2 Complex numbers

Though they don't exist, *per se*, complex numbers pop up all over the place when we do science. They are an especially useful mathematical tool in problems involving:

- oscillations and waves in light, fluids, magnetic fields, electrical circuits, etc.,
- stability problems in fluid flow and structural engineering,
- signal processing (the Fourier transform, etc.),
- quantum physics, e.g. Schroedinger's equation,

$$i\frac{\partial\psi}{\partial t} + \nabla^2\psi + V(x)\psi = 0\,,$$

  which describes the wavefunctions of atomic and molecular systems,

- difficult differential equations.

For your revision: recall that the general quadratic equation, $ax^2 + bx + c = 0$ (solving for x), has two solutions given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,. \tag{85}$$

But what happens when the discriminant is negative?
For instance, consider

$$x^2 - 2x + 2 = 0\,. \tag{86}$$

Its solutions are

$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 4\times 1\times 2}}{2\times 1} = \frac{2 \pm \sqrt{-4}}{2} \\ &= 1 \pm \sqrt{-1}\,. \end{aligned}$$

These two solutions are evidently *not* 'normal' numbers!

If we put that aside, however, and accept the existence of the square root of minus one, written as

$$i = \sqrt{-1}\,, \tag{87}$$

then these two solutions, $1+i$ and $1-i$, can be assigned to the set of *complex numbers*. The quadratic equation now can be factorised as

$$z^2 - 2z + 2 = (z-1-i)(z-1+i) = 0\,,$$

and to be able to factorise every polynomial is very useful, as we shall see.

2.1 Complex algebra

2.1.1 Definitions

A complex number z takes the form

$$\boxed{z = x + iy\ ,} \tag{88}$$

where x and y are real numbers and i is the imaginary unit satisfying

$$i^2 = -1\,. \tag{89}$$

We call x and y the real and imaginary parts of z respectively, and write

$$\boxed{\begin{array}{lllll} x & = & \Re(z) & \text{or} \quad \text{Re}(z) & \text{the real part of } z\,, \\ y & = & \Im(z) & \text{or} \quad \text{Im}(z) & \text{the imaginary part of } z\,. \end{array}}$$

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, i.e. $z_1 = z_2$, then their real and imaginary parts must be equal, that is, both $x_1 = x_2$ and $y_1 = y_2$.

2.1.2 Addition

The sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is also a complex number given by

$$\boxed{z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)\,.} \tag{90}$$

The real part is $\Re(z_1 + z_2) = x_1 + x_2$ and the imaginary part is $\Im(z_1 + z_2) = y_1 + y_2$. The commutativity and associativity of real numbers under addition is therefore also passed on to the complex numbers, e.g. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

2.1.3 Multiplication

We can multiply complex numbers, provided we know how to multiply i by itself. The following table gives the results for the powers of i:

$$\boxed{\begin{array}{lllllll} i & = & i & & & & \\ i^2 & = & -1 & & & & \\ i^3 & = & i^2 \times i & = & -i & & \\ i^4 & = & i^3 \times i & = & -i \times i & = & 1 \\ i^5 & = & i^4 \times i & = & i & & \\ \vdots & & \vdots & & \vdots & & \end{array}} \tag{91}$$

Note the pattern has a fourfold periodicity with $i^{4n+m} = i^m$, where n is any integer.

Okay, now consider the product of the two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2 \\ &= x_1 x_2 + i(x_1 y_2 + y_1 x_2) - y_1 y_2 \end{aligned}$$

where we have used (89). So, collecting real and imaginary parts, we have

$$\boxed{z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)\,.} \tag{92}$$

Complex multiplication inherits commutativity and associativity from the real numbers,

$$z_1 z_2 = z_2 z_1\,, \qquad z_1(z_2 z_3) = (z_1 z_2) z_3\,, \tag{93}$$

and it is also distributive

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3\,. \tag{94}$$

Example 2.1 If $z_1 = 3 + i$ and $z_2 = 1 - i$ calculate $z_1 + z_2$, $z_1 - z_2$ and $z_1 z_2$.

2.1.4 Complex conjugate and modulus

The complex conjugate of $z = x + iy$ is found by changing the sign of its imaginary component. It is denoted by z^* (or often $\overline{z}$) and defined by

$$\boxed{z^* = x - iy\,.} \tag{95}$$

Note that it follows that we must have $z + z^* = 2\Re(z)$ and $z - z^* = 2i\Im(z)$.

If we take the product of z with its complex conjugate z^* we find

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 + i(-xy + xy) = x^2 + y^2 \ ,$$

which is real and non-negative. Thus

$$\boxed{zz^* = x^2 + y^2 \,.} \tag{96}$$

The *modulus* of z, denoted by $|z|$ or mod(z), is defined by

$$\boxed{|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}\,.} \tag{97}$$

2.1.5 Division

It is easiest to compute the division of one complex number by another by using the properties of the conjugate and modulus.

The division of z_1 by z_2 may be manipulated into

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} \,, \tag{98}$$

and the denominator is conveniently a real number. Then we use the rule for multiplication of complex numbers, in the numerator, to compute the result.

The general rule for simplifying expressions involving division by a complex number z_2 is to multiply numerator and denominator by the complex conjugate z_2^*/z_2^* (i.e. the identity), thus making the denominator real $z_2 z_2^* = |z_2|^2$.

Example 2.2 Take $z_1 = 3 + i$ and $z_2 = 1 - i$ again and calculate z_1/z_2.

2.2 The Complex Plane

2.2.1 Argand diagram

The real (x) and imaginary (y) parts of a complex number z are completely independent quantities, so we can think of z as plotting a point in a two-dimensional space, (x, y), where the y axis corresponds to the imaginary part and the x axis corresponds to the real part of the number.

In fact, we can go further and think of a complex number as a two-dimensional vector.

This two-dimensional space is often called the Argand diagram.

Now purely algebraic processes (conjugation, addition, division, etc.) can be represented as geometric operations. For instance

- The complex conjugate z^* is found by merely reflecting any point z about the real (i.e. x) axis.

- Addition and subtraction of complex numbers is essentially equivalent to vector addition and subtraction.

