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Functional Representations of Conformal Symmetry in Quantum Field Theory

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Abstract

Representations of the conformal algebra are constructed in terms of functional differential operators depending a scalar field. For first order differential operators the representation corresponds to a theory of generalised free fields. This representation is extended to second order operators for the scale and special conformal generators. This form depends on a largely arbitrary cut off function and also a functional of the scalar field which is determined by second order functional differential equation. For the generator of scale transformations the result is equivalent to a version of Wilsonian exact RG equations. The eigenvalues of the generator of scale transformations, subject to some assumptions of locality, determine the spectrum and scale dimensions of operators in the theory.

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1 Conformal Symmetry

At any fixed point which is scale invariant the symmetry can usually be enhanced to include also special conformal transformations so that the fixed point theory is invariant under all conformal transformations [1]. A sufficient condition is if the energy momentum tensor at the fixed point can be improved by additional contributions, according to well defined prescription, so as to have vanishing trace. In this case $T_{\mu\nu}v_\nu$ is conserved a vector for v_ν a conformal Killing vector. Although for $d > 2$ the conformal group is finite dimensional the additional symmetries, beyond rotations and translations, allow the dependence of two and three point functions for local conformal primary operators $\mathcal{O}(x)$ on their positions x to be determined in terms of the scale dimensions Δ and spins of of the operators. There is also a one to one correspondence between the operator fields and states forming a basis of the associated Hilbert space. In such conformal field theories there a prescription for a natural norm in this basis is defined by the overall scale of the two point function for the corresponding operator fields. This is necessarily positive in unitary theories. In two dimensional conformal theories this norm played an essential role in the derivation of the c-theorem by Zamolodchikov [2].

The generators of the conformal group $\{\mathcal{M}_{\mu\nu}, \mathcal{P}_\mu, \mathcal{D}, \mathcal{K}_\mu\}$, which generate rotations, translations, scale transformations and special conformal translations, have the non zero commutators

$$\begin{aligned}
&\big[\mathcal{M}_{\mu\nu}, \mathcal{M}_{\sigma\rho}\big] = \delta_{\mu\sigma}\,\mathcal{M}_{\nu\rho} - \delta_{\nu\sigma}\,\mathcal{M}_{\mu\rho} - \delta_{\mu\rho}\,\mathcal{M}_{\nu\sigma} + \delta_{\nu\rho}\,\mathcal{M}_{\mu\sigma}\,, \\
&\big[\mathcal{M}_{\mu\nu}, \mathcal{P}_\sigma\big] = \delta_{\mu\sigma}\,\mathcal{P}_\nu - \delta_{\nu\sigma}\,\mathcal{P}_\mu\,, \qquad \big[\mathcal{M}_{\mu\nu}, \mathcal{K}_\sigma\big] = \delta_{\mu\sigma}\,\mathcal{K}_\nu - \delta_{\nu\sigma}\,\mathcal{K}_\mu\,, \\
&\big[\mathcal{D}, \mathcal{P}_\mu\big] = \mathcal{P}_\mu\,, \qquad \big[\mathcal{D}, \mathcal{K}_\mu\big] = -\mathcal{K}_\mu\,, \qquad \big[\mathcal{K}_\mu, \mathcal{P}_\nu\big] = 2\delta_{\mu\nu}\,\mathcal{D} + 2\mathcal{M}_{\mu\nu}\,.
\end{aligned} \tag{1.1}$$

The conformal algebra becomes more succinct by rewriting

$$\mathcal{M}_{AB} = -\mathcal{M}_{BA} = \begin{pmatrix} \mathcal{M}_{\mu\nu} & -\mathcal{K}_\mu & -\mathcal{P}_\mu \\ \mathcal{K}_\nu & 0 & 2\,\mathcal{D} \\ \mathcal{P}_\nu & -2\,\mathcal{D} & 0 \end{pmatrix}, \tag{1.2}$$

with A, B $(d+2)$-dimensional indices, so that (1.1) is equivalent to

$$\big[\mathcal{M}_{AB}, \mathcal{M}_{CD}\big] = \eta_{AC}\,\mathcal{M}_{BD} - \eta_{BC}\,\mathcal{M}_{AD} - \eta_{AD}\,\mathcal{M}_{BC} + \eta_{BD}\,\mathcal{M}_{AC}\,, \tag{1.3}$$

if

$$\eta_{AB} = \begin{pmatrix} \delta_{\mu\nu} & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \tag{1.4}$$

Assuming ${\mathcal{M}_{AB}}^\dagger = -\mathcal{M}_{AB}$ then $\{\mathcal{M}_{AB}\}$ are generators of $SO(d+1,1)$ (for a Minkowski metric when $\delta_{\mu\nu} \to \eta_{\mu\nu}$ the conformal group is then $SO(d,2)$). The usual momentum, angular momentum operators $P_\mu = -i\mathcal{P}_\mu, M_{\mu\nu} = -i\mathcal{M}_{\mu\nu}$ are then hermitian.

A conformal primary operator scalar field $\mathcal{O}(x)$ with scale dimension Δ is required to satisfy

$$\big[\mathcal{P}_\mu, \mathcal{O}\big] = \partial_\mu\mathcal{O}\,, \quad \big[\mathcal{M}_{\mu\nu}, \mathcal{O}\big] = L_{\mu\nu}\mathcal{O}\,, \quad \big[\mathcal{D}, \mathcal{O}\big] = D^{(\Delta)}\mathcal{O}\,, \quad \big[\mathcal{K}_\mu, \mathcal{O}\big] = {K^{(\Delta)}}_\mu\mathcal{O}\,, \tag{1.5}$$

where

$$L_{x\,\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \,, \qquad D_x^{(\Delta)} = x \cdot \partial + \Delta \,,$$
$$K_x^{(\Delta)}{}_\mu = -x^2 \partial_\mu + 2 x_\mu \, x \cdot \partial + 2\Delta \, x_\mu \,. \tag{1.6}$$

These action of the generators is straightforwardly extended to fields with spin. For a tensorial field $\mathcal{O}_{\alpha_1 \ldots \alpha_l}(x)$ then (1.5) is valid with the modifications

$$\big[\mathcal{M}_{\mu\nu}, \mathcal{O}_{\alpha_1\ldots\alpha_l}(x)\big] = L_{x\,\mu\nu}\, \mathcal{O}_{\alpha_1\ldots\alpha_l}(x) + (S_{\mu\nu}\mathcal{O})_{\alpha_1\ldots\alpha_l}(x) \,,$$
$$(S_{\mu\nu}\mathcal{O})_{\alpha_1\ldots\alpha_l}(x) = \textstyle\sum_{i=1}^{l} \big(\delta_{\mu\alpha_i}\, \mathcal{O}_{\alpha_1\ldots\nu\ldots\alpha_l}(x) - \delta_{\nu\alpha_i}\, \mathcal{O}_{\alpha_1\ldots\mu\ldots\alpha_l}(x)\big) \tag{1.7}$$

and

$$\big[\mathcal{K}_\mu, \mathcal{O}_{\alpha_1\ldots\alpha_l}(x)\big] = K_x^{(\Delta)}{}_\mu\, \mathcal{O}_{\alpha_1\ldots\alpha_l}(x) + 2x_\nu\, (S_{\mu\nu}\mathcal{O})_{\alpha_1\ldots\alpha_l}(x) \,. \tag{1.8}$$

The vacuum state $|0\rangle$ belongs to the trivial singlet representation so that $\mathcal{M}_{AB}|0\rangle = 0$. The non trivial unitary representations of physical interest are lowest weight representations where $\mathcal{D}$ has positive real eigenvalues, since for $|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle$ then $\mathcal{D}|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle$. With this definition it is easy to see that $\mathcal{M}_{\mu\nu}|\mathcal{O}\rangle = \mathcal{K}_\mu|\mathcal{O}\rangle = 0$.

The two point function for a real conformal primary field $\mathcal{O}$, with scale dimension Δ, $\big\langle \mathcal{O}(x)\, \mathcal{O}(y) \big\rangle$ in conformal field theories satisfies

$$\big\langle \big[\mathcal{M}_{AB}, \mathcal{O}(x)\big]\, \mathcal{O}(y) \big\rangle + \big\langle \mathcal{O}(x)\, \big[\mathcal{M}_{AB}, \mathcal{O}(y)\big] \big\rangle = 0 \,. \tag{1.9}$$

Letting $\{L_{\mu\nu}, \partial_\mu, D^{(\Delta)}, K^{(\Delta)}{}_\mu\} \to M^{(\Delta)}{}_{AB}$, and then requiring

$$\big[\mathcal{M}_{AB}, \mathcal{O}(x)\big] = M_x^{(\Delta)}{}_{AB}\, \mathcal{O}(x) \,, \qquad \big[\mathcal{M}_{AB}, \mathcal{O}(y)\big] = M_y^{(\Delta)}{}_{AB}\, \mathcal{O}(y) \,, \tag{1.10}$$

ensures that (1.9) becomes a conformal Ward identity determining the functional form of the two point function. For a scalar field $\mathcal{O}$

$$\big\langle \mathcal{O}(x)\, \mathcal{O}(y) \big\rangle = \frac{c_{\mathcal{O}\mathcal{O}}}{\big((x-y)^2\big)^\Delta} \,. \tag{1.11}$$

1.1 Inversions and Scalar Product

To obtain a scalar product in which $\mathcal{D}$ is hermitian we consider the conformal transformation obtained from an inversion through $x = 0$,

$$x \to \bar{x} = \frac{1}{x^2}\, x \,. \tag{1.12}$$

Conformality follows from

$$\mathrm{d}x^2 \to \mathrm{d}\bar{x}^2 = \frac{1}{(x^2)^2}\, \mathrm{d}x^2 \,. \tag{1.13}$$

Corresponding to such a conformal transformation a scalar conformal primary field $\mathcal{O}$ transforms as

$$\mathcal{O}(x) \to \bar{\mathcal{O}}(x) = (x^2)^{-\Delta}\, \mathcal{O}(\bar{x}) \,, \tag{1.14}$$

and it is easy to check

$$D_x^{(\Delta)}\bar{\mathcal{O}}(x) = -(x^2)^{-\Delta}\, D_{\bar x}^{(\Delta)}\mathcal{O}(\bar x)\,, \quad \partial_{x\mu}\bar{\mathcal{O}}(x) = -(x^2)^{-\Delta}\, K_{\bar x}^{(\Delta)}{}_\mu\mathcal{O}(\bar x)\,,$$
$$L_{x\,\mu\nu}\bar{\mathcal{O}}(x) = (x^2)^{-\Delta}\, L_{\bar x\,\mu\nu}\mathcal{O}(\bar x)\,. \tag{1.15}$$

For complex fields $\mathcal{O} \to \bar{\mathcal{O}}$ is antilinear.

Acting on $\bar{\mathcal{O}}$ we may define

$$\big[\bar{\mathcal{M}}_{AB}, \bar{\mathcal{O}}(x)\big] = M_x^{(\Delta)}{}_{AB}\,\bar{\mathcal{O}}(x)\,, \tag{1.16}$$

This gives, from (1.15),

$$\bar{\mathcal{D}} = -\mathcal{D}\,, \qquad \bar{\mathcal{P}}_\mu = -\mathcal{K}_\mu\,, \qquad \bar{\mathcal{K}}_\mu = -\mathcal{P}_\mu\,, \qquad \bar{\mathcal{M}}_{\mu\nu} = \mathcal{M}_{\mu\nu}\,. \tag{1.17}$$

In terms of the two point function given by (1.11) we may define

$$\big(\mathcal{O}(x), \mathcal{O}(y)\big) \equiv \big\langle \bar{\mathcal{O}}(x)\,\mathcal{O}(y)\big\rangle = \frac{c_{\mathcal{O}\mathcal{O}}}{(1 + x^2\,y^2 - 2\,x\cdot y)^\Delta}\,, \tag{1.18}$$

so that

$$\big(\mathcal{O}(0), \mathcal{O}(0)\big) = c_{\mathcal{O}\mathcal{O}}\,. \tag{1.19}$$

It is straightforward to verify from (1.18)

$$D_x^{(\Delta)}\big(\mathcal{O}(x), \mathcal{O}(y)\big) = D_y^{(\Delta)}\big(\mathcal{O}(x), \mathcal{O}(y)\big)\,,$$
$$\partial_{x\,\mu}\big(\mathcal{O}(x), \mathcal{O}(y)\big) = K_y^{(\Delta)}{}_\mu\big(\mathcal{O}(x), \mathcal{O}(y)\big)\,,$$
$$L_{x\,\mu\nu}\big(\mathcal{O}(x), \mathcal{O}(y)\big) + L_{y\,\mu\nu}\big(\mathcal{O}(x), \mathcal{O}(y)\big) = 0\,. \tag{1.20}$$

Defining a conjugate operator by

$$\mathcal{M}_{AB}{}^{+} = -\bar{\mathcal{M}}_{AB}\,, \tag{1.21}$$

then

$$\big(\,\mathcal{O}(x), \big[\mathcal{M}_{AB}, \mathcal{O}(y)\big]\big) = \big(\,\big[\mathcal{M}_{AB}{}^{+}, \mathcal{O}(x)\big], \mathcal{O}(y)\big)\,. \tag{1.22}$$

Clearly with a scalar product defined by (1.18) $\mathcal{D}^+ = \mathcal{D}$.

2 Free Field Representations

For generalised free fields it is straightforward to construct corresponding representations of the conformal generators in terms of functional derivatives and also local conformal primary fields satisfying (1.6). For such functional representations it is convenient to adopt the notation such that for ψ, ϕ functions on $\mathbb{R}^d$ then

$$\phi\cdot\psi = \psi\cdot\phi = \int \mathrm{d}^d x\ \phi(x)\psi(x)\,, \qquad 1\cdot\phi = \int \mathrm{d}^d x\ \phi(x)\,. \tag{2.1}$$

For any functional $F[\phi]$ functional derivatives are here defined so that $\delta F[\phi] = \delta\phi \cdot \frac{\delta}{\delta\phi} F[\phi]$ and for any bilocal function $G(x,y)$ then $\phi \cdot G \cdot \psi$ is defined as expected from (2.1). It is convenient to define

$$I(x,y) = \delta^d(x-y)\,, \qquad X_\mu(x,y) = x_\mu\, \delta^d(x-y)\,, \tag{2.2}$$

so that $I \cdot \phi = \phi$. Occasionally it is helpful to transform to momentum space by taking $\tilde\varphi(p) = \int \mathrm{d}^d x\, e^{ip\cdot x} \varphi(x)$ and for any $f(x^2)$, $\tilde f(p^2) = \int \mathrm{d}^d x\, e^{ip\cdot x} f(x^2)$.

For functional representations of the conformal generators in terms of a scalar field $\varphi(x)$ on $\mathbb{R}^d$ the construction of local conformal primary operators, as in (1.5), reduces to solving functional differential equations for $\mathcal{O}[\varphi; x]$,

$$\mathcal{P}_\mu\, \mathcal{O}[\varphi;x] = \partial_{x\,\mu} \mathcal{O}[\varphi;x]\,, \qquad \mathcal{M}_{\mu\nu}\, \mathcal{O}[\varphi;x] = \big(L_{x\,\mu\nu} + S_{\mu\nu}\big)\mathcal{O}[\varphi;x]\,, \tag{2.3}$$

and

$$\mathcal{D}\, \mathcal{O}[\varphi;x] = D_x{}^{(\Delta)} \mathcal{O}[\varphi;x]\,, \quad \mathcal{K}_\mu\, \mathcal{O}[\varphi;x] = \big(K_x{}^{(\Delta)}{}_\mu + 2x_\nu\, S_{\mu\nu}\big)\mathcal{O}[\varphi;x]\,, \tag{2.4}$$

for appropriate Δ and spin. Alternatively

$$\mathcal{D}\, \mathcal{O}[\varphi;0] = \Delta\, \mathcal{O}[\varphi;0]\,, \quad \mathcal{M}_{\mu\nu}\, \mathcal{O}[\varphi;0] = S_{\mu\nu}\, \mathcal{O}[\varphi;0]\,, \quad \mathcal{K}_\mu\, \mathcal{O}[\varphi;0] = 0\,, \tag{2.5}$$

which is equivalent to (2.4) where $\mathcal{O}[\varphi;x] = \exp(x\cdot\mathcal{P})\mathcal{O}[\varphi;0]$. For an appropriately restricted class of functionals $\{\mathcal{O}[\varphi;x]\}$ (2.4) or (2.5) are assumed to be eigenvalue equations determining Δ.

