a$ in the plane $z=0$ and centred on $x=y=0$. Discuss whether Stokes's theorem applies in this situation.

5. Show that, in Cartesian coordinates,

$$\nabla^2\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F})\,.$$

This vector identity remains true for all coordinate systems; however, for non-Cartesian coordinates,

$$(\nabla^2\boldsymbol{F})_i \neq \nabla^2 F_i\,.$$

Why is this the case? Illustrate this point by evaluating $\nabla^2\boldsymbol{F}$ for $\boldsymbol{F} = f(\rho)\,\boldsymbol{e}_\phi$ in cylindrical polar coordinates $(\rho,\phi,z)$ and comparing the result with $\nabla^2 f$.

6. Find the general circularly symmetric solution to the fourth-order equation

$$\nabla^4\Psi \equiv \nabla^2(\nabla^2\Psi) = 0\,.$$

*Hint: use plane polar coordinates $(\rho,\phi)$, and do not be too eager to expand everything out.*

Find those circularly symmetric solutions in the unit disc that are equal to unity at the centre $\rho=0$ and vanish on the boundary $\rho=1$. Give a further condition to render the solution unique.

7. Parabolic coordinates $(u,v,\phi)$ are defined in terms of Cartesian coordinates $(x,y,z)$ by

$$x = uv\cos\phi\,,\qquad y = uv\sin\phi\,,\qquad z = \tfrac{1}{2}(u^2-v^2)\,.$$

Show that the surfaces of constant $u$, and those of constant $v$, are surfaces obtained by rotating parabolae about the $z$-axis. What are the surfaces of constant $\phi$? Show that the coordinate surfaces intersect at right angles and hence that these coordinates are orthogonal. Find the scale factors $(h_u,h_v,h_\phi)$ defined by

$$|\mathrm{d}\boldsymbol{r}|^2 = h_u^2\,\mathrm{d}u^2 + h_v^2\,\mathrm{d}v^2 + h_\phi^2\,\mathrm{d}\phi^2\,.$$

Hence obtain the Laplacian in these coordinates using the formula

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right]\,.$$

8. Consider the two-stage transformation of Cartesian coordinates $(x, y, z)$ given by

$$x = ax', \qquad y = by', \qquad z = cz',$$

$$x' = r' \sin\theta' \cos\phi', \qquad y' = r' \sin\theta' \sin\phi', \qquad z' = r' \cos\theta',$$

where $a$, $b$ and $c$ are positive constants. Calculate the Jacobian matrices of the transformations $(x, y, z) \mapsto (x', y', z')$ and $(x', y', z') \mapsto (r', \theta', \phi')$ and verify explicitly that

$$\frac{\partial(x, y, z)}{\partial(r', \theta', \phi')} = \frac{\partial(x, y, z)}{\partial(x', y', z')} \frac{\partial(x', y', z')}{\partial(r', \theta', \phi')}.$$

Are the coordinates $(r', \theta', \phi')$ orthogonal? What range of these coordinates is required to cover the interior of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,?$$

Express the volume element in coordinates $(r', \theta', \phi')$ and hence calculate the volume of the ellipsoid.

9. In a Cartesian coordinate system $(x_1, x_2, x_3)$, $A$ is the point $(0, 0, -1)$, $B$ is the point $(0, 0, 1)$ and $P$ is an arbitrary point $(x_1, x_2, x_3)$. In a curvilinear coordinate system, the coordinates of $P$ are specified by

$$u_1 = \frac{1}{2}(r_1 + r_2) \qquad u_2 = \frac{1}{2}(r_1 - r_2) \qquad u_3 = \phi,$$

where $r_1$ and $r_2$ are the distances $AP$ and $BP$ respectively and $\phi$ is the angle between the planes $ABP$ and $x_2 = 0$. Show that $x_3 = u_1 u_2$ and that the distance $\rho$ from $P$ to the $x_3$-axis is equal to $\sqrt{(u_1^2 - 1)(1 - u_2^2)}$. Next evaluate $\partial x_i / \partial u_j$ (with $i, j$ = 1,2,3). Deduce that the curvilinear coordinates are orthogonal and sketch the coordinate surfaces. Show that the metric coefficients are

$$h_1 = \sqrt{\frac{u_1^2 - u_2^2}{u_1^2 - 1}}, \quad h_2 = \sqrt{\frac{u_1^2 - u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{(u_1^2 - 1)(1 - u_2^2)}\,.$$

Show that if the function $\Psi$ satisfies Laplace's equation and is independent of $u_2$ and $u_3$ then it has the form

$$\Psi = a + b \ln\left(\frac{u_1 - 1}{u_1 + 1}\right)$$

for constant $a$ and $b$.

10. A uniform stretched string of length $L$, mass per unit length $\rho$ and tension $T = \rho c^2$ is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force per unit length being $-2\mu\rho\dot{y}$, where $y(x, t)$ is the transverse displacement and $\dot{y} = \partial y / \partial t$. Generalize the argument given in lectures to show that the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} + 2\mu \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}\,.$$

Find $y(x,t)$ if $\mu = \pi c/L$, $y(x,0) = d\sin(\pi x/L)$ and $\dot{y}(x,0) = 0$.

If an extra transverse force $F\sin(\pi x/L)\cos(\pi ct/L)$ per unit length acts on the string, find the resulting forced oscillation.

11. Show that the solution of Laplace's equation, $\nabla^2\Phi = 0$, in the region $0 < x < a$, $0 < y < b$, $0 < z < c$, with $\Phi = 1$ on the surface $z = 0$ and $\Phi = 0$ on the other surfaces, is

$$\Phi = \frac{16}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin[(2m-1)\pi x/a]\sin[(2n-1)\pi y/b]\sinh[k(c-z)]}{(2m-1)(2n-1)\sinh(kc)}\,,$$

where $k^2 = (2m-1)^2\pi^2/a^2 + (2n-1)^2\pi^2/b^2$.

12. The temperature distribution $\theta(x,t)$ along a thin bar of length $L$ satisfies the one-dimensional diffusion equation

$$\frac{\partial\theta}{\partial t} = \nu\frac{\partial^2\theta}{\partial x^2}\,,$$

where the diffusivity $\nu$ is a constant, $t$ denotes time and $x$ is the distance from one of the ends. Find $\theta(x,t)$ if the bar is insulated at each end (i.e. if $\partial\theta/\partial x = 0$ at each end), and if the initial temperature distribution is given by

$$\theta(x,0) = 2\theta_0\cos^2\left(\frac{\pi x}{L}\right)\,,$$

where $\theta_0$ is a constant. For large times what is the temperature distribution in the bar? Comment.

*This example sheet is available at* `http://www.damtp.cam.ac.uk/user/hl278/NST`
*Hints and answers will be posted on the Cambridge University Moodle.*
*Supervisors may request these early.*

Comments/corrections to `hl278@cam.ac.uk`.