2.2.2 Polar form of complex numbers

We can also use plane polar coordinates in the Argand plane. Simple trigonometry gives us

$$\boxed{r = \sqrt{x^2 + y^2} = |z| \,,} \tag{99}$$

The radius r is, in fact, the modulus of z, met earlier.

The polar angle θ is called the *argument* or *phase* of z. It is the angle subtended by the real axis and the line made by the complex number and the origin. The following formula can help us get the argument

$$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right) \,.} \tag{100}$$

(as can $\cos\theta = x/r$ or $\sin\theta = y/r$.) However, care must be taken because we need to first determine in which quadrant the complex number lies (the above formula does not distinguish between the first and third quadrant!). For complex numbers it is conventional to restrict the argument to the range $-\pi < \theta \leqslant \pi$ in order to make it unique; this is called the *principal range*, and then θ is the *principal argument*.

Th polar representation of z is:

$$\boxed{z = x + iy = r(\cos\theta + i\sin\theta) \,.} \tag{101}$$

For the complex conjugate, it is clear that the argument will become $-\theta$, i.e.,

$$z^* = x - iy = r(\cos\theta - i\sin\theta) = r\left[\cos(-\theta) + i\sin(-\theta)\right] \,. \tag{102}$$

The inverse, on the other hand, is

$$z^{-1} = \frac{z^*}{|z|^2} = r^{-1}\left[\cos(-\theta) + i\sin(-\theta)\right] \tag{103}$$

<u>Example 2.3</u> Calculate the modulus and argument for $z_1 = 3 + i$ and $z_2 = 1 - i$.

Example 2.4 What shape in the Argand diagram is described by the equation $3|z| = |z - i|$? What about $|z| = |z - i|$?

2.3 The complex exponential

2.3.1 Euler's formula

There is a profound relationship between trigonometric functions and the complex exponential function:

$$\cos\theta + i\sin\theta = e^{i\theta}\,. \tag{104}$$

We will prove this later. This is called Euler's formula.

Euler's formula means that we can write any complex number compactly as

$$\boxed{z = r\,e^{i\theta}\,.} \tag{105}$$

As with the standard polar representation, there is some degeneracy here because we can add any integer multiple of 2π onto θ without changing the value of z, since $\exp(2i\pi n) = 1$ for integer n.

2.3.2 Multiplication

The exponential form makes it *easy* to do multiplication and division. If $z_1 = r_1\exp(i\theta_1)$ and $z_2 = r_2\exp(i\theta_2)$ then

$$\boxed{z_1 z_2 = r_1 r_2 \exp\left(i(\theta_1 + \theta_2)\right)\,.} \tag{106}$$

So when multiplying complex numbers the moduli are *multiplied* together and the arguments are *added*.

2.3.3 Division

We looked at division previously (98) and this now becomes

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\exp(i[\theta_1 - \theta_2])\,. \tag{107}$$

So when dividing complex numbers the moduli are divided and the arguments are subtracted.

2.3.4 Geometric manifestations

A geometrical interpretation of multiplication of z_1 by z_2 corresponds to rotation of z_1 by the argument of z_2 and a scaling of z_1's modulus by $|z_2|$. Note the special case of multiplication by i corresponds simply to rotation by 90° anti-clockwise. Taking the complex conjugate of z corresponds to reflection in the x axis.

2.3.5 New expressions for cos and sin

Taking the complex conjugate of (104) yields

$$\exp(-i\theta) = \cos\theta - i\sin\theta \; , \tag{108}$$

and adding (104) to (108) gets us

$$\boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \; .} \tag{109}$$

Similarly, subtracting (108) from (104) gives an expression for sin

$$\boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \; .} \tag{110}$$

2.3.6 Fundamental Theorem of Algebra

We state the fundamental theorem of algebra without proof. The polynomial equation of degree n (a positive integer)

$$a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0 \,, \qquad a_n \neq 0 \,. \tag{111}$$

has n complex roots for any possible (complex) coefficients $a_0, a_1, ...a_n$.

2.3.7 Roots of unity

We want to solve the equation

$$z^n = 1 \;, \tag{112}$$

where n is an integer. One root is $z = 1$, of course, but (112) is a polynomial equation of degree n, so we expect n roots!

The way to grab these is to use the complex exponential notation and allow for degeneracy in the complex argument.

- We recognise that 1 is a complex number with modulus 1 and argument $0 + 2\pi m$, for m an integer:

$$z^n = 1 = \exp(2\pi i m) \qquad m = 0, 1, 2, ... \,. \tag{113}$$

- then take the nth root of both sides

$$z = \exp(2\pi i m/n) \qquad m = 0, 1, 2, ... \quad . \tag{114}$$

However, the root with $m = n$ is $z = \exp(2\pi i) = 1$, which is the same as the root with $m = 0$, so the n *distinct* roots of equation (112) are

$$\boxed{z = \exp(2\pi i m/n) \qquad m = 0, 1, 2, ...n - 1 \,.} \tag{115}$$

Alternatively, we can write $\omega = e^{2\pi i/n}$, and the n roots are 1, ω, ω^2, ..., ω^{n-1}. These roots are distributed around the unit circle (with radius 1 centred on the origin) at regular angles of $2\pi/n$.

Example 2.5 Solve $z^5 = 2i$.

2.4 De Moivre's Theorem

We can use the exponential form of a complex number to derive a very useful result for obtaining trigonometric identities.

First, recall

$$e^{i\theta} = \cos\theta + i\sin\theta \ . \tag{116}$$

We can replace θ by $n\theta$ and write

$$e^{in\theta} = \cos n\theta + i \sin n\theta \ . \tag{117}$$

Also, we know that

$$e^{in\theta} = [e^{i\theta}]^n \ . \tag{118}$$

Combining these results yields

$$\boxed{\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n \ .} \tag{119}$$

This is called *De Moivre's Theorem*. Note that n does not have to be an integer.

De Moivre's Theorem, and the complex exponential more generally, are very useful indeed for working out expressions for multiple angle formulae, such as for $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$, and vice versa.

For instance, rewriting $\cos$ in terms of complex exponentials:

$$\begin{aligned}\cos^3\theta &= \left(\tfrac{1}{2}\right)^3 [\exp(i\theta) + \exp(-i\theta)]^3 \\ &= \tfrac{1}{8}[\exp(3i\theta) + 3\exp(i\theta) + 3\exp(-i\theta) + \exp(-3i\theta)] \\ &= \tfrac{1}{8}(2\cos 3\theta + 6\cos\theta) \\ &= \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta \ .\end{aligned} \tag{120}$$

Another application is to work out sums of trigonometric functions.