Using the notation in (2.1) the generalised free field representation is given by

$$\mathcal{P}_\mu = \partial_\mu \varphi \cdot \frac{\delta}{\delta\varphi}\,, \qquad \mathcal{M}_{\mu\nu} = L_{\mu\nu}\varphi \cdot \frac{\delta}{\delta\varphi}\,, \tag{2.6}$$

as well as

$$\mathcal{D}^F = D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi}\,, \qquad \mathcal{K}^F_\mu = K^{(\delta)}{}_\mu \varphi \cdot \frac{\delta}{\delta\varphi}\,, \tag{2.7}$$

with definitions in (1.6). It is easy to see that the generators given by (2.6) and (2.7) satisfy the conformal algebra (1.1). By integration by parts in general

$$\begin{aligned} \partial_\mu\varphi \cdot \psi &= -\varphi \cdot \partial_\mu \psi\,, & L_{\mu\nu}\varphi \cdot \psi &= -\varphi \cdot L_{\mu\nu}\psi\,, \\ D^{(\delta)}\varphi \cdot \psi &= -\varphi \cdot D^{(d-\delta)}\psi\,, & K^{(\delta)}{}_\mu \varphi \cdot \psi &= -\varphi \cdot K^{(d-\delta)}{}_\mu \psi\,. \end{aligned} \tag{2.8}$$

The action of inversions on free fields φ is defined, in accord with (1.14), by

$$\bar\varphi(x) = (x^2)^{-\delta}\, \varphi(x/x^2) \qquad \Rightarrow \qquad \frac{\delta}{\delta\bar\varphi(x)} = (x^2)^{\delta-d}\, \frac{\delta}{\delta\varphi(x/x^2)}\,. \tag{2.9}$$

and hence it is easy to show from (2.7)

$$D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} = -D^{(\delta)}\bar\varphi \cdot \frac{\delta}{\delta\bar\varphi}\,, \tag{2.10}$$

and also using (2.6)

$$\partial_\mu\varphi\cdot\frac{\delta}{\delta\varphi}=-K^{(\delta)}{}_\mu\bar\varphi\cdot\frac{\delta}{\delta\bar\varphi}\,,\quad K^{(\delta)}{}_\mu\varphi\cdot\frac{\delta}{\delta\varphi}=-\partial_\mu\bar\varphi\cdot\frac{\delta}{\delta\bar\varphi}\,,\qquad L_{\mu\nu}\varphi\cdot\frac{\delta}{\delta\varphi}=L_{\mu\nu}\bar\varphi\cdot\frac{\delta}{\delta\bar\varphi}\,. \tag{2.11}$$

Hence, with the definitions (2.6) and (2.7), (2.10) and (2.11) are in accord with (1.17).

The equations (2.4) are easy to solve in the free case with the representation given by (2.6) and (2.7) if strict locality is imposed. For eigen-operators $\Phi_{n,p}[\varphi]=\mathrm{O}(\varphi^n,\partial^p)$ constructed from φ and its derivatives at the same x then

$$\begin{gathered}\mathcal{D}^F\,\Phi_{n,p}[\varphi]=D^{(\Delta^F{}_{n,p})}\,\Phi_{n,p}[\varphi]\,,\qquad \mathcal{K}^F_\mu\,\Phi_{n,p}[\varphi]=K^{(\Delta^F{}_{n,p})}{}_\mu\,\Phi_{n,p}[\varphi]\,,\\ \Delta^F{}_{n,p}=n\delta+p\,,\quad n,p=0,1,\dots\,.\end{gathered} \tag{2.12}$$

Explicitly, for just a single component scalar field, conformal primary scalar operators with up to two derivatives are given by

$$\Phi_{n,0}[\varphi]=\varphi^n\,,\qquad \Phi_{n,2}[\varphi]=-\partial^2\varphi\,\varphi^{n-1}+\frac{1}{n}\,\frac{\delta-\delta_0}{n\delta-\delta_0}\,\partial^2\varphi^n\,, \tag{2.13}$$

for

$$\delta_0=\tfrac12(d-2)\,. \tag{2.14}$$

When $\delta=\delta_0$, $\partial^2\varphi$ is a conformal primary and may be set to zero by imposing dynamical equations on φ, defining the free theory. With $\delta=\delta_0$, $\Phi_{2,2}[\varphi]=-\partial^2\varphi\,\varphi$ is a marginal operator since then $\Delta^F{}_{2,2}=d$.

The simplest primary tensorial fields with two derivatives, satisfying (1.8), are

$$\Phi_{n,2,\alpha\beta}[\varphi]=-\partial_\alpha\partial_\beta\varphi\,\varphi^{n-1}+\frac{1}{n(n\delta+1)}\Big((\delta+1)\,\partial_\alpha\partial_\beta-\frac{(n-1)\delta}{2(n\delta-\delta_0)}\,\delta_{\alpha\beta}\,\partial^2\Big)\varphi^n\,, \tag{2.15}$$

where $\Phi_{n,2,\alpha\alpha}=\Phi_{n,2}$. For there to be a conserved traceless energy momentum tensor with $\Delta=d$ it is necessary to identify $\delta=\delta_0$ and then

$$\Theta_{\alpha\beta}=\Phi_{2,2,\alpha\beta}=-\partial_\alpha\partial_\beta\varphi\,\varphi+\frac{1}{4(d-1)}\big(d\,\partial_\alpha\partial_\beta-\delta_{\alpha\beta}\,\partial^2\big)\varphi^2\,, \tag{2.16}$$

from which it follows that $\partial_\alpha\Theta_{\alpha\beta}=\Theta_{\alpha\alpha}=0$ on the equations of motion $\partial^2\varphi=0$.

In general for the representation of the conformal generators given by the first order functional operators (2.6) and (2.7) conformal primaries are easily constructed in terms of products of conformal primaries and their descendants. For two conformal primaries $\mathcal{O},\mathcal{O}'$ with scale dimensions Δ,Δ' then other scalar conformal primaries with two and four derivatives are given, using $K^{(\Delta+2)}{}_\mu\,\partial^2-\partial^2K^{(\Delta)}{}_\mu=-4(\Delta-\delta_0)\,\partial_\mu$,

$$\begin{aligned}\mathcal{O}_2&=(\Delta'-\delta_0)\,\partial^2\mathcal{O}\,\mathcal{O}'+(\Delta-\delta_0)\,\mathcal{O}\,\partial^2\mathcal{O}'-\frac{(\Delta-\delta_0)(\Delta'-\delta_0)}{\Delta+\Delta'-\delta_0}\,\partial^2(\mathcal{O}\,\mathcal{O}')\,,\\ \mathcal{O}_4&=\partial^2\mathcal{O}\,\partial^2\mathcal{O}'+\frac{\Delta'-\delta_0}{2(\Delta+1-\delta_0)}\,\partial^2\partial^2\mathcal{O}\,\mathcal{O}'+\frac{\Delta-\delta_0}{2(\Delta'+1-\delta_0)}\,\mathcal{O}\,\partial^2\partial^2\mathcal{O}'\\ &\quad-\frac{1}{\Delta+\Delta'+2-\delta_0}\,\partial^2\mathcal{O}_2-\frac{(\Delta-\delta_0)(\Delta'-\delta_0)}{2(\Delta+\Delta'-\delta_0)(\Delta+\Delta'+1-\delta_0)}\,\partial^2\partial^2(\mathcal{O}\,\mathcal{O}')\,,\end{aligned} \tag{2.17}$$

which have scale dimensions $\Delta + \Delta' + 2$, $\Delta + \Delta' + 4$ respectively. This result implies that $\Phi_{n,4}$ has a threefold degeneracy for $n \geq 4$. For $\delta = \delta_0$ and $\partial^2\varphi \to 0$ there is just one conformal primary which can be obtained starting from $\partial^2\varphi^2\partial^2\varphi^{n-2}$.

Simple constructions are also possible for operators constructed from $\mathcal{O}, \mathcal{O}'$ which have maximal spin ℓ with ℓ derivatives. If t_μ is a null vector, $t^2 = 0$, then

$$\mathcal{O}_{(\ell)}(t) = \mathcal{O}_{\alpha_1 \dots \alpha_\ell}\, t_{\alpha_1} \dots t_{\alpha_\ell} = \sum_{n=0}^{\ell} \binom{\ell}{n} \frac{(-1)^n}{(\Delta)_n (\Delta')_{\ell-n}}\, (t\cdot\partial)^n \mathcal{O}\, (t\cdot\partial)^{\ell-n}\mathcal{O}' \,, \tag{2.18}$$

defines a symmetric traceless tensor which is a conformal primary with spin ℓ and scale dimension $\Delta+\Delta'+\ell$. Of course $(t\cdot\partial)^n\mathcal{O}\,(t\cdot\partial)^{\ell-n}\mathcal{O}'$ may be expressed as a linear combination of $\mathcal{O}_{(\ell)}(t)$ and descendants $(t\cdot\partial)^r\mathcal{O}_{(\ell-r)}(t)$ for $r = 1, \dots, \ell$. For $\ell = 2$ and $\mathcal{O}, \mathcal{O}' \to \varphi$ $\mathcal{O}_{\alpha\beta}$ is proportional to $\Phi_{2,2,\alpha\beta}$ in (2.15). For $\mathcal{O} = \mathcal{O}'$, $\Delta = \Delta'$, $\mathcal{O}_{(\ell)} = 0$ for ℓ odd.

The two point function of a scalar field with scale dimension δ

$$\big\langle \varphi(x)\,\varphi(y)\big\rangle = \mathcal{G}_\delta(s)\,, \qquad s = (x-y)^2\,, \tag{2.19}$$

is determined up to a constant by the conformal identities

$$L_{\mu\nu}\,\mathcal{G}_\delta + \mathcal{G}_\delta \overleftarrow{L}_{\mu\nu} = 0\,, \quad \partial_\mu\,\mathcal{G}_\delta + \mathcal{G}_\delta\overleftarrow{\partial}_\mu = 0\,, \quad D^{(\delta)}\mathcal{G}_\delta + \mathcal{G}_\delta\overleftarrow{D}^{(\delta)} = 0\,, \quad K^{(\delta)}{}_\mu\,\mathcal{G}_\delta + \mathcal{G}_\delta\overleftarrow{K}^{(\delta)}{}_\mu = 0\,. \tag{2.20}$$

It is convenient here to take

$$\mathcal{G}_\delta(s) = \frac{\Gamma(\delta)}{(4\pi)^{\frac{1}{2}d}\Gamma(\frac{1}{2}d-\delta)} \left(\frac{4}{s}\right)^{\delta}\,, \tag{2.21}$$

which satisfies

$$\mathcal{G}_\delta \cdot \mathcal{G}_{\delta'} = \mathcal{G}_{\delta+\delta'-\frac{1}{2}d}\,, \quad -\partial^2\mathcal{G}_\delta = \mathcal{G}_{\delta+1}\,, \quad \mathcal{G}_{\frac{1}{2}d} = I\,, \quad \mathcal{G}_\delta{}^{-1} = \mathcal{G}_{d-\delta}\,, \tag{2.22}$$

and $\tilde{\mathcal{G}}_\delta(p^2) = (p^2)^{\delta-\frac{1}{2}d}$. For any local field $\Phi[\varphi;x]$ formed from φ and its derivatives at x the corresponding two point function is given by

$$\begin{aligned}\big\langle \Phi(x)\,\Phi(y)\big\rangle &= e^{\frac{\delta}{\delta\varphi}\cdot\mathcal{G}_\delta\cdot\frac{\delta}{\delta\varphi'}}\; \Phi[\varphi;x]\;\Phi[\varphi',y]\,\Big|_{\varphi,\varphi'=0} \\ &= e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathcal{G}_\delta\cdot\frac{\delta}{\delta\varphi}}\, \mathcal{N}_{\mathcal{G}_\delta}\Phi[\varphi;x]\; \mathcal{N}_{\mathcal{G}_\delta}\Phi[\varphi;y]\,\Big|_{\varphi=0}\,,\end{aligned} \tag{2.23}$$

where $\mathcal{N}_\mathcal{G}$ denotes normal ordering with respect to a symmetric kernel $\mathcal{G}(x,y)$ and is formally defined by

$$\mathcal{N}_\mathcal{G}\Phi[\varphi;x] = e^{-\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\;\Phi[\varphi;x]\,. \tag{2.24}$$

The result (2.23) can also be expressed as functional integral

$$\big\langle \Phi(x)\,\Phi(y)\big\rangle = \int \mathrm{d}[\varphi]\; e^{-\frac{1}{2}\varphi\cdot\mathcal{G}_\delta{}^{-1}\cdot\varphi}\; \mathcal{N}_{\mathcal{G}_\delta}\Phi[\varphi;x]\;\mathcal{N}_{\mathcal{G}_\delta}\Phi[\varphi;y]\,, \tag{2.25}$$

assuming a normalisation such that $\int \mathrm{d}[\varphi]\; e^{-\frac{1}{2}\varphi\cdot\mathcal{G}_\delta{}^{-1}\cdot\varphi} = 1$. The normal ordering in (2.25) for $\mathcal{G} = \mathcal{G}_\delta$ is formal since $\mathcal{G}_\delta(s)$ is singular as $s \to 0$ but in a diagrammatic expansion it removes contractions involving fields at the same point. Conformal invariance is reflected by

$$\mathcal{M}^F{}_{AB}\,\varphi\cdot\mathcal{G}_\delta{}^{-1}\cdot\varphi = 0\,, \tag{2.26}$$

and from (2.22)

$$\varphi\cdot\mathcal{G}_{\delta_0}{}^{-1}\cdot\varphi = -\varphi\cdot\partial^2\varphi\,. \tag{2.27}$$

Since $\frac{\delta}{\delta\varphi}\cdot\mathcal{G}_\delta\cdot\frac{\delta}{\delta\varphi}$ commutes with the conformal generators (2.6), (2.7) the action on conformal primaries is unmodified by normal ordering

$$\begin{aligned}\mathcal{D}^F\,\mathcal{N}_{\mathcal{G}_\delta}\Phi_{n,p}[\varphi;x] &= D^{(n\delta+p)}\,\mathcal{N}_{\mathcal{G}_\delta}\Phi_{n,p}[\varphi;x]\,,\\ \mathcal{K}^F_\mu\,\mathcal{N}_{\mathcal{G}_\delta}\Phi_{n,p}[\varphi;x] &= K^{(n\delta+p)}{}_\mu\,\mathcal{N}_{\mathcal{G}_\delta}\Phi_{n,p}[\varphi;x]\,.\end{aligned} \tag{2.28}$$

Assuming

$$\bar\Phi[\varphi;x] = \Phi[\bar\varphi,x]\,, \tag{2.29}$$

then from $\varphi\cdot\mathcal{G}_\delta{}^{-1}\cdot\varphi = \bar\varphi\cdot\mathcal{G}_\delta{}^{-1}\cdot\bar\varphi$, $\frac{\delta}{\delta\varphi}\cdot\mathcal{G}_\delta\cdot\frac{\delta}{\delta\varphi} = \frac{\delta}{\delta\bar\varphi}\cdot\mathcal{G}_\delta\cdot\frac{\delta}{\delta\bar\varphi}$, then from (2.25) and (1.18)

$$\big(\Phi(x),\Phi(y)\big) = \int \mathrm{d}[\varphi]\; e^{-\frac{1}{2}\,\varphi\cdot\mathcal{G}_\delta{}^{-1}\cdot\varphi}\;\mathcal{N}_{\mathcal{G}_\delta}\Phi[\bar\varphi;x]\,\mathcal{N}_{\mathcal{G}_\delta}\Phi[\varphi;y]\,. \tag{2.30}$$

In (2.30) we suppose $\mathrm{d}[\varphi]=\mathrm{d}[\bar\varphi]$ so as to ensure symmetry under $\varphi\leftrightarrow\bar\varphi$, the measure $\mathrm{d}[\varphi]$ then invariant under conformal transformations on φ.

