

Example Sheet 2

1. Define $\delta_\epsilon(x)$ for $\epsilon > 0$ by

$$\delta_\epsilon(x) = \begin{cases} (x + \epsilon)/\epsilon^2 & -\epsilon < x < 0 \\ (\epsilon - x)/\epsilon^2 & 0 \leq x < \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Evaluate

$$\int_{-\infty}^{\infty} \delta_\epsilon(x) dx$$

- (b) Argue that for a ‘good’ function f and a constant ξ

$$\lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \delta_\epsilon(x - \xi) f(x) dx = f(\xi).$$

Hint: Consider the substitution $x - \xi = \epsilon t$.

- (c) Sketch $\delta_\epsilon(x)$ and comment.

2. (a) Starting from the definition that $\delta(x)$ is the generalized function such that for all ‘good’ functions $f(x)$

$$\int_{-\infty}^{\infty} \delta(x - \xi) f(x) dx = f(\xi),$$

show that, for constant $a \neq 0$,

$$\delta(ax) = \frac{1}{|a|} \delta(x).$$

- (b) Evaluate

$$\int_{-\infty}^{\infty} |x| \delta(x^2 - a^2) dx,$$

where a is a non-zero constant. *Hint: the answer is not $2a$. Consider a substitution such as $t = x^2$ or one similar.*

3. Show that the equation

$$y'' + py' + qy = f(x),$$

where p and q are constants, can be written in the form

$$z' - az = f, \quad y' - by = z,$$

for suitable choices of the constants a and b . Solve these first-order equations using integrating factors, subject to the initial conditions $y(0) = y'(0) = 0$, to obtain the solution

$$y(x) = e^{bx} \int_0^x \int_0^\eta f(\xi) e^{-a\xi} e^{(a-b)\eta} d\xi d\eta.$$

By changing the order of integration and carrying out the integration with respect to η , show that

$$y(x) = \frac{1}{a-b} \int_0^x f(\xi) [e^{a(x-\xi)} - e^{b(x-\xi)}] d\xi$$

and interpret this result.

4. The differential equation

$$y'' + y = H(x) - H(x - \epsilon),$$

where H is the Heaviside step function and ϵ is a positive parameter, represents a simple harmonic oscillator subject to a constant force for a finite time. By solving the equation in the three intervals of x separately and applying appropriate matching conditions, show that the solution that vanishes for $x < 0$ is

$$y = \begin{cases} 0, & x < 0, \\ 1 - \cos x, & 0 < x < \epsilon, \\ \cos(x - \epsilon) - \cos x, & x > \epsilon. \end{cases}$$

Hence show that the solution of

$$y'' + y = \frac{H(x) - H(x - \epsilon)}{\epsilon}$$

that vanishes for $x < 0$ agrees, in the limit $\epsilon \rightarrow 0$, with the appropriate solution of $y'' + y = \delta(x)$, namely $y = H(x) \sin x$.

5. The function $G(x, \xi)$ is defined by

$$G(x, \xi) = \begin{cases} x(\xi - 1), & 0 \leq x \leq \xi, \\ \xi(x - 1), & \xi \leq x \leq 1. \end{cases}$$

If $f(x)$ is continuous for $0 \leq x \leq 1$, and

$$y(x) = \int_0^1 f(\xi) G(x, \xi) d\xi,$$

show by direct calculation that $y''(x) = f(x)$ and find $y(0)$ and $y(1)$.

Hint: use the definition of $G(x, \xi)$ to write $y(x)$ as the sum of two integrals, one with $\xi \leq x$ and the other with $x \leq \xi$.

6. Use the method of Green's function to solve

(a)

$$y'' - y = x^2 \quad \text{with} \quad y(0) = y(1) = 0,$$

(b)

$$y'' + \omega^2 y = x \quad \text{with} \quad y'(0) = y(\pi/\omega) = 0,$$

(c)

$$y'' + \alpha y' = e^{-\beta x} \quad \text{with} \quad x \geq 0 \text{ and } y(0) = y'(0) = 0$$

(d)

$$y'''' = f(x) \quad \text{with} \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

7. Let α and β be positive constants, and let $H(x)$ denote the Heaviside step function. Find the Fourier transforms of

(a) the odd function $f_o(x)$, where f_o is defined for $x > 0$ by

$$f_o(x) = \begin{cases} 1, & 0 < x \leq 1, \\ 0, & x > 1. \end{cases}$$

(b) the even function $f_e(x) = e^{-|x|}$.

(c) the even function $g(x)$, where

$$g(x) = \begin{cases} 1, & |x| < \alpha, \\ 0, & |x| \geq \alpha. \end{cases}$$

(d) the function

$$h(x) = H(x) \sinh(\alpha x) e^{-\beta x}, \quad \text{where} \quad \alpha < \beta.$$

8. (a) Use Parseval's theorem and the result of question 7a to show that

$$\int_{-\infty}^{\infty} \left(\frac{1 - \cos x}{x} \right)^2 dx = \pi.$$

(b) Use Parseval's theorem and the result of question 7b to evaluate the integral

$$\int_0^{\infty} \frac{dk}{(1 + k^2)^2}.$$

9. For $g(x)$ as given in question 7c define

$$G(x) = \int_{-\infty}^{\infty} g(x - \xi) g(\xi) d\xi.$$

Find an expression for $G(x)$. Explicitly demonstrate that the Fourier transforms of $G(x)$ and $g(x)$ satisfy the convolution theorem.

10. Show that, if a function f and its Fourier transform \tilde{f} are both real, then f is even. Show also that, if a function f is real and its Fourier transform \tilde{f} is purely imaginary, then f is odd.

11. By taking the Fourier transform of the equation

$$\frac{d^2\phi}{dx^2} - m^2\phi = f(x)$$

show that its solution $\phi(x)$ can be written as

$$\phi(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx} \tilde{f}(k)}{k^2 + m^2} dk$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$.

*This example sheet is available at <http://www.damtp.cam.ac.uk/user/h1278/NST>
Please send any comments and corrections to h1278@cam.ac.uk*