For example, if we wish to sum the series

$$\sum_{r=0}^{N} \cos r\theta \ ,$$

then the thing to do is to write

$$\sum_{r=0}^{N} \cos r\theta = \Re\left[\sum_{r=0}^{N} \exp(ir\theta)\right] \ . \tag{121}$$

The series

$$\sum_{r=0}^{N} \exp(ir\theta)$$

is actually a geometric progression with first term 1 and common ratio $\exp(\mathrm{i}\theta)$, for which we can write down the answer, and then the cosine series we want follows by taking the real part. We do this in detail in Example 2.7.

Example 2.6 Use De Moivre's Theorem to find expressions for $\cos 4\theta$ and $\sin 4\theta$.

Example 2.7 Evaluate $\sum_{k=0}^{N} \cos k\theta$.

2.5 Complex logarithms

Having discussed the complex exponential, the obvious thing to do next is to consider the inverse function, i.e. the complex natural logarithm $\ln z$. We first write z in the exponential form $z = |z| \exp(i\theta)$, and then:

$$\boxed{\begin{aligned} \ln z &= \ln(|z| \exp(i\theta)) \\ &= \ln(|z|) + \ln(\exp(i\theta)) \\ &= \ln(|z|) + i\theta \end{aligned}} \tag{122}$$

Of course, as we have already noted, the argument θ of z is really multi-valued, in the sense that we can add any integer multiple of 2π onto θ without changing the value of z. This means that $\ln z$ is a multi-valued function.

Often the *principal value* of $\ln z$ is defined by choosing just one of these possible values of θ, and the usual convention with $\ln z$ is to choose $-\pi < \theta < \pi$.

As an example, we work out $\ln(2i)$. First,

$$2i = 2\exp(i\pi/2 + 2n\pi i) \quad \text{for } n = 0, \pm 1, \pm 2, \dots \, , \tag{123}$$

and then using (122) we see that

$$\ln(2i) = \ln 2 + i\left(\frac{\pi}{2} + 2\pi n\right) \, . \tag{124}$$

Example 2.8 Use complex logarithms to re-express 2^i and i^i.

2.6 Oscillation problems

Complex numbers are especially useful in problems which involve oscillatory or periodic motion, such as when describing the motion of a simple pendulum, alternating electrical circuits, or any sort of wave motion in air and water.

To be specific, let us consider a simple pendulum swinging under gravity with angular frequency ω.

The angular displacement, $x(t)$, of the pendulum about the vertical then takes the general form

$$x(t) = a\cos\omega t + b\sin\omega t \, , \tag{125}$$

where a and b are real constants.

Using complex numbers we can write this as

$$x(t) = \Re\left[A\exp(i\omega t)\right] \ , \tag{126}$$

where A is a *complex* constant. In fact, by comparing (125) and (126) we find that

$$A = a - ib \ . \tag{127}$$

The big advantage of the complex representation (126) is that differentiation is very easy. For example, the velocity $v(t)$ is given by

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Re\left[A\exp(i\omega t)\right] = \Re\left[i\omega A\exp(i\omega t)\right] \ . \tag{128}$$

In other words, to differentiate we simply multiply by $i\omega$. This idea leads to various transform methods of solving differential equations.

3 Hyperbolic Functions

So far you've met a small number of important and commonly used functions, such as $\sin(x)$, $\cos(x)$, $\tan(x)$, $\exp(x)$, and $\ln(x)$. But there is, in fact, an entire zoo of interesting functions that emerge when solving various physical problems (Bessel functions, spherical harmonics, hypergeometric functions, elliptic integrals, etc.). In this section, we will introduce a class of new functions, that are related to the usual circular functions (cos, sin, tan). These are the *hyperbolic functions*.

3.1 Definitions

The hyperbolic functions are denoted and defined through:

$$\boxed{\begin{aligned} \cosh x &= \tfrac{1}{2}\left(e^x + e^{-x}\right), \\ \sinh x &= \tfrac{1}{2}\left(e^x - e^{-x}\right), \\ \tanh x &= \sinh x / \cosh x = \tfrac{e^x - e^{-x}}{e^x + e^{-x}}. \end{aligned}} \tag{129}$$

and are pronounced 'cosh', 'shine', and 'tansh'. In the same way as with circular trigonometric functions, we also define

$$\boxed{\begin{aligned} \operatorname{sech} x &= 1/\cosh x \\ \operatorname{cosech} x &= 1/\sinh x \\ \coth x &= 1/\tanh x \,. \end{aligned}} \tag{130}$$

There is in fact a very close relationship between circular and hyperbolic trigonometric functions that involves complex numbers. Recall equation (109),

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \ .$$

Now consider

$$\cos(iz) = \frac{1}{2}\left[\exp(i^2 z) + \exp(-i^2 z)\right] = \frac{1}{2}\left(e^{-z} + e^{z}\right) \ ,$$

and comparing this result to equation (129) tells us that

$$\boxed{\cos(iz) = \cosh z\ .} \tag{131}$$

Similarly, recalling equation (110) we see that

$$\begin{aligned}\sin iz = \frac{1}{2i}\left[\exp(i^2 z) - \exp(-i^2 z]\right) &= \frac{1}{2i}\left(e^{-z} - e^{z}\right) \\ &= \frac{i}{2}\left(-e^{-z} + e^{z}\right)\ ,\end{aligned}$$

so that

$$\boxed{\sin(iz) = i\sinh z\ .} \tag{132}$$

Dividing (132) by (131) gives

$$\boxed{\tan(iz) = i\tanh z\ .} \tag{133}$$

3.2 Identities

All the identities we know for circular trigonometric functions can be converted to corresponding identities for hyperbolic functions.

For example, consider $\cos^2 x + \sin^2 x = 1$. For hyperbolic functions, take

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \frac{1}{4}\left(\exp(x) + \exp(-x)\right)^2 - \frac{1}{4}\left(\exp(x) - \exp(-x)\right)^2 \\ &= \frac{1}{4}\left[\exp(2x) + \exp(-2x) + 2\right] - \frac{1}{4}\left[\exp(2x) + \exp(-2x) - 2\right] \\ &= \frac{1}{4}(4)\ ,\end{aligned}$$

so that

$$\boxed{\cosh^2 x - \sinh^2 x = 1}\ . \tag{134}$$

Note the different sign compared to the classical trig relationship!