3 Extensions to Non Free Theories

For framework of the exact non linear RG equations describing the flow of non polynomial actions S under a change of scale provides a method of determining non trivial fixed points, at least for for scalar theories. At the fixed points there is exact scale invariance and the linearised RG equations in the neighbourhood of the fixed point define a functional differential operator, depending on the fixed point action S_*, whose eigenvalues determine the scale dimensions of the fields. It is natural to consider the extension to conformal symmetry where there are additional functional differential operators satisfying the algebra of the conformal group. Such a discussion was undertaken some time ago by Schafer [3] in the context of the original Wilsonian exact RG equations.

With this motivation we therefore consider extensions of the free field representation to accommodate such non trivial IR fixed points. The expressions for $\mathcal{P}_\mu, \mathcal{M}_{\mu\nu}$ remain the same as in (2.6) so that rotations and translations are realised by linear action on the fields in the standard fashion. The results for the generators of scale and special conformal transformations in (2.7) are however extended to second order functional differential operators of the form

$$\mathcal{D} = D^{(\delta)}\varphi\cdot\frac{\delta}{\delta\varphi} + \frac{\delta}{\delta\varphi}S_*[\varphi]\cdot G\cdot\frac{\delta}{\delta\varphi} - \frac{1}{2}\,\frac{\delta}{\delta\varphi}\cdot G\cdot\frac{\delta}{\delta\varphi}\,, \tag{3.1}$$

and

$$\mathcal{K}_\mu = K^{(\delta)}{}_\mu\varphi\cdot\frac{\delta}{\delta\varphi} + \frac{\delta}{\delta\varphi}S_*[\varphi]\cdot F_\mu\cdot\frac{\delta}{\delta\varphi} - \frac{1}{2}\,\frac{\delta}{\delta\varphi}\cdot F_\mu\cdot\frac{\delta}{\delta\varphi}\,, \tag{3.2}$$

for $G(x,y), F_\mu(x,y)$ symmetric in x,y and $S_*[\varphi]$ an as yet undetermined functional of φ subject to

$$\mathcal{M}_{\mu\nu} S_*[\varphi] = \mathcal{P}_\mu S_*[\varphi] = 0\,. \tag{3.3}$$

This ensures that $[\mathcal{M}_{\mu\nu}, \mathcal{D}] = 0$ and also $[\mathcal{P}_\mu, \mathcal{D}]$ given in (1.1) so long as

$$L_{\mu\nu}\, G + G\, \overleftarrow{L}_{\mu\nu} = 0\,, \qquad \partial_\mu\, G + G\, \overleftarrow{\partial}_\mu = 0\,. \tag{3.4}$$

These conditions are trivially solved by taking

$$G(x,y) \to G(s)\,, \qquad s = (x-y)^2\,. \tag{3.5}$$

By considering $[\mathcal{M}_{\mu\nu}, \mathcal{K}_\sigma]$ and $[\mathcal{P}_\mu, \mathcal{K}_\nu]$ we also obtain

$$L_{\mu\nu}\, F_\sigma + F_\sigma\, \overleftarrow{L}_{\mu\nu} = -\delta_{\mu\sigma}\, F_\nu + \delta_{\nu\sigma}\, F_\mu\,, \qquad \partial_\mu\, F_\nu + F_\nu\, \overleftarrow{\partial}_\mu = 2\delta_{\mu\nu}\, G\,. \tag{3.6}$$

Here the solution is also straightforward

$$F_\mu(x,y) = (x+y)_\mu\, G(s)\,, \tag{3.7}$$

or $F_\mu = X_\mu \cdot G + G \cdot X_\mu$. Locality requires

$$G(s) \to 0 \quad \text{as} \quad s \to \infty\,, \tag{3.8}$$

where it is convenient to assume that the limit is approached faster than any inverse power of s. To a large extent G, F_μ are arbitrary but their presence is crucial in regularising the functional Laplacians in (3.1).

The form for (3.1) is of course motivated by similar expressions that arise in exact RG equations, (3.2) is then essentially determined by imposing $[\mathcal{P}_\mu, \mathcal{K}_\nu] = -2\delta_{\mu\nu}\, \mathcal{D} - 2\mathcal{M}_{\mu\nu}$.

The remaining commutators $[\mathcal{D}, \mathcal{K}_\mu] = -\mathcal{K}_\mu$ and $[\mathcal{K}_\mu, \mathcal{K}_\nu] = 0$ are non trivial in this representation. After some calculation

$$\begin{aligned}
\big[\mathcal{D}, \mathcal{K}_\mu\big] = {} & - K^{(\delta)}{}_\mu \varphi \cdot \frac{\delta}{\delta\varphi} \\
& - \frac{\delta}{\delta\varphi} S_*[\varphi] \cdot \big(D^{(\delta)} F_\mu + F_\mu \overleftarrow{D}^{(\delta)} - K^{(\delta)}{}_\mu\, G - G \overleftarrow{K}^{(\delta)}{}_\mu \big) \cdot \frac{\delta}{\delta\varphi} \\
& + \frac{1}{2}\, \frac{\delta}{\delta\varphi} \cdot \big(D^{(\delta)} F_\mu + F_\mu \overleftarrow{D}^{(\delta)} - K^{(\delta)}{}_\mu\, G - G \overleftarrow{K}^{(\delta)}{}_\mu \big) \cdot \frac{\delta}{\delta\varphi} \\
& + \frac{\delta}{\delta\varphi} E^{(\delta)}{}_{S_*}[\varphi] \cdot F_\mu \cdot \frac{\delta}{\delta\varphi} - \frac{\delta}{\delta\varphi} E^{(\delta)}{}_{S_* \mu}[\varphi] \cdot G \cdot \frac{\delta}{\delta\varphi}\,,
\end{aligned} \tag{3.9}$$

where we define

$$E^{(\delta)}{}_S[\varphi] = D^{(\delta)} \varphi \cdot \frac{\delta}{\delta\varphi} S[\varphi] + \frac{1}{2}\, \frac{\delta}{\delta\varphi} S[\varphi] \cdot G \cdot \frac{\delta}{\delta\varphi} S[\varphi] - \frac{1}{2}\, \frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi} S[\varphi]\,, \tag{3.10a}$$

$$E^{(\delta)}{}_{S\mu}[\varphi] = K^{(\delta)}{}_\mu \varphi \cdot \frac{\delta}{\delta\varphi} S[\varphi] + \frac{1}{2}\, \frac{\delta}{\delta\varphi} S[\varphi] \cdot F_\mu \cdot \frac{\delta}{\delta\varphi} S[\varphi] - \frac{1}{2}\, \frac{\delta}{\delta\varphi} \cdot F_\mu \cdot \frac{\delta}{\delta\varphi} S[\varphi]\,. \tag{3.10b}$$

Using (3.5) and (3.7) it is easy to show that

$$D^{(\delta)} F_\mu + F_\mu \overleftarrow{D}^{(\delta)} - K^{(\delta)}{}_\mu \, G - G \overleftarrow{K}^{(\delta)}{}_\mu = F_\mu \, . \tag{3.11}$$

Applying this in (3.9) then $[\mathcal{D}, \mathcal{K}_\mu] = -\mathcal{K}_\mu$ follows so long as we require $S_*[\varphi]$ to satisfy

$$E^{(\delta)}{}_{S_*}[\varphi] = C \, , \qquad E^{(\delta)}{}_{S_* \mu}[\varphi] = C_\mu \, , \tag{3.12}$$

for C, C_μ independent of φ.

In a similar fashion

$$\begin{aligned} \big[\mathcal{K}_\mu, \mathcal{K}_\nu\big] = & - \frac{\delta}{\delta\varphi} S_*[\varphi] \cdot \big(K^{(\delta)}{}_\mu \, F_\nu + F_\nu \overleftarrow{K}^{(\delta)}{}_\mu - K^{(\delta)}{}_\nu \, F_\mu - F_\mu \overleftarrow{K}^{(\delta)}{}_\nu \big) \cdot \frac{\delta}{\delta\varphi} \\ & + \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot \big(K^{(\delta)}{}_\mu \, F_\nu + F_\nu \overleftarrow{K}^{(\delta)}{}_\mu - K^{(\delta)}{}_\nu \, F_\mu - F_\mu \overleftarrow{K}^{(\delta)}{}_\nu \big) \cdot \frac{\delta}{\delta\varphi} \\ & + \frac{\delta}{\delta\varphi} E^{(\delta)}{}_{S_* \mu}[\varphi] \cdot F_\nu \cdot \frac{\delta}{\delta\varphi} - \frac{\delta}{\delta\varphi} E^{(\delta)}{}_{S_* \nu}[\varphi] \cdot F_\mu \cdot \frac{\delta}{\delta\varphi} \, , \end{aligned} \tag{3.13}$$

so that the conformal algebra requires, as well as (3.12).

$$K^{(\delta)}{}_\mu \, F_\nu + F_\nu \overleftarrow{K}^{(\delta)}{}_\mu = K^{(\delta)}{}_\nu \, F_\mu + F_\mu \overleftarrow{K}^{(\delta)}{}_\nu \, . \tag{3.14}$$

This follows directly from (3.7).

With the definitions (3.10) the equations (3.12) are assumed to determine $S_*[\varphi]$ at a fixed point with conformal symmetry subject to conditions on the functional form of $S_*[\varphi]$. If $E^{(\delta)}{}_{S_*}[\varphi] = C$ without $E^{(\delta)}{}_{S_* \mu}[\varphi] = C_\mu$ at the same time the critical point determined by S_* is just scale invariant. Note that from (3.3) $\mathcal{P}_\mu E^{(\delta)}{}_{S_*}[\varphi] = \mathcal{M}_{\mu\nu} E^{(\delta)}{}_{S_*}[\varphi] = 0$ and also

$$\mathcal{P}_\mu E^{(\delta)}{}_{S_* \nu}[\varphi] = -2\delta_{\mu\nu} \, E^{(\delta)}{}_{S_*}[\varphi] \, , \qquad \mathcal{M}_{\mu\nu} E^{(\delta)}{}_{S_* \sigma}[\varphi] = \delta_{\mu\sigma} \, E^{(\delta)}{}_{S_* \nu}[\varphi] - \delta_{\nu\sigma} \, E^{(\delta)}{}_{S_* \mu}[\varphi] \, . \tag{3.15}$$

The integrability conditions necessary to consistently solve (3.12) follow from

$$\mathcal{K}_\mu E^{(\delta)}{}_{S_*}[\varphi] - \mathcal{D} E^{(\delta)}{}_{S_* \mu}[\varphi] = E^{(\delta)}{}_{S_* \mu}[\varphi] \, , \qquad \mathcal{K}_\mu E^{(\delta)}{}_{S_* \nu}[\varphi] - \mathcal{K}_\nu E^{(\delta)}{}_{S_* \mu}[\varphi] = 0 \, . \tag{3.16}$$

The crucial conditions which are presumed to ensure that (3.12) has relevant solutions are that $S_*[\varphi]$ should be essentially local. This is equivalent to requiring that in a functional Taylor expansion of $S_*[\varphi]$ the coefficient functions should all fall off rapidly as any mutual separations become large. The existence of solutions $S_*[\varphi]$ subject to this condition are only possible for particular choices of δ and are then restricted to a discrete set although in each case $S_*[\varphi]$ may depend on one or more continuously variable parameters.

At a scale invariant fixed point $E^{(\delta)}{}_{S_*}[\varphi] = 0$. In this case (3.15), (3.16) imply

$$\mathcal{D} E^{(\delta)}{}_{S_* \mu}[\varphi] = -E^{(\delta)}{}_{S_* \mu}[\varphi] \, , \qquad \mathcal{P}_\mu E^{(\delta)}{}_{S_* \nu}[\varphi] = 0 \, . \tag{3.17}$$

If $\mathcal{D}$ does not have non trivial translation invariant eigenfunctionals with eigenvalue -1 then $E^{(\delta)}{}_{S_* \mu}[\varphi] = 0$ in which case, as recognised by Delamotte *et al* [7], the theory is conformally invariant.

3.1 Examples of Local Primary Operators

Although finding conformal primary operators and their associated scaling dimensions by solving, for the representations (3.1) and (3.2), (2.4) becomes a non trivial functional eigenvalue problem assuming $\mathcal{O}[\varphi; x]$ is quasi-local in the sense of depending on φ at points in the neighbourhood of x. For a few special cases the operators are simply given in terms of S_*. From (3.10) it is easy to see that

$$\begin{aligned} \frac{\delta}{\delta\varphi} E^{(\delta)}{}_{S_*}[\varphi] &= \mathcal{D}\,\frac{\delta}{\delta\varphi} S_*[\varphi] - D^{(d-\delta)}\,\frac{\delta}{\delta\varphi} S_*[\varphi]\,, \\ \frac{\delta}{\delta\varphi} E^{(\delta)}{}_{S_*\mu}[\varphi] &= \mathcal{K}_\mu\,\frac{\delta}{\delta\varphi} S_*[\varphi] - K^{(d-\delta)}{}_\mu \frac{\delta}{\delta\varphi} S_*[\varphi]\,, \end{aligned} \tag{3.18}$$

as well as from (3.3)

$$\mathcal{M}_{\mu\nu}\,\frac{\delta}{\delta\varphi} S_*[\varphi] = L_{\mu\nu} \frac{\delta}{\delta\varphi} S_*[\varphi]\,, \qquad \mathcal{P}_\mu\,\frac{\delta}{\delta\varphi} S_*[\varphi] = \partial_\mu \frac{\delta}{\delta\varphi} S_*[\varphi]\,. \tag{3.19}$$

Hence so long as (3.12) are satisfied

$$\Phi_R[\varphi] = \frac{\delta}{\delta\varphi} S_*[\varphi]\,, \tag{3.20}$$

is a quasi-local scalar conformal primary operator which has scaling dimension $\Delta = d - \delta$.

