The use of Green's functions for time-dependent linear PDEs

This document aims to provide some more background on Green's functions we used to solve the linear diffusion equation. It is by no means a complete description of the subject, nor is the material presented examinable.

Consider the following partial differential equation (PDE) for a scalar function $u(x,t)$ on a bounded domain D:

$$\frac{\partial u}{\partial t} = L(u), \tag{1}$$

where L is a linear, self-adjoint operator (containing spatial derivatives only). Boundary conditions are applied on ∂D, and the initial condition is denoted by $u_0(x)$. L being self-adjoint means that there exists an orthonormal basis of eigenfunctions e_n of L, satisfying the boundary conditions, such that

$$L(e_n) = \lambda_n e_n, \tag{2}$$

where λ_n is the associated eigenvalue. We can use these eigenfunctions to construct a solution to our PDE, by noting that

$$u_n = c_n \exp(\lambda_n t) e_n, \tag{3}$$

where c_n is an arbitrary constant, is a solution to (1). The general solution to our PDE is a sum over all possible n. To satisfy the initial condition, we must have that $u_0 = \sum_n c_n e_n$. Exploiting the orthonormality of the e_n (i.e. using Fourier's trick), we see that we must choose

$$c_n = \int_D e_n^*(x') u_0(x') dx', \tag{4}$$

where a superscript $*$ denotes complex conjugation. The general solution to (1) is then

$$u(x,t) = \int_D u_0(x') \left[\sum_n e_n^*(x') \exp(\lambda_n t) e_n(x) \right] dx'. \tag{5}$$

The quantity between square brackets is the Green's function $G(x, x', t)$. (Note that we have changed the order of summation and integration, which may not always be allowed. We'll skip over this, and just note that this issue can be overcome by interpreting G as a *generalised function*.)

Applying the operator $\partial/\partial t - L_x$, where the subscript x indicates that spatial derivatives are to be taken with respect to x, on both sides of (5), we get that

$$0 = \int_D u_0(x') \left[\frac{\partial G}{\partial t} - L_x(G) \right] dx'. \tag{6}$$

Since this must be true for *any* function u_0, we must conclude that G itself is a solution to (1) when spatial derivatives are taken with respect to x. Also note that at $t = 0$

$$u_0(x) = \int_D G(x, x', 0) u_0(x') dx', \tag{7}$$

which again must be true for *any* function u_0. This means that $G(x, x', 0) = \delta(x - x')$.

So in the end, our Green's function $G(x, x', t)$ is defined by

$$\frac{\partial G}{\partial t} = L_x(G), \tag{8}$$

$$G(x, x', 0) = \delta(x - x'), \tag{9}$$

together with suitable boundary conditions on ∂D. Although we have worked with a finite domain D, it can be shown that for well-enough behaved u we may extend the domain to infinity.

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