To derive multi-angle formulas we exploit the relationships between 'cosh' and 'cos' etc. For instance, start with the trigonometric identity

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \ .$$

This identity is actually true for all *complex* A and B, not just real values, so we could equally well have

$$\cos(iA+iB) = \cos iA \cos iB - \sin iA \sin iB \ ,$$

and then using (131) & (132) we find that

$$\cos(iA+iB) = \cosh(A+B) = \cosh A \cosh B - i^2 \sinh A \sinh B \ ,$$

so that

$$\boxed{\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B} \ . \tag{135}$$

Here are some more identities:

$$\boxed{\begin{array}{rcl} \sinh(A+B) & = & \sinh A \cosh B + \sinh B \cosh A \\ \sinh(A-B) & = & \sinh A \cosh B - \sinh B \cosh A \\ \cosh(A-B) & = & \cosh A \cosh B - \sinh A \sinh B \\ 1-\tanh^2 z & = & \mathrm{sech}^2 z \\ \coth^2 z - 1 & = & \mathrm{cosech}^2 z \end{array} \ .}$$

Example 3.1 Find an identity for $\tanh(A+B)$.

3.3 Graphs of hyperbolic functions

When sketching the functions, note the following:

- $\cosh 0 = 1$, $\sinh 0 = \tanh 0 = 0$.
- The graph of $\cosh x$ is symmetric about $x = 0$, i.e. $\cosh x = \cosh(-x)$, while $\sinh x$ and $\tanh x$ are antisymmetric,
- As x approaches infinity through positive values, $y = \cosh x$ and $y = \sinh x$ approach the same curve $\frac{1}{2}\exp(x)$. This is because as x gets bigger $\exp(-x)$ is much less than $\exp(x)$,
- As x gets more and more negative, the term $\exp(-x)$ dominates. Hence $\sinh x$ approaches $-\frac{1}{2}\exp(-x)$, and $\cosh x$ approaches $\frac{1}{2}\exp(-x)$
- The previous two points explain why $\tanh x \to \pm 1$ as $x \to \pm\infty$.

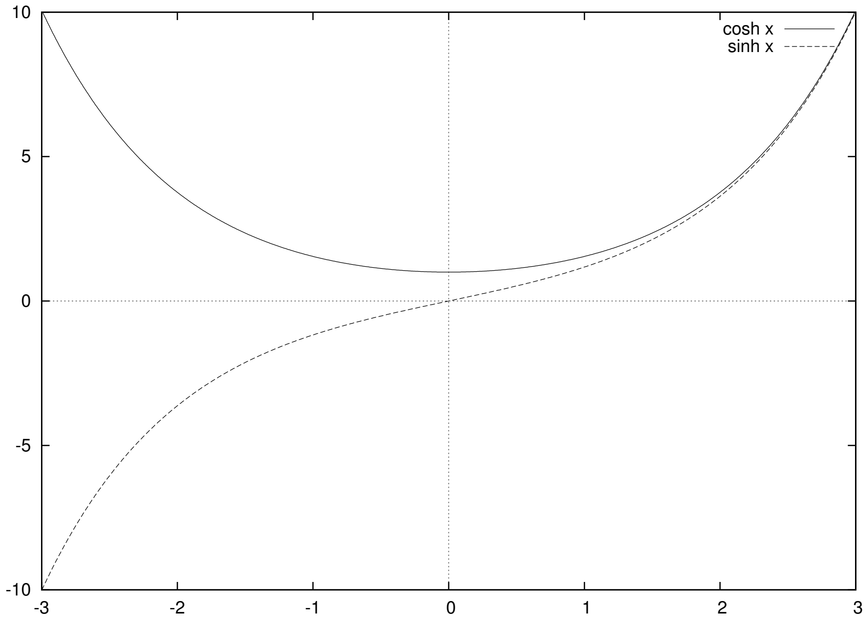

Figure 11: Plots of $\cosh x$ and $\sinh x$.

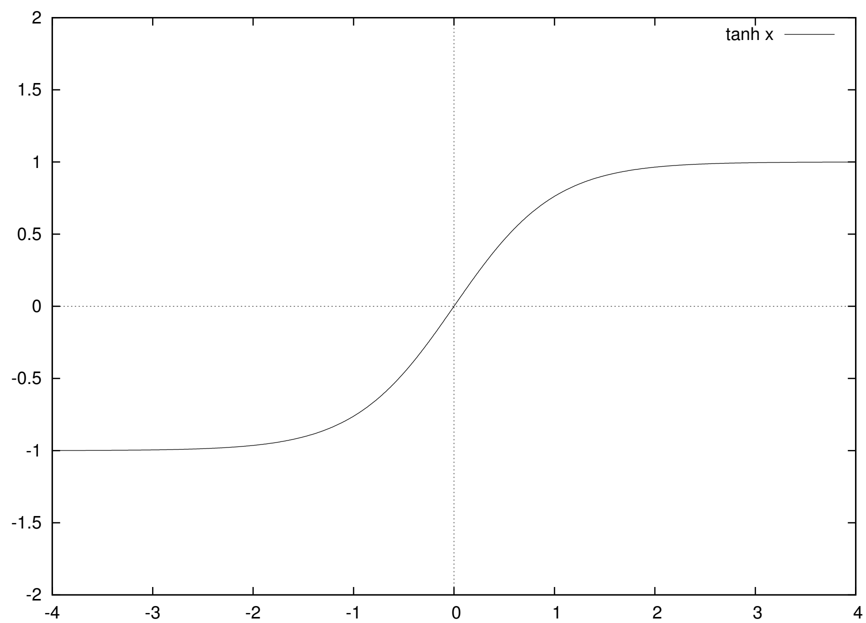

Figure 12: Plot of $\tanh x$. This is often used to represent a smooth step function.

- Note that $\cosh x \geqslant 1$ and $-1 < \tanh x < 1$ for all real x.