Furthermore if

$$\Phi_\varphi[\varphi] = \varphi + \mathcal{H} \cdot \frac{\delta}{\delta\varphi} S_*[\varphi]\,, \tag{3.21}$$

then, using (3.18), Φ_φ is a conformal primary operator with $\Delta = \delta$,

$$\mathcal{D}\,\Phi_\varphi = D^{(\delta)}\,\Phi_\varphi\,, \qquad \mathcal{K}_\mu\,\Phi_\varphi = K^{(\delta)}{}_\mu \Phi_\varphi\,, \tag{3.22}$$

so long as $\mathcal{H}(x, y)$ satisfies

$$D^{(\delta)} \mathcal{H} + \mathcal{H} \overleftarrow{D}^{(\delta)} = G\,, \qquad K^{(\delta)}{}_\mu\,\mathcal{H} + \mathcal{H} \overleftarrow{K}^{(\delta)}{}_\mu = F_\mu\,, \tag{3.23}$$

and we assume also

$$L_{\mu\nu}\,\mathcal{H} + \mathcal{H}\,\overleftarrow{L}_{\mu\nu} = 0\,, \qquad \partial_\mu\,\mathcal{H} + \mathcal{H}\,\overleftarrow{\partial}_\mu = 0\,. \tag{3.24}$$

This ensures that (3.3) implies

$$\mathcal{M}_{\mu\nu} \Phi_\varphi = L_{\mu\nu} \Phi_\varphi\,, \qquad \mathcal{P}_\mu \Phi_\varphi = \partial_\mu \Phi_\varphi\,. \tag{3.25}$$

The rotational and translation invariance constraint (3.24) is trivially satisfied so long as $\mathcal{H}(x, y) \equiv \mathcal{H}(s)$, $s = (x - y)^2$, and then (3.23) reduces to just

$$s\,\mathcal{H}'(s) + \delta\,\mathcal{H}(s) = \tfrac{1}{2}\,G(s)\,. \tag{3.26}$$

The requirement that Φ_φ is quasi-local then necessitates that the solution $\mathcal{H}(s)$, should fall off rapidly for large s, or $\tilde{\mathcal{H}}(p^2)$ is regular as $p^2 \to 0$. This condition may be achieved by taking

$$\mathcal{H}(s) = -\frac{1}{2s^\delta}\int_s^\infty \mathrm{d}u\, u^{\delta-1} G(u)\,, \qquad \tilde{\mathcal{H}}(p^2) = -\frac{1}{2(p^2)^{\frac{1}{2}d-\delta}}\int_0^{p^2} \mathrm{d}x\, x^{\frac{1}{2}d-\delta-1}\tilde{G}(x)\,. \tag{3.27}$$

Writing

$$\mathcal{H}(s) = -k\,\mathcal{G}_\delta(s) + \mathcal{G}(s)\,, \tag{3.28}$$

then from the integral representation in (3.27) if $G(u)$ is analytic at $u = 0$

$$\mathcal{G}(s) \underset{s\to 0}{\sim} \frac{1}{2}\sum_{n\geq 0}\frac{G^{(n)}(0)}{n!\,(\delta+n)}\, s^n\,, \qquad k = \frac{1}{2}\int_0^\infty \mathrm{d}x\, x^{\frac{1}{2}d-\delta-1}\tilde{G}(x)\,, \tag{3.29}$$

for $G^{(n)}$ the n'th derivative. Since we assume $G(u) > 0$ then $k > 0$. We also assume here that $\delta < \frac{1}{2}d$ so that the integral in (3.21) is non singular.

In quantum field theories local operators are determined modulo the equations of motion, operators which vanish subject to the equations of motion of the theory are redundant. In the framework described here redundant quasi-local operators $\mathcal{O}_\psi(x)$ are defined in terms bilocal functionals $\psi(x,y)$ which fall off rapidly for large $(x-y)^2$ and are of the form

$$\mathcal{O}_\psi = \psi\cdot\frac{\delta}{\delta\varphi}S_* - \psi\cdot\frac{\overleftarrow{\delta}}{\delta\varphi}\,, \tag{3.30}$$

or $\mathcal{O}_\psi\, e^{-S_*} = -\big(\psi\, e^{-S_*}\big)\cdot\frac{\overleftarrow{\delta}}{\delta\varphi}$. For operators of the form (3.30)

$$(\mathcal{P}_\mu - \partial_\mu)\mathcal{O}_\psi = \mathcal{O}_{\mathcal{P}_\mu\psi - \partial_\mu\psi - \psi\overleftarrow{\partial}_\mu}\,, \qquad (\mathcal{M}_{\mu\nu} - L_{\mu\nu})\mathcal{O}_\psi = \mathcal{O}_{\mathcal{M}_{\mu\nu}\psi - L_{\mu\nu}\psi - \psi\overleftarrow{L}_{\mu\nu}}\,, \tag{3.31}$$

and also, using the explicit forms in (3.1), (3.2),

$$\big(\mathcal{D} - D^{(\Delta)}\big)\mathcal{O}_\psi = \mathcal{O}_{\mathcal{D}\,\psi - D^{(\Delta)}\psi - \psi\overleftarrow{D}^{(\delta)}}\,, \qquad \big(\mathcal{K}_\mu - K^{(\Delta)}{}_\mu\big)\mathcal{O}_\psi = \mathcal{O}_{\mathcal{K}_\mu\,\psi - K^{(\Delta)}{}_\mu\psi - \psi\overleftarrow{K}^{(\delta)}{}_\mu}\,. \tag{3.32}$$

By virtue of (3.31), (3.32) the space of redundant operators $\{\mathcal{O}_\psi\}$ forms an invariant subspace under the action of the conformal generators. If ψ is assumed to satisfy

$$\mathcal{P}_\mu\,\psi(x,y) = \big(\partial_{x\,\mu} + \partial_{y\,\mu}\big)\psi(x,y)\,, \qquad \mathcal{M}_{\mu\nu}\,\psi(x,y) = \big(L_{x\,\mu\nu} + L_{y\,\mu\nu}\big)\psi(x,y)\,, \tag{3.33}$$

then any such $\mathcal{O}_\psi$ satisfies $\mathcal{P}_\mu\,\mathcal{O}_\psi = \partial_\mu\mathcal{O}_\psi$, $\mathcal{M}_{\mu\nu}\,\mathcal{O}_\psi = L_{\mu\nu}\mathcal{O}_\psi$ and if

$$\big(\mathcal{D}\; - D^{(\Delta)}\big)\psi - \psi\overleftarrow{D}^{(\delta)} = \big(\mathcal{K}_\mu - K^{(\Delta)}{}_\mu\big)\psi - \psi\overleftarrow{K}^{(\delta)}{}_\mu = 0\,, \tag{3.34}$$

for some choice of Δ then $\mathcal{O}_\psi$ is a redundant conformal primary local operator with scale dimension Δ. Redundant conformal primaries do not belong to the spectrum of mutually local conformal primaries in a CFT.

Quasi-local solutions of (3.34), as well as (3.33) may be obtained for any scalar primary operator $\mathcal{O}$ satisfying (2.4) as well as (2.3) by taking

$$\psi_{\mathcal{O}}(x,y) = \delta^d(x-y)\,\mathcal{O}(x)\,. \tag{3.35}$$

In this case

$$\mathcal{O}'(x) = \mathcal{O}_{\psi_\mathcal{O}}(x) = \mathcal{O}(x)\frac{\delta}{\delta\varphi(x)}S_*[\varphi] - \frac{\delta}{\delta\varphi(x)}O(x)\,, \tag{3.36}$$

is a redundant conformal primary with $\Delta_{\mathcal{O}'} = \Delta_\mathcal{O} + d - \delta$. To verify this we use

$$\begin{aligned} D_x{}^{(\Delta_\mathcal{O}+d-\delta)}\psi(x,y) + \psi(x,y)\,\overleftarrow{D}_y{}^{(\delta)} &= \big(D_x{}^{(\Delta_\mathcal{O})}\mathcal{O}(x)\big)\,\delta^d(x-y)\,, \\ K_x{}^{(\Delta_\Phi+d-\delta)}{}_\mu\,\psi(x,y) + \psi(x,y)\,\overleftarrow{K}_y{}^{(\delta)}{}_\mu &= \big(K_x{}^{(\Delta_\mathcal{O})}{}_\mu\,\mathcal{O}(x)\big)\,\delta^d(x-y)\,. \end{aligned} \tag{3.37}$$

As particular examples taking $\mathcal{O} = 1$, $\Delta_1 = 0$ gives Φ_R while

$$\psi_Z(x,y) = \delta^d(x-y)\,\Phi_\varphi(x)\,, \tag{3.38}$$

generates a redundant marginal operator $Z = \mathcal{O}_{\psi_Z}$ as in (3.36)

$$Z(x) = \Phi_\varphi(x)\frac{\delta}{\delta\varphi(x)}S_*[\varphi] - \frac{\delta}{\delta\varphi(x)}\big(\Phi_\varphi(x) - \varphi(x)\big)\,, \tag{3.39}$$

where φ is subtracted from Φ_δ in the second term to remove the singular $\delta^d(0)$ contribution otherwise resulting from the functional derivative. Z satisfies

$$\mathcal{D}\,Z = D^{(d)}\,Z\,, \qquad \mathcal{K}_\mu\,Z = K^{(d)}{}_\mu\,Z\,. \tag{3.40}$$

From (2.4), for any quasi-local primary functional $\mathcal{O}[\varphi;x] = \mathcal{O}(x)$ the corresponding functional $1\cdot\mathcal{O} = \int \mathrm{d}^d x\, O(x)$ satisfies

$$\mathcal{D}\,1\cdot\mathcal{O} = (\Delta - d)\,1\cdot\mathcal{O}\,, \tag{3.41}$$

and

$$\mathcal{K}_\mu\,1\cdot\mathcal{O} = 2(\Delta-d)\,1\cdot X_\mu\cdot\mathcal{O}\,, \qquad 1\cdot X_\mu\cdot\mathcal{O} = \int \mathrm{d}^d x\;x_\mu\,\mathcal{O}(x)\,. \tag{3.42}$$

Hence from (3.40) $1\cdot Z$ is an exact zero mode. In general for marginal operators with $\Delta = d$ we may let $S_* \to S_* + \epsilon\,1\cdot\mathcal{O}$ for infinitesimal ϵ. This deformation may in some cases be integrated so that S_* depends on a continuous parameter corresponding to a line of fixed points.

3.2 Energy Momentum Tensor

The energy momentum tensor $\Theta_{\mu\nu}$ is a local conformal primary with $\Delta = d$ and spin two, in terms of which the conformal generators can be constructed. Nevertheless in a general CFT there is no general reason for a local energy momentum tensor to be present, as illustrated by the example of generalised free fields constructed in terms of an elementary free scalar φ with dimension $\delta \neq \frac{1}{2}(d-2)$. Such mean field theories arise if the elementary theory has long range interactions which violate the assumptions of a quasi-local underlying S_*. Recently constructions for the energy momentum tensor in exact RG frameworks have been described by Rosten [8] and Sonoda [9]. We extend such discussions to the framework of functional representations for the conformal generators presented here using the redundant operator Z as well as a corresponding vector operator V_ν. These play a similar role to equation of motion operators E, E_μ introduced in perturbative treatments [18].

The redundant quasi-local vector operator is defined by

$$V_\nu = \mathcal{O}_{\psi_{V,\nu}} , \qquad \psi_{V,\nu}(x,y) = \delta^d(x-y)\,\partial_\nu \Phi_\varphi(x) \, . \tag{3.43}$$

With this definition of $\psi_{V,\nu}$

$$\begin{aligned} \mathcal{D}\,\psi_{V,\nu}(x,y) &= D_x{}^{(d+1)}\psi_{V,\nu}(x,y) + \psi_{V,\nu}(x,y)\,\overleftarrow{D}_y{}^{(\delta)} \, , \\ \mathcal{K}_\mu\,\psi_{V,\nu}(x,y) &= K_x{}^{(d+1)}{}_\mu\,\psi_{V,\nu}(x,y) + \psi_{V,\nu}(x,y)\,\overleftarrow{K}_y{}^{(\delta)}{}_\mu + 2\delta\,\delta_{\mu\nu}\,\psi_Z(x,y) \, , \end{aligned} \tag{3.44}$$

so that

$$\mathcal{D}\,V_\nu = D^{(d+1)}V_\nu \, , \qquad \mathcal{K}_\mu\,V_\nu = K^{(d+1)}{}_\mu\,V_\nu + 2\delta\,\delta_{\mu\nu}\,Z \, . \tag{3.45}$$

For $\Theta_{\mu\nu}$ a conformal primary of scale dimension d, so that $\mathcal{D}\,\Theta_{\mu\nu} = D^{(d)}\Theta_{\mu\nu}$, $\mathcal{K}_\sigma\,\Theta_{\mu\nu} = K^{(d)}{}_\sigma\Theta_{\mu\nu}$, it follows that

$$\mathcal{D}\,\partial_\sigma\Theta_{\sigma\nu} = D^{(d+1)}\partial_\sigma\Theta_{\sigma\nu} \, , \qquad \mathcal{K}_\mu\,\partial_\sigma\Theta_{\sigma\nu} = K^{(d+1)}{}_\mu\,\partial_\sigma\Theta_{\sigma\nu} + 2\,\delta_{\mu\nu}\,\Theta_{\rho\rho} \, . \tag{3.46}$$

Comparison with (3.45) suggests the identifications [8, 9]

$$\partial_\mu\Theta_{\mu\nu} = -V_\nu \, , \qquad \Theta_{\mu\mu} = -\delta\,Z \, , \tag{3.47}$$

although any expression for $\Theta_{\mu\nu}$ has an inherent arbitrariness up to redundant contributions so that

$$\Theta_{\mu\nu} \sim \Theta_{\mu\nu} + b\,\delta_{\mu\nu}\,Z \, , \tag{3.48}$$

for any b. Assuming (3.47) requires various consistency conditions. First we have

$$\begin{aligned} \int \mathrm{d}^d x\; V_\nu = {} & \int \mathrm{d}^d x\; \partial_\nu\varphi(x) \frac{\delta S_*}{\delta\varphi(x)} \\ & + \int \mathrm{d}^d x\,\mathrm{d}^d y\, \left(\frac{\delta S_*}{\delta\varphi(x)}\,\partial_\nu\mathcal{H}(x,y)\,\frac{\delta S_*}{\delta\varphi(y)} - \frac{\delta}{\delta\varphi(x)}\,\partial_\nu\mathcal{H}(x,y)\,\frac{\delta}{\delta\varphi(y)} S_* \right) \\ = {} & 0 \, , \end{aligned} \tag{3.49}$$

which follows from translation invariance of S_* in (3.3) and also from the antisymmetry of $\partial_\nu\mathcal{H}(x,y)$ following from (3.24). In a similar fashion

$$\int \mathrm{d}^d x\; (x_\mu V_\nu - x_\nu V_\mu) = 0 \, , \tag{3.50}$$

since in the corresponding equation to (3.49) $\partial_\nu\varphi \to L_{\mu\nu}\varphi$, $\partial_\nu\mathcal{H} \to L_{\mu\nu}\mathcal{H}$ which vanishes as a consequence of rotational invariance of S_*, as given by (3.3), and also (3.24) implies $L_{\mu\nu}\mathcal{H}$ is antisymmetric. This condition is necessary for $\int \mathrm{d}^d x\,\Theta_{\mu\nu} = \int \mathrm{d}^d x\,\Theta_{\nu\mu}$. We may also obtain

$$\int \mathrm{d}^d x\; x_\nu V_\nu + \delta \int \mathrm{d}^d x\; Z(x) = E^{(\delta)}{}_{S_*} \to 0 \, , \tag{3.51}$$

using now in (3.49) $\partial_\nu\varphi \to (D^{(\delta)} - \delta)\varphi$, $\partial_\nu\mathcal{H} \to (D^{(\delta)} - \delta)\mathcal{H}$ and, from (3.23), in (3.49) $D^{(\delta)}\mathcal{H} \to \frac{1}{2}G$. This is consistent with (3.47), since $\int \mathrm{d}^d x\, x_\nu \partial_\mu \Theta_{\mu\nu} + \int \mathrm{d}^d x\, \Theta_{\mu\mu} = 0$. Finally we have, from (3.10b) and (3.23),

$$\int \mathrm{d}^d x \,\big(- x^2\, V_\nu + 2\, x_\nu\, x_\mu V_\mu\big) + 2\delta \int \mathrm{d}^d x\; x_\nu\; Z(x) = E^{(\delta)}{}_{S_*\nu} \to 0\,, \tag{3.52}$$

which corresponds to the identity $\int \mathrm{d}^d x\,(-x^2 \partial_\mu \Theta_{\mu\nu} + 2 x_\nu x_\mu\, \partial_\sigma \Theta_{\sigma\mu}) + 2\int \mathrm{d}^d x\, x_\nu\, \Theta_{\mu\mu} = 0$.

As a consequence of the above

$$T_{\mu\nu} = \int \mathrm{d}^d x\; x_\mu V_\nu = T_{\nu\mu}\,, \tag{3.53}$$

is a marginal operator,

$$\mathcal{P}_\sigma\, T_{\mu\nu} = \mathcal{D}\, T_{\mu\nu} = 0\,, \qquad \mathcal{M}_{\sigma\rho}\, T_{\mu\nu} = \delta_{\sigma\mu}\, T_{\rho\nu} - \delta_{\rho\mu}\, T_{\sigma\nu} - \delta_{\rho\nu}\, T_{\mu\sigma} + \delta_{\sigma\nu}\, T_{\mu\rho}\,. \tag{3.54}$$

There is in general no guarantee of the existence of a quasi-local conformal primary $\Theta_{\mu\nu}$ satisfying (3.47) but if valid, since V_ν, Z are redundant operators, this is consistent with the conservation and tracelessness of the energy momentum tensor in conformal field theories.

3.3 Multi-local Functionals

To extend the discussion of conformal primary operators to a prescription for determining their correlation functions it is necessary to consider functionals depending on two or more points.This is non trivial since the exact RG equation is second order in functional derivatives. A related discussion is given in [10]. For the two point case (2.4) is extended for two spinless operators to

$$\begin{aligned}
\mathcal{P}_\mu\, \mathcal{E}_{12}[\varphi; x, y] &= \big(\partial_{x\,\mu} + \partial_{y\,\mu}\big)\mathcal{E}_{12}[\varphi; x, y]\,,\\
\mathcal{M}_{\mu\nu}\, \mathcal{E}_{12}[\varphi; x, y] &= \big(L_{x\,\mu\nu} + L_{y\,\mu\nu}\big)\mathcal{E}_{12}[\varphi; x, y]\,,\\
\mathcal{D}\, \mathcal{E}_{12}[\varphi; x, y] &= \big(D_x{}^{(\Delta_1)} + D_y{}^{(\Delta_2)}\big)\mathcal{E}_{12}[\varphi; x, y]\,,\\
\mathcal{K}_\mu\, \mathcal{E}_{12}[\varphi; x, y] &= \big(K_x{}^{(\Delta_1)}{}_\mu + K_y{}^{(\Delta_2)}{}_\mu\big)\mathcal{E}_{12}[\varphi; x, y]\,,
\end{aligned} \tag{3.55}$$

with obvious generalisations for three or more points. Since $\mathcal{D}, \mathcal{K}_\mu$ involve second order functional derivatives with the representation (3.1), (3.2) $\mathcal{E}_{12}[\varphi; x, y] \neq \mathcal{O}_1[\varphi, x]\, \mathcal{O}_2[\varphi, y]$ but locality is imposed by requiring $\mathcal{E}_{12}$ factorises for large separations so that

$$\mathcal{E}_{12}[\varphi; x, y] = \mathcal{O}_1[\varphi; x]\, \mathcal{O}_2[\varphi; y] + \mathrm{o}\big(s^{-\max(\Delta_1,\Delta_2)}\big) \quad \text{as} \quad s = (x-y)^2 \to \infty\,, \tag{3.56}$$

assuming $\mathcal{O}_1, \mathcal{O}_2$ are here conformal primary operators with scale dimensions Δ_1, Δ_2. This boundary conditions excludes the freedom to add trivial homogeneous solutions of (3.55) for $\Delta_1 = \Delta_2$ and zero left hand side. However, if $\Delta_1 = \Delta_2$, at short distances $\mathcal{E}_{12}$ has a leading φ-independent contribution such that we may define the correlator of $\mathcal{O}_1, \mathcal{O}_2$ by

$$\mathcal{E}_{12}[\varphi; x, y] \sim \frac{c_{12}}{s^{\,\Delta_1}}\, \delta_{\Delta_1\Delta_2} = \big\langle \mathcal{O}_1(x)\, \mathcal{O}_2(y)\big\rangle\,, \qquad x \sim y\,. \tag{3.57}$$

where the explicit form is determined by conformal identities (3.55) and of course the restriction to $\Delta_1 = \Delta_2$ in (3.57) depends on the special conformal identity arising from $\mathcal{K}_\mu \, \mathcal{E}_{12}$. The normalisation of $\mathcal{E}_{12}$ is determined by the large distance form in (3.56) although of course the normalisation of $\mathcal{O}_1, \mathcal{O}_2$ may be chosen at will. The short distance form in (3.57) is extended in appendix B to a form equivalent to the operator product expansion so that $\mathcal{E}_{12}$ in principle determines the coefficients which determine the overall scale of three point functions $\langle \mathcal{O}_1 \, \mathcal{O}_2 \, \mathcal{O} \rangle$. It is of course crucial that for physical operators two point functions are positive although this does not apply if they are redundant.

Exact results are also possible for any bilocal functional containing Φ_φ since the conformal identities are satisfied, using (3.59), by

$$\mathcal{E}_{\Phi_\varphi \mathcal{O}}(x,y) = \Phi_\varphi(x) \, \mathcal{O}(y) - \mathcal{H}(x) \cdot \frac{\delta}{\delta\varphi} \mathcal{O}(y) \, , \tag{3.58}$$

defining $\mathcal{H}(x) \cdot \frac{\delta}{\delta\varphi} = \int \mathrm{d}^d u \; \mathcal{H}((x-u)^2) \frac{\delta}{\delta\varphi(u)}$ with $\mathcal{H}$ given by (3.27). To verify (3.58) we use

$$\begin{aligned} \left[\mathcal{D}, \, \frac{\delta}{\delta\varphi} \right] &= D^{(d-\delta)} \frac{\delta}{\delta\varphi} - S_*{}^{(2)} \cdot G \cdot \frac{\delta}{\delta\varphi} \, , \\ \left[\mathcal{K}_\mu, \, \frac{\delta}{\delta\varphi} \right] &= K^{(d-\delta)}{}_\mu \frac{\delta}{\delta\varphi} - S_*{}^{(2)} \cdot F_\mu \cdot \frac{\delta}{\delta\varphi} \, , \end{aligned} \tag{3.59}$$

to evaluate the action of the conformal generators on $\frac{\delta}{\delta\varphi}\mathcal{O}$. Here the bilocal functional $S_*{}^{(2)}$ is defined by

$$S_*{}^{(2)}[\varphi; x, y] = \frac{\delta^2}{\delta\varphi(x) \, \delta\varphi(y)} \, S_*[\varphi] \, . \tag{3.60}$$

From (3.18) this satisfies the identities

$$\begin{aligned} \mathcal{D} \, S_*{}^{(2)} &= D^{(d-\delta)} \, S_*{}^{(2)} + S_*{}^{(2)} \overleftarrow{D}^{(d-\delta)} - S_*{}^{(2)} \cdot G \cdot S_*{}^{(2)} \, , \\ \mathcal{K}_\mu \, S_*{}^{(2)} &= K^{(d-\delta)}{}_\mu \, S_*{}^{(2)} + S_*{}^{(2)} \overleftarrow{K}^{(d-\delta)}{}_\mu - S_*{}^{(2)} \cdot F_\mu \cdot S_*{}^{(2)} \, , \end{aligned} \tag{3.61}$$

and since S_* is quasi-local then for large or small separations, when the G, F_μ terms in (3.61) can be neglected,

$$S_*{}^{(2)}[\varphi; x, y] \sim 0 \quad \text{as} \quad s \to \infty \, , \qquad S_*{}^{(2)}[\varphi; x, y] \sim c \, \mathcal{G}_\delta{}^{-1}(s) \quad \text{as} \quad s \to 0 \, . \tag{3.62}$$

Multi-local operators are redundant if they can be expressed in the form (3.30). In this case for two points the bilocal functional $\mathcal{E}_{12}$ is determined in terms of $\psi_{12}(x,y;z)$ and has the form, with $\mathcal{E}_{\psi_{12}}[\varphi; x, y] = \mathcal{E}_{\psi_{12}}(x,y)$,

$$\mathcal{E}_{\psi_{12}}(x,y) = \int \mathrm{d}^d z \, \left(\psi_{12}(x,y;z) \frac{\delta S_*}{\delta\varphi(z)} - \psi_{12}(x,y;z) \frac{\overleftarrow{\delta}}{\delta\varphi(z)} \right) . \tag{3.63}$$

For this to satisfy (3.55) there are corresponding equations for ψ_{12} analogous to (3.33), (3.34),

$$\begin{aligned} \big(\mathcal{D} \; - D_x{}^{(\Delta_1)} - D_y{}^{(\Delta_2)}\big) \psi_{12}(x,y;z) &= \psi_{12}(x,y;z) \overleftarrow{D}_z{}^{(\delta)} \, , \\ \big(\mathcal{K}_\mu - K_x{}^{(\Delta_1)}{}_\mu - K_y{}^{(\Delta_2)}{}_\mu\big) \psi_{12}(x,y;z) &= \psi_{12}(x,y;z) \overleftarrow{K}_z{}^{(\delta)}{}_\mu \, . \end{aligned} \tag{3.64}$$

For $\mathcal{O}_1 \to \Phi_R$ with $\Delta_1 \to d-\delta$ and $\mathcal{O}_2 \to \mathcal{O}$, $\Delta_2 \to \Delta$ it is sufficient to take just

$$\psi_{\Phi_R\mathcal{O}}(x,y;z) = \delta^d(x-z)\,\mathcal{O}(y) \quad \Rightarrow \quad \mathcal{E}_{\Phi_R\mathcal{O}}(x,y) = \Phi_R(x)\,\mathcal{O}(y) - \frac{\delta}{\delta\varphi(x)}\mathcal{O}(y)\,. \tag{3.65}$$

Taking $\mathcal{O}_1 \to Z, \Delta_1 \to d$ as well as $\mathcal{O}_2 \to \mathcal{O}$, $\Delta_2 \to \Delta$ the equations (3.64) for $\psi_{Z\mathcal{O}}$ are solved using (3.59), extending (3.38), if

$$\psi_{Z\mathcal{O}}(x,y;z) = \delta^d(x-z)\Big(\Phi_\varphi(x)\,\mathcal{O}(y) - \mathcal{H}(x)\cdot\frac{\delta}{\delta\varphi}\mathcal{O}(y)\Big)\,. \tag{3.66}$$

In a similar fashion to (3.66) for the redundant vector operator defined in (3.43)

$$\psi_{V\mathcal{O},\nu}(x,y;z) = \delta^d(x-z)\Big(\partial_\nu\Phi_\varphi(x)\,\mathcal{O}(y) - \partial_\nu\mathcal{H}(x)\cdot\frac{\delta}{\delta\varphi}\mathcal{O}(y)\Big)\,. \tag{3.67}$$

(3.66) and (3.67) determine

$$\begin{aligned}\mathcal{E}_{Z\mathcal{O}}(x,y) &= Z(x)\,\mathcal{O}(y) - \mathcal{N}(x)\,\mathcal{O}(y)\,,\\ \mathcal{E}_{V\mathcal{O},\nu}(x,y) &= V_\nu(x)\,\mathcal{O}(y) - \mathcal{V}_\nu(x)\,\mathcal{O}(y)\,,\end{aligned} \tag{3.68}$$

where $\mathcal{N}, \mathcal{V}_\nu$ are local functional differential operators defined by

$$\begin{aligned}\mathcal{N}(x) &= \varphi(x)\frac{\delta}{\delta\varphi(x)} + \frac{\delta S_*}{\delta\varphi(x)}\,\mathcal{H}(x)\cdot\frac{\delta}{\delta\varphi} + \mathcal{H}(x)\cdot\frac{\delta S_*}{\delta\varphi}\,\frac{\delta}{\delta\varphi(x)} - \frac{\delta}{\delta\varphi(x)}\,\mathcal{H}(x)\cdot\frac{\delta}{\delta\varphi}\,,\\ \mathcal{V}_\nu(x) &= \partial_\nu\varphi(x)\frac{\delta}{\delta\varphi(x)} + \frac{\delta S_*}{\delta\varphi(x)}\,\partial_\nu\mathcal{H}(x)\cdot\frac{\delta}{\delta\varphi} + \partial_\nu\mathcal{H}(x)\cdot\frac{\delta S_*}{\delta\varphi}\,\frac{\delta}{\delta\varphi(x)} - \frac{\delta}{\delta\varphi(x)}\,\partial_\nu\mathcal{H}(x)\cdot\frac{\delta}{\delta\varphi}\,.\end{aligned} \tag{3.69}$$

With the same arguments as used to obtain (3.49), (3.50), (3.51), (3.52)

$$\begin{aligned}&\int \mathrm{d}^d x\,\big(x_\mu\mathcal{V}_\nu(x) - x_\nu\mathcal{V}_\mu(x)\big) = \mathcal{M}_{\mu\nu}\,, \qquad \int \mathrm{d}^d x\,\big(x_\nu\mathcal{V}_\nu(x) + \delta\,\mathcal{N}(x)\big) = \mathcal{D}\,,\\ &\int \mathrm{d}^d x\;\mathcal{V}_\mu(x) = \mathcal{P}_\mu\,, \qquad \int \mathrm{d}^d x\,\big(-x^2\mathcal{V}_\mu(x) + 2\,x_\mu x_\nu\mathcal{V}_\nu(x) + 2\delta\,x_\mu\mathcal{N}(x)\big) = \mathcal{K}_\mu\,.\end{aligned} \tag{3.70}$$

(3.68) has no direct interpretation for arbitrary x but from (3.70), for $\mathcal{O}$ conformal primary with scale dimension Δ, then using (3.70) as well as (3.49), (3.50), (3.51), (3.52)

$$\begin{aligned}\int \mathrm{d}^d x\;\mathcal{E}_{V\mathcal{O},\nu}(x,y) &= -\,\partial_{y\,\nu}\mathcal{O}(y)\,,\\ \int \mathrm{d}^d x\,\big(x_\mu\mathcal{E}_{V\mathcal{O},\nu}(x,y) - x_\nu\mathcal{E}_{V\mathcal{O},\mu}(x,y)\big) &= -\,L_{y\,\mu\nu}\mathcal{O}(y)\,,\\ \int \mathrm{d}^d x\,\big(x_\nu\mathcal{E}_{V\mathcal{O},\nu}(x,y) + \delta\,\mathcal{E}_{Z\mathcal{O}}(x,y)\big) &= -\,D_y^{(\Delta)}\mathcal{O}(y)\,,\\ \int \mathrm{d}^d x\,\big(-x^2\mathcal{E}_{V\mathcal{O},\mu}(x,y) + 2\,x_\mu x_\nu\mathcal{E}_{V\mathcal{O},\nu}(x,y) + 2\delta\,x_\mu\mathcal{E}_{Z\mathcal{O}}(x,y)\big) &= -\,K_y{}^{(\Delta)}{}_\mu\mathcal{O}(y)\,.\end{aligned} \tag{3.71}$$

The results in (3.71) are as expected from conformal Ward identities with the identifications (3.47). Implicitly in (3.71) $\mathcal{O}$ is spinless but the result extends easily to non zero spin. Up to terms which vanish on integration on (3.71)

$$\begin{aligned}
&\mathcal{E}_{V\mathcal{O},\nu}(x,y) \sim \; -\mathcal{V}_\nu(x)\mathcal{O}(y) \sim \partial_\nu \delta^d(x-y)\,\mathcal{O}(y)\,, \\
&\delta\,\mathcal{E}_{Z\mathcal{O}}(x,y) \sim \; -\delta\,\mathcal{N}(x)\mathcal{O}(y) \sim -(\Delta-d)\,\delta^d(x-y)\,O(y)\,.
\end{aligned} \tag{3.72}$$