3.4 Inverse hyperbolic functions

Having defined the hyperbolic functions, we now want to introduce their inverses. Consider first the inverse of $\sinh x$, denoted

$$y = \sinh^{-1} x \ ,$$

which means

$$\sinh y = x \ . \tag{136}$$

Using the definition of sinh gives

$$\frac{1}{2}\left(e^{y} - e^{-y}\right) = x \ , \tag{137}$$

and multiplying through by $2e^y$ gives

$$e^{2y} - 2xe^{y} - 1 = 0 \ , \tag{138}$$

a quadratic equation in $\exp y$. The solutions are

$$e^{y} = x \pm \sqrt{x^2 + 1} \ . \tag{139}$$

Of these two roots, the one with the minus sign is negative, and must be thrown away (the exponential of a real number can never be negative).

Taking the log of the remaining root yields

$$\boxed{y = \sinh^{-1} x \equiv \ln\left(x + \sqrt{x^2 + 1}\right)} \ . \tag{140}$$

To obtain inverse cosh, set $y = \cosh^{-1} x$, and after almost the same algebra, we arrive at the equation

$$e^{2y} - 2xe^{y} + 1 = 0 \ , \tag{141}$$

with the two roots

$$e^{y} = x \pm \sqrt{x^2 - 1} \ . \tag{142}$$

Both roots are positive, so both must be kept.

Taking logs

$$y = \cosh^{-1} x \equiv \ln\left(x \pm \sqrt{x^2-1}\right) ,$$

our answer.

But the $\pm$ can be brought outside the log. First note that

$$x - \sqrt{x^2-1} = \frac{1}{x+\sqrt{x^2-1}} , \tag{143}$$

so that

$$\ln(x-\sqrt{x^2-1}) = \ln\left[\frac{1}{x+\sqrt{x^2-1}}\right] = -\ln(x+\sqrt{x^2-1}) .$$

We are therefore finally left with the answer

$$\boxed{y = \cosh^{-1} x \equiv \pm\ln\left(x+\sqrt{x^2-1}\right) .} \tag{144}$$

Obviously, $\cosh^{-1}$ is not defined when $x < 1$. That is because $\cosh(x) \geq 1$.

The reason that the inverse cosh is multivalued (i.e. possesses the $\pm$) is that cosh is symmetric ($\cosh x = \cosh(-x)$). So for every possible value of y on the $y = \cosh x$ curve there is a positive *and* a negative value of x.

<u>Example 3.2</u> Find an identity for $\tanh^{-1} x$.

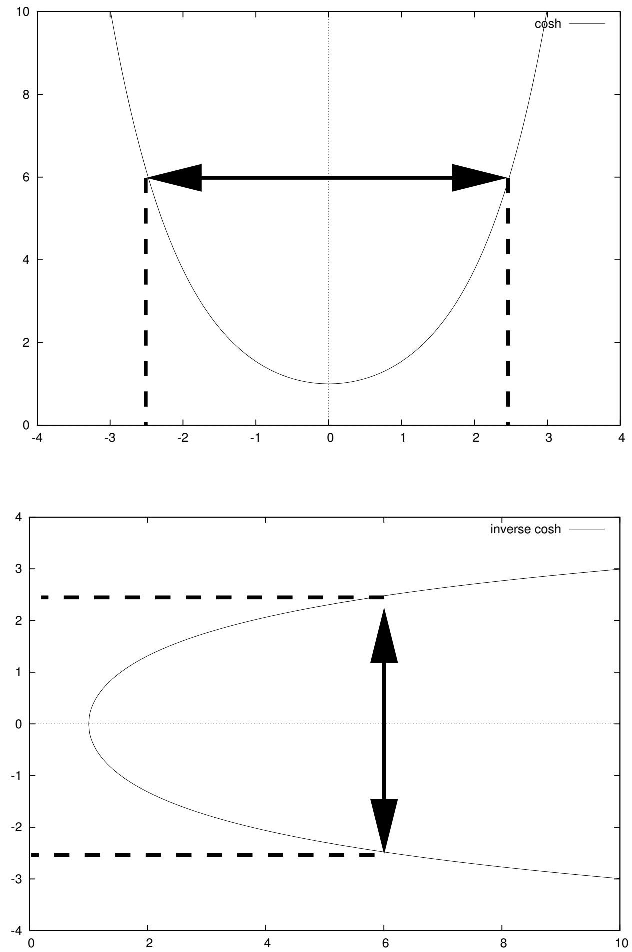

Figure 13: Plot of (a) $y = \cosh x$ and (b) $y = \cosh^{-1} x$.

Example 3.3 Find all the roots of $\cos z = 2$.

3.5 Circles, ellipses, and hyperbolae

3.5.1 Circles

As you all know, a circle in the xy plane with centre at the origin and with radius a has equation:

$$x^2 + y^2 = a^2. \tag{145}$$

But the curve can also be represented parametrically via

$$x = a\cos\theta, \qquad y = a\sin\theta, \tag{146}$$

where θ is the polar angle.

3.5.2 Ellipses

One way to generate an ellipse is to take a circle and then stretch one or both of the x and y axes. We then obtain the equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{147}$$

where now a is called the *semi-major axis*, and b is the *semi-minor* axis. An ellipse need not have its semi-major and semi-minor axes aligned with the x and y axes, but in the canonical form above this is the case.

The curve has the parametric representation:

$$x = a\cos\theta, \qquad y = b\sin\theta, \tag{148}$$

where again θ is the polar angle.

An important quantity is the *eccentricity* $e = \sqrt{1 - b^2/a^2}$, which measures the degree of the ellipse's 'distortion'.