If an energy momentum tensor satisfying (3.47) exists then $\mathcal{E}_{\Theta\mathcal{O},\mu\nu}(x,y)$ of course requires non contact contributions but, with the prescription (3.57), correlation functions involving V_ν, Z are just given by

$$\begin{aligned}
\big\langle V_\nu(x)\ldots\mathcal{O}_i(y_i)\ldots\big\rangle &= \textstyle\sum_i \partial_\nu\delta^d(x-y_i)\,\big\langle\ldots\mathcal{O}_i(y_i)\ldots\big\rangle\,, \\
\delta\,\big\langle Z(x)\ldots\mathcal{O}_i(y_i)\ldots\big\rangle &= \; -\textstyle\sum_i\,(d-\Delta_i)\,\delta^d(x-y_i)\,\big\langle\ldots\mathcal{O}_i(y_i)\ldots\big\rangle\,.
\end{aligned} \tag{3.73}$$

Using (3.58) as well as (3.63), (3.65) we may obtain

$$\begin{aligned}
\mathcal{E}_{\Phi_\varphi\Phi_\varphi} &= \Phi_\varphi\,\Phi_\varphi{}^T - \mathcal{H}\cdot S_*^{(2)}\cdot\mathcal{H} - \mathcal{H}\,, \\
\mathcal{E}_{\Phi_R\Phi_\varphi} &= \Phi_R\,\Phi_\varphi{}^T - S_*^{(2)}\cdot\mathcal{H} - ck\,I\,, \\
\mathcal{E}_{\Phi_R\Phi_R} &= \Phi_R\,\Phi_R{}^T - S_*^{(2)}\,,
\end{aligned} \tag{3.74}$$

vhere we take $\Phi_\varphi(x)\,\Phi_\varphi(y)\to\Phi_\varphi\,\Phi_\varphi{}^T$. In the result for $\mathcal{E}_{R\phi}$ conformal identities allow an arbitrary additional term proportional to I since $D^{(d-\delta)}\,I + I\,\overleftarrow{D}^{(\delta)} = 0$, $K^{(d-\delta)}{}_\mu\,I + I\,\overleftarrow{K}^{(\delta)}{}_\mu = 0$. The resulting freedom in the expression for $\mathcal{E}_{\Phi_R\Phi_\varphi}[\varphi;x,y]$ can be used to ensure that any $\delta^d(x-y)$ contributions are absent. The coefficient of I in (3.74) ensures this by virtue of (3.29), (3.62). In the absence of such singular contributions there is a well defined limit $y\to x$ giving

$$Z[\varphi;x] = \mathcal{E}_{\Phi_R\Phi_\varphi}[\varphi;x,x]\,, \tag{3.75}$$

which is identical with (3.39). Using (3.29) and (3.62) as $s\to 0$

$$\begin{aligned}
\mathcal{E}_{\Phi_\varphi\Phi_\varphi}(x,y) &\sim k(1-ck)\,\mathcal{G}_\delta(s) = \big\langle\Phi_\varphi(x)\,\Phi_\varphi(y)\big\rangle\,, \\
\mathcal{E}_{\Phi_R\phi_R}(x,y) &\sim \; -c\,\mathcal{G}_\delta{}^{-1}(s) = \big\langle\Phi_R(x)\,\Phi_R(y)\big\rangle\,,
\end{aligned} \tag{3.76}$$

with the correlation functions defined according to (3.57).

3.4 Variations in G, δ

The function G, which plays the role of a cut off, introduced in the expressions for the conformal generators $\mathcal{D}, \mathcal{K}_\mu$ (3.1), (3.2) with (3.5), (3.7) is to a large extent arbitrary. It is crucial that physical results should be independent of the detailed form of G (previous discussions may be found in [11] and more recently in [6]). However δ which also appears in the conformal generators $\mathcal{D}, \mathcal{K}_\mu$ plays the role of a physical scaling dimension and should not be freely variable.

To discuss the freedom in G we first determine an expression for the variation in S_*, $\mathrm{d}_G S_*$, due to an infinitesimal change $\mathrm{d}G$ with δ fixed. By considering the change in ${E^{(\delta)}}_{S_*}$, ${E^{(\delta)}}_{S_*\mu}$ given by (3.10a), (3.10b) this is determined by

$$\mathcal{D}\,\mathrm{d}_G S_* + 1\cdot O_\psi = 0\,, \qquad \mathcal{K}_\mu\,\mathrm{d}_G S_* + 1\cdot O_{\psi_\mu} = 0\,, \tag{3.77}$$

where $\mathcal{O}_\psi$ is given by (3.30) with here, since $\mathrm{d}F_\mu$ is given by (3.7),

$$\psi(x,y) = \frac{1}{2}\,\frac{\delta S_*}{\delta\varphi(x)}\,\mathrm{d}G(s)\,, \qquad \psi_\mu(x,y) = \frac{1}{2}\,\frac{\delta S_*}{\delta\varphi(x)}\,(x+y)_\mu\,\mathrm{d}G(s)\,. \tag{3.78}$$

To solve (3.77) it is sufficient by virtue of (3.32) to find a quasi-local $\lambda(x,y)$ such that $(\mathcal{D}-D^{(d)})\lambda - \lambda\overleftarrow{D}^{(\delta)} + \psi = 0$ and $(\mathcal{K}_\mu - {K^{(d)}}_\mu)\lambda - \lambda{\overleftarrow{K}^{(\delta)}}_\mu + \psi_\mu = 0$. Making use of

$$\begin{aligned}
&{D_x}^{(\Delta_\mathcal{O}+\delta)}\big(\mathcal{O}(x)\,f(s)\big) + \big(\mathcal{O}(x)\,f(s)\big)\,{\overleftarrow{D}_y}^{(\delta)} - \big({D_x}^{(\Delta_\mathcal{O})}\mathcal{O}(x)\big)\,f(s) \\
&\qquad\qquad = -\,2\,\mathcal{O}(x)\big(sf'(s) + \delta\,f(s)\big)\,, \\
&{{K_x}^{(\Delta_\mathcal{O}+\delta)}}_\mu\,\big(\mathcal{O}(x)\,f(s)\big) + \big(\mathcal{O}(x)\,f(s)\big)\,{{\overleftarrow{K}_y}^{(\delta)}}_\mu - \big({{K_x}^{(\Delta_\mathcal{O})}}_\mu\,\mathcal{O}(x)\big)\,f(s) \\
&\qquad\qquad = -\,2(x+y)_\mu\,\mathcal{O}(x)\big(sf'(s) + \delta\,f(s)\big)\,.
\end{aligned} \tag{3.79}$$

and taking $\lambda(x,y) = \frac{\delta S_*}{\delta\varphi(x)}\,f(s)$ the equations for λ then reduce to just a single equation for $f(s)$ in terms of $\mathrm{d}G(s)$ so that, for $\mathcal{O}_\lambda$ the redundant operator given by λ,

$$\mathrm{d}_G S_* = 1\cdot\mathcal{O}_\lambda = \frac{1}{2}\bigg(\frac{\delta S_*}{\delta\varphi}\cdot\mathrm{d}\mathcal{H}\cdot\frac{\delta S_*}{\delta\varphi} - \frac{\delta}{\delta\varphi}\cdot\mathrm{d}\mathcal{H}\cdot\frac{\delta}{\delta\varphi}S_*\bigg)\,, \qquad \lambda(x,y) = \frac{1}{2}\,\frac{\delta S_*}{\delta\varphi(x)}\,\mathrm{d}\mathcal{H}(s)\,, \tag{3.80}$$

with $\mathrm{d}\mathcal{H}$ determined in terms of $\mathrm{d}G$ by (3.27).

To show that the spectrum of non redundant quasi-local operators is invariant under smooth changes in G we consider the variation of the eigenvalue equations for a quasi-local conformal primary scalar operator $\mathcal{O}_\Delta$ of scale dimension Δ,[1]

$$\big(\mathcal{D} - D^{(\Delta)}\big)\mathcal{O}_\Delta = 0\,, \qquad \big(\mathcal{K}_\mu - {\mathcal{K}^{(\Delta)}}_\mu\big)\mathcal{O}_\Delta = 0\,. \tag{3.81}$$

Under a variation $\mathrm{d}G$ then $\mathcal{D}\to\mathcal{D}+\mathrm{d}_G\mathcal{D}$, $\mathcal{K}_\mu\to\mathcal{K}_\mu+\mathrm{d}_G\mathcal{K}_\mu$ which are determined in the representations (3.1), (3.2) by using (3.80) and (3.7). To verify that Δ is invariant we first show that taking

$$\mathrm{d}_1\mathcal{O}_\Delta = \frac{1}{2}\,\mathcal{O}_\Delta\,\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathrm{d}\mathcal{H}\cdot\frac{\delta}{\delta\varphi}S_*\,. \tag{3.82}$$

then

$$\big(\mathcal{D} - D^{(\Delta)}\big)\mathrm{d}_1\mathcal{O}_\Delta + \mathrm{d}_G\mathcal{D}\,\mathcal{O}_\Delta = \mathcal{O}_\chi\,, \qquad \big(\mathcal{K}_\mu - {K^{(\Delta)}}_\mu\big)\mathrm{d}_1\mathcal{O}_\Delta + \mathrm{d}_G\mathcal{K}_\mu\,O_\Delta = \mathcal{O}_{\chi_\mu}\,. \tag{3.83}$$

for $\mathcal{O}_\chi$, $\mathcal{O}_{\chi_\mu}$ redundant operators determined by $\chi(x,y)$, $\chi_\mu(x,y)$ linear in $\mathrm{d}G$. The action of $\mathcal{D}-D^{(\Delta)}$, $\mathcal{K}_\mu - {\mathcal{K}^{(\Delta)}}_\mu$ on $\mathrm{d}_1\mathcal{O}_\Delta$ may be calculated using (3.81) and (3.59) with ${S_*}^{(2)}$ as in (3.60). This is sufficient to show that (3.83) are satisfied for

$$\chi = \frac{1}{2}\,\mathcal{O}_\Delta\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\big(\mathrm{d}G + G\cdot{S_*}^{(2)}\cdot\mathrm{d}\mathcal{H}\big)\,, \qquad \chi_\mu = \frac{1}{2}\,\mathcal{O}_\Delta\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\big(\mathrm{d}F_\mu + F_\mu\cdot{S_*}^{(2)}\cdot\mathrm{d}\mathcal{H}\big)\,. \tag{3.84}$$

[1]This discussion is an adaptation of that given in appendix A of [12].

Using (3.59) we may solve

$$(\mathcal{D} - D^{(\Delta)})\xi - \xi\overleftarrow{D}^{(\delta)} = -\chi\,, \qquad (\mathcal{K}_\mu - K^{(\Delta)}{}_\mu)\xi - \xi\overleftarrow{K}^{(\delta)}{}_\mu = -\chi_\mu\,, \tag{3.85}$$

by

$$\xi = \frac{1}{2}\,\mathcal{O}_\Delta\,\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot \mathrm{d}\mathcal{H}\,, \tag{3.86}$$

and hence

$$\mathrm{d}_G\mathcal{O}_\Delta = \mathrm{d}_1\mathcal{O}_\Delta + \mathcal{O}_\xi = \tilde{\mathcal{D}}\,\mathcal{O}_\Delta\,, \quad \tilde{\mathcal{D}} = \frac{\delta S_*}{\delta\varphi}\cdot \mathrm{d}\mathcal{H}\cdot\frac{\delta}{\delta\varphi} - \frac{1}{2}\,\frac{\delta}{\delta\varphi}\cdot \mathrm{d}\mathcal{H}\cdot\frac{\delta}{\delta\varphi}\,, \tag{3.87}$$

ensures that $\mathcal{O}_\Delta + \mathrm{d}_G\mathcal{O}_\Delta$ solves the eigenvalue equations (3.81) perturbed to first order in $\mathrm{d}G$ for fixed Δ. This shows that Δ is invariant under small changes in G. However we need to restrict the variations in G to satisfy

$$\int_0^\infty \mathrm{d}u\; u^{\delta-1}\mathrm{d}G(u) = 0\,, \tag{3.88}$$

so that, from (3.27), $\mathrm{d}\mathcal{H}(s)$ is not singular as $s\to 0$. This ensures in (3.28) $\mathrm{d}_G k = 0$.

For an infinitesimal change in δ, $\mathrm{d}\delta$, then $S_*[\varphi] \to S_*[\varphi] + \mathrm{d}_\delta S_*[\varphi]$ is determined from (3.12) by

$$\mathrm{d}\delta\,\varphi\cdot\frac{\delta}{\delta\varphi}S_*[\varphi] + \mathcal{D}\,\mathrm{d}_\delta S_*[\varphi] = 0\,, \quad \mathrm{d}\delta\,\varphi\cdot X_\mu\cdot\frac{\delta}{\delta\varphi}S_*[\varphi] + \mathcal{K}_\mu\,\mathrm{d}_\delta S_*[\varphi] = 0\,. \tag{3.89}$$

Assuming $\{\mathcal{O}_\Delta[\varphi;x]\}$ form a basis of quasi-local functionals then $\mathcal{D},\mathcal{K}_\mu$ have non trivial cokernels spanned by $1\cdot Z$, $1\cdot X_\mu\cdot Z$ as a consequence of (3.41), (3.42). Hence (3.89) are not soluble in general.[2]

4 Transformation to First Order Generators

With the representation of the conformal generators provided by (3.1) and (3.2) the construction of quasi-local conformal primary operators $\mathcal{O}[\varphi;x]$ satisfying (2.4) becomes a non trivial eigenvalue problem determining the spectrum of conformal primary operators and their scale dimensions Δ. Here we describe how to obtain the conformally covariant conformal correlation functions for these operators.

The form of the conformal generators can be reduced to first order functional differential operators by considering a non invertible transformation $\mathcal{O}[\varphi]\to\mathcal{P}[\varphi]$. For this purpose we require a symmetric kernel $\mathcal{G}(x,y)$ such that

$$\begin{aligned}\Big[D^{(\delta)}\varphi\cdot\frac{\delta}{\delta\varphi},\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}\Big] &= -\,\frac{\delta}{\delta\varphi}\cdot G\cdot\frac{\delta}{\delta\varphi}\,,\\ \Big[K^{(\delta)}{}_\mu\varphi\cdot\frac{\delta}{\delta\varphi},\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}\Big] &= -\,\frac{\delta}{\delta\varphi}\cdot F_\mu\cdot\frac{\delta}{\delta\varphi}\,,\end{aligned} \tag{4.1}$$

[2] A similar argument is given in [13].

as well as $\big[\partial_\mu\varphi\cdot\frac{\delta}{\delta\varphi},\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}\big]=\big[L_{\mu\nu}\varphi\cdot\frac{\delta}{\delta\varphi},\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}\big]=0$. (4.1) requires

$$D^{(\delta)}\mathcal{G}+\mathcal{G}\overleftarrow{D}^{(\delta)}=G\,,\qquad {K^{(\delta)}}_\mu\,\mathcal{G}+\mathcal{G}{\overleftarrow{K}^{(\delta)}}_\mu=F_\mu\,,\tag{4.2}$$

which are identical to (3.23). For $\mathcal{G}(x,y)=\mathcal{G}(s)$, $s=(x-y)^2$ then both equations (4.2) are satisfied, for F_μ given by (3.7), if $\mathcal{G}(s)$ is determined by an identical equation to (3.26). However we now require that $\mathcal{G}$ has a form which is regularised at short distances, $x\to y$, so that the necessary solution becomes

$$\mathcal{G}(s)=\frac{1}{2s^\delta}\int_0^s \mathrm{d}u\; u^{\delta-1}G(u)\,,\quad \tilde{\mathcal{G}}(p^2)=\frac{1}{2(p^2)^{\frac12 d-\delta}}\int_{p^2}^\infty \mathrm{d}x\; x^{\frac12 d-\delta-1}\tilde G(x)\,.\tag{4.3}$$

$\mathcal{G}(s),\mathcal{H}(s)$ are related as in (3.28) which demonstrates that $\mathcal{G}(s)$ is regular as $s\to 0$. For applications below we make extensive use of $\exp\big(\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}\big)$ so that $\mathcal{G}$ is required to define a positive definite kernel, or that $\tilde{\mathcal{G}}(p^2)>0$.