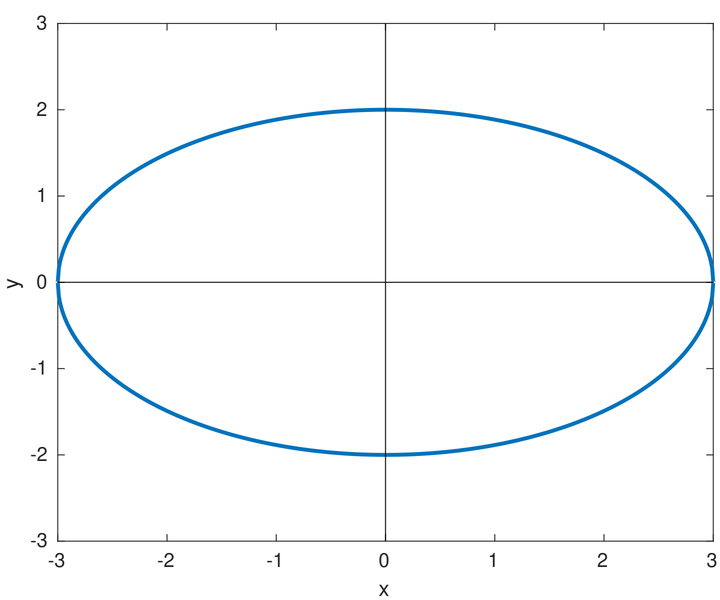

Figure 14: An ellipse with semi-major axis of $a = 3$ and semi-minor axis of $b = 2$.

3.5.3 Hyperbolae

The equation of a hyperbola centred on the origin and aligned with the x and y axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{149}$$

Its parametric representation is

$$x = \pm a \cosh\theta, \qquad y = b \sinh\theta, \tag{150}$$

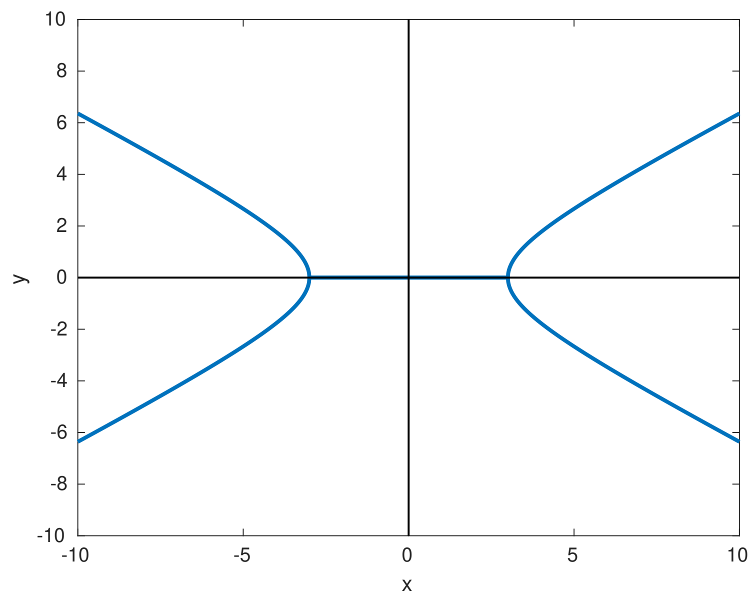

Figure 15: A hyperbola with semi-major axis of $a = 3$ and semi-minor axis of $b = 2$.

4 Differentiation

Rates of change, usually with respect to time and space, underpin so many of our scientific theories of the world. They regularly appear in the governing *differential* equations of a theory. There are almost too many examples to list, but prominent equations include: Newton's second law (dynamics), Schroedinger's equation (quantum mechanics), Einstein's field equations (general relativity), the Navier-Stokes equation (fluid dynamics), the Malthus and logistic models (population growth), Fisher's equation (gene propagation), and chemical reaction kinetics (chemistry).

In this part of the course we revise the basics of *differentiation*, which provides the mathematical foundations of change. We focus only on functions of a single variable.

4.1 First Principles

The derivative of a function $y(x)$ at a given point x will be denoted dy/dx and is defined by the limiting process:

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} \equiv \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x}\ .} \tag{151}$$

Geometrically, the derivative is the gradient of the tangent line to the curve given by $y(x)$ at the point x. The tangent line has a slope such that it only just touches the curve at this point.

Example: differentiate $y = x^3$ from first principles:

$$\begin{aligned} y(x+\delta x) &= (x+\delta x)^3 = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 \\ y(x+\delta x) - y(x) &= 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 \ . \end{aligned} \tag{152}$$

Now

$$\frac{y(x+\delta x) - y(x)}{\delta x} = 3x^2 + 3x(\delta x) + (\delta x)^2 \ .$$

Take the limit $\delta x \to 0$ and the second and third terms on the right disappear, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} \equiv \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x} = 3x^2 \ . \tag{153}$$

4.1.1 Differentiability

Functions are not necessarily differentiable everywhere.

Example 1: consider the *Heaviside step function* $\mathrm{H}(x)$, defined to be $\mathrm{H}(x) = 0$ for $x < 0$ and $\mathrm{H}(x) = 1$ for $x > 0$. What is its derivative at $x = 0$?

The difference $y(\delta x) - y(-\delta x) = 1$ measured about $x = 0$ is unchanged even as $\delta x \to 0$, so the derivative at $x = 0$ using (151) is not finite.

This is an example of a **discontinuous function**; these are not differentiable at their discontinuities.

Example 2: consider the absolute value function $y(x) = |x|$, a function which is continuous but **not smooth** at $x = 0$.

Taking the derivative, we find

$$\frac{\mathrm{d}|x|}{\mathrm{d}x} = \begin{cases} -1 & \text{if } x < 0, \\ +1 & \text{if } x > 0. \end{cases}$$

The derivative at $x = 0$ is clearly not well-defined, as it depends on the direction from which you approach the origin.

For a function $y(x)$ to be differentiable at a point x, the function must be both *continuous* and *smooth*.

4.1.2 Higher order derivatives

The derivative dy/dx is a function of x, so we can differentiate it again (assuming it is smooth and continuous). This is the *second derivative*, which is denoted by

$$\frac{d^2y}{dx^2} \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right) , \tag{154}$$

It measures the rate of change of the slope, i.e. its *curvature*.

The notation for going further and taking the nth derivative is

$$\frac{d^ny}{dx^n} \equiv \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) . \tag{155}$$

So, for the example with $y = x^3$, we have

$$\begin{aligned} \frac{\mathrm{d}x^3}{\mathrm{d}x} &= 3x^2 , \\ \frac{\mathrm{d}^2(x^3)}{\mathrm{d}x^2} &= \frac{\mathrm{d}(3x^2)}{\mathrm{d}x} = 6x , \\ \frac{\mathrm{d}^3(x^3)}{\mathrm{d}x^3} &= \frac{\mathrm{d}(6x)}{\mathrm{d}x} = 6 , \\ \frac{\mathrm{d}^4(x^3)}{\mathrm{d}x^4} &= \frac{\mathrm{d}(6)}{\mathrm{d}x} = 0 , \end{aligned} \tag{156}$$

where all derivatives of higher order than the fourth are zero.