It follows from (4.1) that $\exp\big(\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}\big)$ satisfies

$$\begin{aligned} D^{(\delta)}\varphi\cdot\frac{\delta}{\delta\varphi}\,e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}&=e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\Big(D^{(\delta)}\varphi\cdot\frac{\delta}{\delta\varphi}-\frac12\frac{\delta}{\delta\varphi}\cdot G\cdot\frac{\delta}{\delta\varphi}\Big)\,,\\ {K^{(\delta)}}_\mu\varphi\cdot\frac{\delta}{\delta\varphi}\,e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}&=e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\Big({K^{(\delta)}}_\mu\varphi\cdot\frac{\delta}{\delta\varphi}-\frac12\frac{\delta}{\delta\varphi}\cdot F_\mu\cdot\frac{\delta}{\delta\varphi}\Big)\end{aligned}\tag{4.4}$$

In consequence, with $\mathcal{D},\mathcal{K}_\mu$ given (3.1) and (3.2),

$$\begin{aligned} \mathcal{D}^F\,e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\Big(e^{-S_*[\varphi]}\,\mathcal{O}[\varphi]\Big)&=e^{\frac12\frac{\delta}{\delta\varphi}\cdot G\cdot\frac{\delta}{\delta\varphi}}\Big(e^{-S_*[\varphi]}\big(-{E^{(\delta)}}_{S_*}[\varphi]\,\mathcal{O}[\varphi]+\mathcal{D}\,\mathcal{O}[\varphi]\big)\Big)\,,\\ \mathcal{K}^F_\mu\,e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\Big(e^{-S_*[\varphi]}\,\mathcal{O}[\varphi]\Big)&=e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\Big(e^{-S_*[\varphi]}\big(-{E^{(\delta)}}_{S_*\mu}[\varphi]\,\mathcal{O}[\varphi]+\mathcal{K}_\mu\,\mathcal{O}[\varphi]\big)\Big)\,.\end{aligned}\tag{4.5}$$

Assuming (3.12) with $C,C_\mu=0$ and for $\mathcal{O}=1$ defining

$$e^{T[\varphi]}=e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\,e^{-S_*[\varphi]}\,,\tag{4.6}$$

then we must have

$$\mathcal{D}^F\,T[\varphi]=\mathcal{K}^F_\mu\,T[\varphi]=0\,.\tag{4.7}$$

For any primary operator $\mathcal{O}$ we then determine the transformation $\mathcal{O}\to\mathcal{P}_\mathcal{O}$ by

$$\mathcal{P}_\mathcal{O}[\varphi]=e^{-T[\varphi]}\,e^{\frac12\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\big(e^{-S_*[\varphi]}\,\mathcal{O}[\varphi]\big)\,.\tag{4.8}$$

By construction $1\to 1$. As a consequence of this definition, assuming (2.4),

$$\mathcal{D}^F\,\mathcal{P}_\mathcal{O}[\varphi;x]=D^{(\Delta)}\mathcal{P}_\mathcal{O}[\varphi;x]\,,\qquad \mathcal{K}^F_\mu\,\mathcal{P}_\mathcal{O}[\varphi;x]={K^{(\Delta)}}_\mu\mathcal{P}_\mathcal{O}[\varphi;x]\,.\tag{4.9}$$

The scale invariance equations can be integrated for $\mathcal{P}_\mathcal{O}$ to give

$$\mathcal{P}_\mathcal{O}[\varphi_\lambda,\lambda x]=\lambda^\Delta\,\mathcal{P}_\mathcal{O}[\varphi;x]\,,\qquad \varphi_\lambda(x)=\lambda^\delta\,\varphi(\lambda^{-1}x)\,,\tag{4.10}$$

and inversions acting on $\mathcal{P}_\mathcal{O}$ can be defined by

$$\bar{\mathcal{P}}_\mathcal{O}[\varphi;x]=(x^2)^{-\Delta}\,\mathcal{P}_\mathcal{O}[\varphi;x/x^2]=\mathcal{P}_\mathcal{O}[\bar\varphi,x]\,,\qquad \bar\varphi(x)=(x^2)^{-\delta}\varphi(x/x^2)\,.\tag{4.11}$$

For any redundant operator $\mathcal{O}_\psi$ of the form (3.30)

$$\mathcal{P}_{\mathcal{O}_\psi} = -\mathcal{P}_\psi \cdot \frac{\delta}{\delta\varphi} T - \mathcal{P}_\psi \cdot \frac{\overleftarrow{\delta}}{\delta\varphi} \, . \tag{4.12}$$

Unlike (2.4), subject to $\mathcal{O}[\varphi;x]$ being constructed from φ an essentially local form, equations (4.9) no longer determine Δ, since the locality constraint is hidden in $\mathcal{P}_{\mathcal{O}}[\varphi;x]$. The functional operator $e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}$ acting on e^{-S_*} has a perturbative expansion that is expressible in terms of Feynman graphs with vertices determined by the expansion of S_* lines represented by $\mathcal{G}$. For this to generate finite Feynman integrals $\mathcal{G}$ is required to be such that no short distance singularities are generated. This requires that $\mathcal{G}(s)$ should not be singular as $s \to 0$, or the Fourier transform $\tilde{\mathcal{G}}(p^2)$ should vanish rapidly for large p^2. This is satisfied by the solution given in (4.3). It is also crucial that $e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}$, for positive definite $\mathcal{G}$, acting on general non polynomial functionals is not invertible. The definition (4.8) shows that $\mathcal{P}_{\mathcal{O}}[\varphi;x]$ is not a quasi-local functional depending on φ restricted to the neighbourhood of x.

In the subsequent treatment $T[\varphi]$ given by (4.6) and also $\mathcal{P}_{\mathcal{O}}[\varphi;x]$ given by (4.8) are assumed to be analytic functionals of φ in the vicinity of $\varphi = 0$ in the sense that they are expressible in terms of an expansion in terms of multinomials in φ. Hence with the definition $\mathcal{G}_\delta(x)\cdot\frac{\delta}{\delta\varphi} = \int \mathrm{d}^d u\, \mathcal{G}_\delta((x-u)^2)\frac{\delta}{\delta\varphi(u)}$.

$$\big\langle \varphi(x_1)\dots\varphi(x_n)\big\rangle = \prod_{r=1}^{n} \mathcal{G}_\delta(x_r)\cdot\frac{\delta}{\delta\varphi}\, T[\varphi]\,\Big|_{\varphi=0} \, , \tag{4.13}$$

defines a conformally covariant correlation function.

Using in (4.8)

$$\varphi\, e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}} = e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\bigg(\varphi - \mathcal{G}\cdot\frac{\delta}{\delta\varphi}\bigg) \, , \tag{4.14}$$

and also (3.28) we may obtain from (3.20), (3.21)

$$\mathcal{P}_{\Phi_R}[\varphi] = -\frac{\delta}{\delta\varphi}T[\varphi] \, , \qquad \mathcal{P}_{\Phi_\varphi}[\varphi] = \varphi + k\,\mathcal{G}_\delta\cdot\frac{\delta}{\delta\varphi}T[\varphi] \, . \tag{4.15}$$

Using now (4.12) for $\psi_Z, \psi_{V,\nu}$ given by (3.38), (3.43)

$$\begin{aligned} \mathcal{P}_Z(x) &= \mathcal{P}_{\Phi_\varphi}(x)\,\mathcal{P}_{\Phi_R}(x) - \frac{\delta}{\delta\varphi(x)}\mathcal{P}_{\Phi_\varphi}(x) \, , \\ \mathcal{P}_{V,\nu}(x) &= \partial_\nu\mathcal{P}_{\Phi_\varphi}(x)\,\mathcal{P}_{\Phi_R}(x) - \frac{\delta}{\delta\varphi(x)}\partial_\nu\mathcal{P}_{\Phi_\varphi}(x) \, . \end{aligned} \tag{4.16}$$

If an energy momentum tensor exists then (3.47) extends to $\mathcal{P}_Z, \mathcal{P}_{V,\nu}$ and $\mathcal{P}_{\Theta,\mu\nu}$.

The transformation defined by (4.8) may be extended to the bilocal functional $\mathcal{E}_{12}[\varphi;x,y]$ by writing

$$e^{-T[\varphi]}\, e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathcal{G}\cdot\frac{\delta}{\delta\varphi}}\Big(e^{-S_*[\varphi]}\,\mathcal{E}_{12}[\varphi;x,y]\Big) = \mathcal{P}_1[\varphi;x]\,\mathcal{P}_2[\varphi;y] + \mathcal{F}_{12}[\varphi;x,y] \, , \tag{4.17}$$

where $\mathcal{P}_i \equiv \mathcal{P}_{\mathcal{O}_i}$, $i = 1, 2$. The conformal generators after the transformation $\mathcal{O} \to \mathcal{P}_O$ are now first order functional differential operators as in (2.6), (2.7). In consequence $\mathcal{P}_1 \, \mathcal{P}_2$ and $\mathcal{F}_{12}$, which represent disconnected and connected two point functionals, separately satisfy homogeneous conformal identities so in particular

$$\begin{aligned} \mathcal{D}^F \, \mathcal{F}_{12}[\varphi; x, y] &= \big(D_x{}^{(\Delta_1)} + D_y{}^{(\Delta_2)} \big) \mathcal{F}_{12}[\varphi; x, y] \, , \\ \mathcal{K}^F{}_\mu \, \mathcal{F}_{12}[\varphi; x, y] &= \big(K_x{}^{(\Delta_1)}{}_\mu + K_y{}^{(\Delta_2)}{}_\mu \big) \mathcal{F}_{12}[\varphi; x, y] \, . \end{aligned} \tag{4.18}$$

As a consequence, for general operators $\mathcal{O}_1, \mathcal{O}_2$, we may therefore define for arbitrary x, y

$$\big\langle \mathcal{O}_1(x) \, \mathcal{O}_2(y) \big\rangle = \mathcal{F}_{12}[0; x, y] \, , \tag{4.19}$$

since (4.18) ensures that this is conformally covariant. Since $\mathcal{G}(s)$ is regular for small s this matches (3.57). In a similar fashion the corresponding one point functions

$$\big\langle \mathcal{O}(x) \big\rangle = \mathcal{P}[0; x] = 0 \quad \text{if} \quad \Delta > 0 \, , \tag{4.20}$$

necessarily vanish except for the identity operator.

As particular examples from (3.58)

$$\mathcal{F}_{\Phi_\varphi \mathcal{O}}(x, y) = k \, \mathcal{G}_\delta(x) \cdot \frac{\delta}{\delta \varphi} \mathcal{P}_{\mathcal{O}}(y) \, , \tag{4.21}$$

and from (3.66), (3.67)

$$\mathcal{F}_{Z\mathcal{O}}(x, y) = - \hat{\mathcal{N}}(x) \, \mathcal{P}_{\mathcal{O}}(y) \, , \qquad \mathcal{F}_{V\mathcal{O},\nu}(x, y) = - \hat{\mathcal{V}}_\nu(x) \, \mathcal{P}_{\mathcal{O}}(y) \, , \tag{4.22}$$

where $\hat{\mathcal{N}}, \hat{\mathcal{V}}_\nu$ are the functional differential operators

$$\begin{aligned} \hat{\mathcal{N}}(x) &= \varphi(x) \frac{\delta}{\delta \varphi(x)} \\ &\quad + \frac{\delta T}{\delta \varphi(x)} \, k \, \mathcal{G}_\delta(x) \cdot \frac{\delta}{\delta \varphi} + k \, \mathcal{G}_\delta(x) \cdot \frac{\delta T}{\delta \varphi} \, \frac{\delta}{\delta \varphi(x)} - \frac{\delta}{\delta \varphi(x)} \, k \, \mathcal{G}_\delta(x) \cdot \frac{\delta}{\delta \varphi} \, , \\ \hat{\mathcal{V}}_\nu(x) &= \partial_\nu \varphi(x) \frac{\delta}{\delta \varphi(x)} \\ &\quad + \frac{\delta T}{\delta \varphi(x)} \, k \, \partial_\nu \mathcal{G}_\delta(x) \cdot \frac{\delta}{\delta \varphi} + k \, \partial_\nu \mathcal{G}_\delta(x) \cdot \frac{\delta T}{\delta \varphi} \, \frac{\delta}{\delta \varphi(x)} - \frac{\delta}{\delta \varphi(x)} \, k \, \partial_\nu \mathcal{G}_\delta(x) \cdot \frac{\delta}{\delta \varphi} \, . \end{aligned} \tag{4.23}$$

Just as in (3.70) $\hat{\mathcal{V}}_\nu, \hat{\mathcal{N}}$ can be integrated using (2.20) to give the conformal generators in the representation (2.6), (2.7) and (3.71) holds for $\mathcal{E} \to \mathcal{F}$.

Applying the transformation (4.17) to (3.74), and using (4.14), we may also obtain

$$\mathcal{F}_{\Phi_\varphi \Phi_\varphi} = k^2 \, \mathcal{G}_\delta \cdot T^{(2)} \cdot \mathcal{G}_\delta + k \, \mathcal{G}_\delta \, , \quad \mathcal{F}_{\Phi_R \Phi_\phi} = k \, T^{(2)} \cdot \mathcal{G}_\delta - ck \, I \, , \quad \mathcal{F}_{\Phi_R \Phi_R} = T^{(2)} \, , \tag{4.24}$$

for

$$T^{(2)}[\varphi; x, y] = \frac{\delta^2}{\delta \varphi(x) \, \delta \varphi(y)} \, T[\varphi] \, . \tag{4.25}$$

From (3.62)

$$T^{(2)}[0; x, y] = - c \, \mathcal{G}_\delta{}^{-1}(s) \, , \tag{4.26}$$

so that (4.19) with (4.24) give identical results to (3.76).

5 Functional Integrals

Determining $T[\varphi]$ as in (4.6) is tantamount to a full solution of the theory. Equivalently it may be written as a functional integral [5, 6]

$$e^{T[\varphi]} = \int \mathrm{d}[f]\, e^{-\frac{1}{2} f\cdot \mathcal{G}^{-1}\cdot f - S_*[\varphi+f]} \,. \tag{5.1}$$

Correspondingly from (4.8)

$$\mathcal{P}_{\mathcal{O}}[\varphi] = e^{-T[\varphi]} \int \mathrm{d}[f]\, \mathcal{O}[\varphi+f]\, e^{-\frac{1}{2} f\cdot \mathcal{G}^{-1}\cdot f - S_*[\varphi+f]} \,. \tag{5.2}$$

Although $S_*[\varphi]$ and $\mathcal{O}[\varphi]$ are required to be quasi-local $\varphi\cdot\mathcal{G}^{-1}\cdot\varphi$ is not for $\delta\neq\delta_0$.

The prescription (4.19) with (4.17) and (4.20) is therefore equivalent to the functional integral representation

$$\big\langle \mathcal{O}_1(x)\,\mathcal{O}_2(y)\big\rangle = \int \mathrm{d}[\varphi]\, e^{-\frac{1}{2}\varphi\cdot\mathcal{G}^{-1}\cdot\varphi - S_*[\varphi]}\, \mathcal{E}_{12}[\varphi;x,y] \,, \tag{5.3}$$

assuming the normalisation of $\mathrm{d}[\varphi]$ is chosen so that $T[0]=0$ and $\langle 1\rangle = 1$. The conformal Ward identities for the two point function may also be obtained directly from the functional integral expression (5.3).