4.1.3 Alternative notations

The dy/dx notation for the derivative of $y(x)$ was proposed by Leibniz. However, Newton originally had a more compact notation using dots (or primes):

$$\dot{y} = \frac{dy}{dx} \quad \text{or} \quad y' = \frac{dy}{dx}\,,$$
$$\ddot{y} = \frac{d^2y}{dx^2} \quad \text{or} \quad y'' = \frac{d^2y}{dx^2}\,.$$

For higher order derivatives it can be unwieldy to employ dots and dashes. Generally we use the more compact notation for the n^{th} derivative

$$y^{(n)}(x) = \frac{d^ny}{dx^n}.$$

Note that some people use Roman numerals with this convention, so that $d^4y/dx^4 = y^{\text{iv}}(x)$ and $d^5y/dx^5 = y^{\text{v}}(x)$.

4.2 Derivatives of elementary functions

Little progress is possible in calculus without knowing the basic derivatives of elementary functions, including powers of x, trigonometric, exponential and logarithmic functions.

You should have the following on automatic recall:

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}\,,$$
$$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x,$$
$$y = \ln x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}\,,$$
$$y = \sin x \quad \Rightarrow \quad \frac{dy}{dx} = \cos x\,,$$
$$y = \cos x \quad \Rightarrow \quad \frac{dy}{dx} = -\sin x\,,$$
$$y = \tan x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos^2 x}\,.$$

You may not be as familiar with the derivatives of the hyperbolic functions introduced in Section 3:

$$y = \sinh x \quad \Rightarrow \quad \frac{dy}{dx} = \cosh x\,,$$
$$y = \cosh x \quad \Rightarrow \quad \frac{dy}{dx} = \sinh x,$$
$$y = \tanh x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cosh^2 x}\,.$$

However, these are easy to derive using the definitions of the hyperbolic functions. For example, differentiating $\sinh x$ we get

$$\frac{d \sinh x}{dx} = \frac{d}{dx}\left[\frac{1}{2}\left(e^x - e^{-x}\right)\right] = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x\,.$$

The hyperbolic and trigonometric cases are similar, but note the sign difference between the derivatives of $\cosh x$ and $\cos x$.

4.3 Rules for differentiation

4.3.1 The product rule

Sometimes we are given a product of functions in the form

$$y(x) = u(x)v(x)\ , \tag{157}$$

where we know how to differentiate the factors $u(x)$ and $v(x)$ individually.

The rule for differentiating this product of functions is the following:

$$\boxed{\frac{\mathrm{d}(uv)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}} \tag{158}$$

This result can be produced from first principles relatively quickly:

$$\begin{aligned}
\frac{y(x+\delta x)-y(x)}{\delta x} &= \frac{u(x+\delta x)v(x+\delta x)-u(x)v(x)}{\delta x} \\
&= \frac{u(x+\delta x)v(x+\delta x)-u(x)v(x+\delta x)}{\delta x} \\
&\qquad + \frac{u(x)v(x+\delta x)-u(x)v(x)}{\delta x} \\
&= \left[\frac{u(x+\delta x)-u(x)}{\delta x}\right] v(x+\delta x) + u(x)\left[\frac{v(x+\delta x)-v(x)}{\delta x}\right] .
\end{aligned}$$

We now take the limit $\delta x \to 0$ and get the result.

Example 4.1 Differentiate $y = \ln x \sin x$.

4.3.2 The chain rule

Often we are given complicated expressions in which we have a function $y = f(u)$ with $u = u(x)$ itself being a function of x (e.g. $y = f(u) = \sin u$ and $u(x) = x^2$, so that $y = \sin x^2$). The method for differentiating a 'function of a function' is called the *chain rule* and is given by

$$\boxed{\frac{\mathrm{d}(f(u(x)))}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}} \tag{159}$$

We can understand why this rule arises by writing

$$\frac{f(u(x+\delta x))-f(u(x))}{\delta x} = \left[\frac{f(u(x+\delta x))-f(u(x))}{u(x+\delta x)-u(x)}\right]\left[\frac{u(x+\delta x)-u(x)}{\delta x}\right] . \tag{160}$$

Next write $\delta u = u(x+\delta x) - u(x)$, i.e. the accompanying small change in the function u due to the small change δx in x.

We then have:

$$\frac{f(u(x+\delta x)) - f(u(x))}{\delta x} = \left[\frac{f(u+\delta u) - f(u)}{\delta u}\right]\left[\frac{u(x+\delta x) - u(x)}{\delta x}\right], \tag{161}$$

and we now take the limit $\delta x \to 0$ (so that necessarily $\delta u \to 0$ as well). The first factor becomes df/du and the second factor du/dx.

Example 4.2 Differentiate $\sin x^2$ and $\ln(\cos x)$ with respect to x.

4.3.3 The quotient rule

We have already seen how to differentiate the product uv, now we consider the quotient u/v. In this case, we can find the derivative from the formula:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v(\mathrm{d}u/\mathrm{d}x) - u(\mathrm{d}v/\mathrm{d}x)}{v^2}} \tag{162}$$

This result comes about via the product and chain rules. First, write the quotient u/v as the product

$$\frac{u}{v} = u \times \left(\frac{1}{v}\right) . \tag{163}$$

Now differentiate (163) using the product rule (158) to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{\mathrm{d}u}{\mathrm{d}x} \times \left(\frac{1}{v}\right) + u\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) . \tag{164}$$

Next use the chain rule (159) to calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) = \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\right)\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} . \tag{165}$$

Finally substitute this result back into (164) to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} , \tag{166}$$

and group terms over the common denominator v^2.

<u>Example 4.3</u> Differentiate $(\sin x)/x$ with respect to x.

4.3.4 Implicit differentiation

It is also possible to find the derivative of y with respect to x from an equation of the form

$$g(y) = f(x) , \tag{167}$$

where g and f are given functions. For example: $\mathrm{e}^y \cos y = x \cos x$. Here, the exact dependence of y on x may not be known explicitly at all. Rather, it is *implicit*.

Using the chain rule:

$$\frac{\mathrm{d}g(y)}{\mathrm{d}x} = \frac{\mathrm{d}g(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}\,, \tag{168}$$

so that differentiating (167) with respect to x we have

$$\frac{\mathrm{d}g(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}\,. \tag{169}$$

Rearranging, we find that

$$\boxed{g(y) = f(x) \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f/\mathrm{d}x}{\mathrm{d}g/\mathrm{d}y}\,.} \tag{170}$$

An important special case gives the 'reciprocal rule'. Suppose we want to differentiate $y(x)$ but only know the derivative of its inverse function $x(y)$, i.e. dx/dy.

Okay, set $f(x) = x$ in (167), so now $x = g(y)$. Then we have immediately

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}}\,. \tag{171}$$

<u>Example 4.4</u> Find the derivative of $y = \tan^{-1} x$ with respect to x.

<u>Example 4.5</u> A circle has equation $y^2 = 9 - (x-1)^2$. Find the gradient.

Example 4.6 [2006 paper 2, Question 1A]. If

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

find dy/dx as a function of x.

4.4 Stationary points

A stationary point (also called a 'turning point') of the curve $y = f(x)$ is a point where $\mathrm{d}y/\mathrm{d}x = 0$.

Stationary points can be classified using the following rules:

- If $\mathrm{d}^2y/\mathrm{d}x^2 > 0$ at the stationary point, then it is a **minimum**.

  This is because at a minimum the gradient *increases* through the turning point.

- If $\mathrm{d}^2y/\mathrm{d}x^2 < 0$ at the stationary point, then it is a **maximum**.

  This is because at a maximum the gradient *decreases* through the turning point.

- If $\mathrm{d}^2y/\mathrm{d}x^2 = 0$ at the stationary point, then further investigation is required:

1. if the first nonzero derivative $\mathrm{d}^ny/\mathrm{d}x^n \neq 0$ has n odd, then the stationary point is a **point of inflection**;
2. If the first nonzero derivative $\mathrm{d}^ny/\mathrm{d}x^n \neq 0$ has n even and it is positive, then the stationary point is a **minimum**;
3. If the first nonzero derivative $\mathrm{d}^ny/\mathrm{d}x^n \neq 0$ has n even and it is negative, then the stationary point is a **maximum**.

As examples, consider $y = x^3$ and $y = x^4$. Both have a stationary point at $x = 0$ with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0\,.$$

For $y = x^3$, the point $x = 0$ is a point of inflection because

$$\frac{\mathrm{d}^3(x^3)}{\mathrm{d}x^3} = 6 \neq 0\,. \tag{172}$$

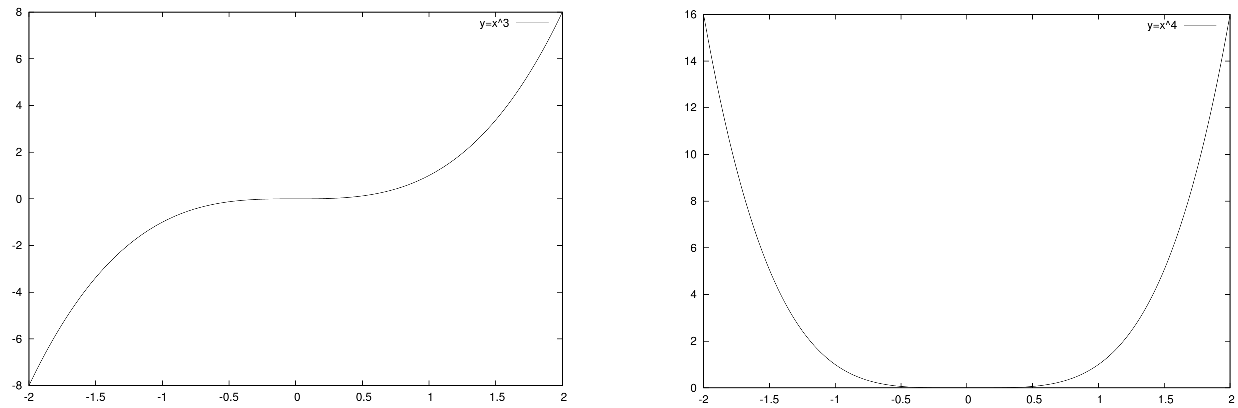

Figure 16: Graphs of $y = x^3$ and $y = x^4$.

For $y = x^4$, this is a minimum because the first non-zero derivative is even and positive,

$$\frac{\mathrm{d}^4 x^4}{\mathrm{d}x^4} = 24 > 0\,. \tag{173}$$

4.4.1 More on points of inflection

An inflection point is where $d^2y/dx^2 = 0$ and also d^2y/dx^2 changes sign.

- It is where the curve changes from being concave up to concave down or vice versa.
- It need not be a stationary point (i.e. where $dy/dx = 0$).
- If $dy/dx > 0$ at the inflection point it is a 'rising point of inflection'; if $dy/dx < 0$ it is a 'falling point of inflection'.
- Between an adjacent maximum and a minimum there must be a point of inflection.

4.5 Curve sketching

Basic curve sketching techniques are very useful for detemining the *main features* of the overall shape of a function $y = f(x)$. It means we can understand the behaviour of the function without the need to compute it everywhere. In other words, we can get a qualitative idea of what it is about.

When sketching curves there are a number of things to consider:

1. Where does the curve **intercept** the x and y axes – i.e. what is the value of $f(0)$ and what are solutions for $f(x) = 0$?

2. Is there any **symmetry**? Is the function *even*, $f(x) = f(-x)$, or is the function *odd*, $f(x) = -f(-x)$?

3. What are the **asymptotes**? In other words, what is the behaviour as $x \to \pm\infty$ or at any boundaries?

4. Are there any **singularities**, that is, points where the function becomes infinite? These create *vertical asymptotes* about the singular point.

5. What are the **stationary points** (i.e. where does $dy/dx = 0$)? What is their nature – minimum, maximum or point of inflection?

The following examples illustrate important aspects of curve sketching techniques.

Example 4.7 Sketch $y = \exp x - \sin x$.

Example 4.8 Sketch $y = \frac{\ln x}{x}$. Show that $e^\pi > \pi^e$.