To this end we note that, with the representation (3.1) and (3.2) for the generators of scale and special conformal transformations $\mathcal{D},\mathcal{K}_\mu$, by functional integration by parts, for any functional $\mathcal{O}[\varphi]$,

$$\begin{aligned}
\int \mathrm{d}[\varphi]\, e^{-\frac{1}{2}\varphi\cdot\mathcal{G}^{-1}\cdot\varphi - S_*[\varphi]}\, \mathcal{D}\,\mathcal{O}[\varphi] &= \int \mathrm{d}[\varphi]\, e^{-\frac{1}{2}\varphi\cdot\mathcal{G}^{-1}\cdot\varphi - S_*[\varphi]}\, e[\varphi]\,\mathcal{O}[\varphi] \,,\\
\int \mathrm{d}[\varphi]\, e^{-\frac{1}{2}\varphi\cdot\mathcal{G}^{-1}\cdot\varphi - S_*[\varphi]}\, \mathcal{K}_\mu\,\mathcal{O}[\varphi] &= \int \mathrm{d}[\varphi]\, e^{-\frac{1}{2}\varphi\cdot\mathcal{G}^{-1}\cdot\varphi - S_*[\varphi]}\, e_\mu[\varphi]\,\mathcal{O}[\varphi] \,,
\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
e[\varphi] &= -\tfrac{1}{2}\,\varphi\cdot\big(\mathcal{G}^{-1}\cdot G\cdot\mathcal{G}^{-1} - D^{(d-\delta)}\mathcal{G}^{-1} - \mathcal{G}^{-1}\overleftarrow{D}^{(d-\delta)}\big)\cdot\varphi + E^{(\delta)}{}_{S_*}[\varphi] + \tfrac{1}{2}\,\mathrm{tr}(\mathcal{G}^{-1}\cdot G) \,,\\
e_\mu[\varphi] &= -\tfrac{1}{2}\,\varphi\cdot\big(\mathcal{G}^{-1}\cdot F_\mu\cdot\mathcal{G}^{-1} - K^{(d-\delta)}{}_\mu\mathcal{G}^{-1} - \mathcal{G}^{-1}\overleftarrow{K}^{(d-\delta)}{}_\mu\big)\cdot\varphi\\
&\qquad + E^{(\delta)}{}_{S_*\mu}[\varphi] + \tfrac{1}{2}\,\mathrm{tr}(\mathcal{G}^{-1}\cdot F_\mu) \,,
\end{aligned} \tag{5.5}$$

with $E^{(\delta)}{}_{S_*}, E^{(\delta)}{}_{S_*\mu}[\varphi]$ given in (3.10). Crucially

$$e[\varphi] = e_\mu[\varphi] = 0 \,, \tag{5.6}$$

by virtue of

$$\begin{aligned}
D^{(d-\delta)}\mathcal{G}^{-1} + \mathcal{G}^{-1}\overleftarrow{D}^{(d-\delta)} &= -\,\mathcal{G}^{-1}\cdot G\cdot\mathcal{G}^{-1} \,,\\
K^{(d-\delta)}{}_\mu\,\mathcal{G}^{-1} + \mathcal{G}^{-1}\overleftarrow{K}^{(d-\delta)}{}_\mu &= -\,\mathcal{G}^{-1}\cdot F_\mu\cdot\mathcal{G}^{-1} \,,
\end{aligned} \tag{5.7}$$

which follow from (4.2), and assuming S_* is constrained by (3.12) with $C = -\frac{1}{2}\,\mathrm{tr}(\mathcal{G}^{-1}\cdot G)$, $C_\mu = -\frac{1}{2}\,\mathrm{tr}(\mathcal{G}^{-1}\cdot F_\mu)$.

Using (5.4) with (5.6) for $\mathcal{O} \to \mathcal{E}_{12}$ ensures that applying (3.55) to (5.3) leads directly to the conformal identities

$$\begin{aligned} \big(D_x{}^{(\Delta_1)} + D_y{}^{(\Delta_2)}\big)\big\langle \mathcal{O}_1(x)\,\mathcal{O}_2(y)\big\rangle &= 0\,, \\ \big(K_x{}^{(\Delta_1)}{}_\mu + K_y{}^{(\Delta_2)}{}_\mu\big)\big\langle \mathcal{O}_1(x)\,\mathcal{O}_2(y)\big\rangle &= 0\,. \end{aligned} \tag{5.8}$$

As $x \to y$ the φ-independent boundary condition (3.57) ensures that the two point function given by the functional integral (5.3) is identical with the result in (3.57).

6 Gaussian Solution

Although essentially trivial and corresponding to free fields considering a Gaussian S_*^G, quadratic in φ, provides an illustration of the general formalism. For this case

$$S_*^G[\varphi] = \tfrac{1}{2}\,\varphi\cdot\mathcal{Z}\cdot\varphi + \alpha\,\mathcal{V}\,, \tag{6.1}$$

where α is independent of φ and, for a spatial cut off, $\mathcal{V}$ is the overall volume. For our purposes the constant term can be neglected. Rotational and translation invariance, (3.3), are satisfied if $\mathcal{Z}(x,y) \to \mathcal{Z}(s)$ for $s = (x-y)^2$. Imposing (3.12) leads to the conditions on $\mathcal{Z}$

$$\begin{aligned} D^{(d-\delta)}\,\mathcal{Z} + \mathcal{Z}\,\overleftarrow{D}^{(d-\delta)} &= \mathcal{Z}\cdot G\cdot\mathcal{Z}\,, \\ K^{(d-\delta)}{}_\mu\,\mathcal{Z} + \mathcal{Z}\,\overleftarrow{K}^{(d-\delta)}{}_\mu &= \mathcal{Z}\cdot F_\mu\cdot\mathcal{Z}\,. \end{aligned} \tag{6.2}$$

These equations can be rewritten as

$$D^{(\delta)}\mathcal{Z}^{-1} + \mathcal{Z}^{-1}\overleftarrow{D}^{(\delta)} = -G\,, \qquad K^{(\delta)}{}_\mu\,\mathcal{Z}^{-1} + \mathcal{Z}^{-1}\overleftarrow{K}^{(\delta)}{}_\mu = -F_\mu\,, \tag{6.3}$$

which are the same, up to a sign, as (3.23) or (4.2). In this case we have a general solution, arbitrary up to solutions of the homogeneous equation (2.20),

$$\mathcal{Z}^{-1} = z\,\mathcal{G}_\delta - \mathcal{H} = \hat{z}\,\mathcal{G}_\delta - \mathcal{G}\,, \qquad \hat{z} = z + k\,. \tag{6.4}$$

$z \neq 0$ is restricted so that $S_*^G[\varphi]$ is a positive definite quadratic form without singularities. Otherwise z is arbitrary and parameterises a line of equivalent Gaussian fixed points. At large distances (6.4) gives

$$\mathcal{Z} \sim \frac{1}{z}\,\mathcal{G}_\delta{}^{-1}\,, \tag{6.5}$$

so that it is necessary that $z > 0$. Requiring S_*^G to be quasi-local restricts $\mathcal{G}_\delta{}^{-1}$ to be proportional to I up to derivatives. This determines $\delta = \delta_0$ as in (2.14) (other choices [14] are possible but these lead to non physical theories) and then, with the conventions in (2.22),

$$\mathcal{G}_{\delta_0}{}^{-1} = -\partial^2 I\,. \tag{6.6}$$

The short distance form of $S_*^{(2)} = \mathcal{Z}$ is compatible with (3.62) with $c = 1/\hat{z}$.

For the Gaussian action (6.1), (3.20) and (3.21) give

$$\Phi_R[\varphi] = \mathcal{Z} \cdot \varphi \,, \qquad \Phi_\varphi[\varphi] = z\, \mathcal{G}_\delta \cdot \Phi_R[\varphi] \,. \tag{6.7}$$

When $\delta = \delta_0$ this gives $-\partial^2 \Phi_\varphi = z\, \Phi_R$. From (3.39), (3.43)

$$Z = \frac{1}{z}\, \Phi_\varphi\, \Phi_\varphi \cdot \mathcal{G}_\delta^{-1} \underset{\delta=\delta_0}{\to} -\frac{1}{z}\, \Phi_\varphi\, \partial^2 \Phi_\varphi \,, \quad V_\nu = \frac{1}{z}\, \partial_\nu \Phi_\varphi\, \Phi_\varphi \cdot \mathcal{G}_\delta^{-1} \underset{\delta=\delta_0}{\to} -\frac{1}{z}\, \partial_\nu \Phi_\varphi\, \partial^2 \Phi_\varphi \,. \tag{6.8}$$

For $\delta = \delta_0$ (3.47) can be 'solved' [8] giving

$$\begin{aligned} \Theta_{\mu\nu} &= \frac{1}{z}\Big(\partial_\mu \Phi_\varphi\, \partial_\nu \Phi_\varphi - \frac{1}{2}\, \delta_{\mu\nu}\, \partial_\lambda \Phi_\varphi\, \partial_\lambda \Phi_\varphi - \frac{d-2}{4(d-1)} \big(\partial_\mu \partial_\nu - \delta_{\mu\nu}\, \partial^2 \big)\, \Phi_\varphi^2 \Big) \\ &= \frac{1}{z}\Big(- \Phi_\varphi\, \partial_\mu \partial_\nu \Phi_\varphi + \frac{1}{4(d-1)} \big(d\, \partial_\mu \partial_\nu - \delta_{\mu\nu}\, \partial^2 \big)\, \Phi_\varphi^2 \Big) + \frac{1}{2}\, \delta_{\mu\nu}\, Z \,, \end{aligned} \tag{6.9}$$

which is equivalent to (2.16) by virtue of (3.48).

The bilocal functionals $\mathcal{E}_{\Phi_\varphi \Phi_\varphi}, \mathcal{E}_{\Phi_R \Phi_\varphi}, \mathcal{E}_{\Phi_R \Phi_R}$ are also simply obtained from (3.74)

$$\begin{aligned} \mathcal{E}_{\Phi_\varphi \Phi_\varphi} &= \Phi_\varphi\, {\Phi_\varphi}^T - z^2\, \mathcal{G}_\delta \cdot \mathcal{Z} \cdot \mathcal{G}_\delta + z\, \mathcal{G}_\delta \,, \\ \mathcal{E}_{\Phi_R \Phi_\varphi} &= \Phi_R\, {\Phi_\varphi}^T - z\, \mathcal{Z} \cdot \mathcal{G}_\delta + z \hat{z}^{-1}\, I \,, \\ \mathcal{E}_{\Phi_R \Phi_R} &= \Phi_R\, \Phi_R^T - \mathcal{Z} \,, \end{aligned} \tag{6.10}$$

since in (3.62) $S_*^{(2)} = \mathcal{Z}$. In the short distance limit (3.28) and (6.4) give, as in (3.76),

$$\mathcal{E}_{\Phi_\varphi \Phi_\varphi}(x,y) \sim z k \hat{z}^{-1}\, \mathcal{G}_\delta(s) \,, \qquad \mathcal{E}_{RR}(x,y) \sim -\hat{z}^{-1}\, \mathcal{G}_\delta^{-1}(s) \,. \tag{6.11}$$

With (6.10) the functional integral (5.3) for $\langle \Phi_\varphi(x)\, \Phi_\varphi(y) \rangle$ and $\langle \Phi_R(x)\, \Phi_R(y) \rangle$ is easily evaluated in the Gaussian case assuming (6.1). With (6.7), (6.10) and using $\mathcal{G}^{-1} + \mathcal{Z} = \hat{z}\, \mathcal{G}^{-1} \cdot \mathcal{G}_\delta \cdot \mathcal{Z}$ then

$$\begin{aligned} \big\langle \Phi_\varphi\, {\Phi_\varphi}^T \big\rangle &= \int \mathrm{d}[\varphi]\, e^{-\frac{1}{2}\varphi \cdot (\mathcal{G}^{-1} + \mathcal{Z}) \cdot \varphi}\, \mathcal{E}_{\Phi_\varphi \Phi_\varphi} \\ &= z^2\, \mathcal{G}_\delta \cdot \mathcal{Z} \cdot (\mathcal{G}^{-1} + \mathcal{Z})^{-1} \cdot \mathcal{Z} \cdot \mathcal{G}_\delta - z^2\, \mathcal{G}_\delta \cdot \mathcal{Z} \cdot \mathcal{G}_\delta + z\, \mathcal{G}_\delta = \frac{zk}{\hat{z}}\, \mathcal{G}_\delta \,, \\ \big\langle \Phi_R\, \Phi_R^T \big\rangle &= \int \mathrm{d}[\varphi]\, e^{-\frac{1}{2}\varphi \cdot (\mathcal{G}^{-1} + \mathcal{Z}) \cdot \varphi}\, \mathcal{E}_{\Phi_R \Phi_R} = \mathcal{Z} \cdot (\mathcal{G}^{-1} + \mathcal{Z})^{-1} \cdot \mathcal{Z} - \mathcal{Z} = -\frac{1}{\hat{z}}\, \mathcal{G}_\delta^{-1} \,. \end{aligned} \tag{6.12}$$

These results are identical to (3.76) although locality here requires $\delta = \delta_0$.

Applying the definition (4.6) to (6.1), with (6.4) and using (3.28), or evaluating the functional integral in (5.1) by completing the square, gives, up to an additive constant,

$$T^G[\varphi] = -\tfrac{1}{2}\, \varphi \cdot (\mathcal{G} + \mathcal{Z}^{-1})^{-1} \cdot \varphi = -\frac{1}{\hat{z}}\, \tfrac{1}{2}\, \varphi \cdot \mathcal{G}_\delta^{-1} \cdot \varphi \,. \tag{6.13}$$

This is conformally invariant. The transformation $\mathcal{O} \to \mathcal{P}$ given by (4.8), or the functional integral (5.2), becomes in the Gaussian case [12]

$$\begin{aligned} \mathcal{P}_{\mathcal{O}}[\varphi] &= e^{\frac{1}{2}\frac{\delta}{\delta\varphi'}\cdot(\mathcal{G}^{-1}+\mathcal{Z})^{-1}\cdot\frac{\delta}{\delta\varphi'}}\,\mathcal{O}\big[\phi'\big]\big|_{\phi'=\mathcal{Z}^{-1}\cdot\mathcal{G}_\delta{}^{-1}\cdot\varphi/\hat z} \\ &= e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\hat{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi}}\,\mathcal{O}\big[\mathcal{Z}^{-1}\cdot\mathcal{G}_\delta{}^{-1}\cdot\varphi/\hat z\big]\,, \end{aligned} \tag{6.14}$$

for

$$\hat{\mathcal{G}} = -\hat z\,\mathcal{G}_\delta + \hat z^2\,\mathcal{G}_\delta\cdot\mathcal{Z}\cdot\mathcal{G}_\delta\,. \tag{6.15}$$

At short distances $\mathcal{Z} \sim \mathcal{G}_\delta{}^{-1}/\hat z$ so that $\hat{\mathcal{G}}(s)$ is not singular as $s \to 0$ and

$$\hat{\mathcal{G}}(s) \sim \frac{\hat z k}{z}\,\mathcal{G}_\delta(s) \quad \text{as} \quad s\to\infty\,. \tag{6.16}$$

Hence in (6.14) the action of $\exp(\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\hat{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi})$ does not generate short distance divergencies. For $\mathcal{O}\to\Phi_R,\Phi_\varphi$, as in (6.7), then (6.14) gives $\mathcal{P}_{\Phi_R}[\varphi] = \mathcal{G}_\delta{}^{-1}\cdot\varphi/z$ and $\mathcal{P}_{\Phi_\varphi}[\varphi]=\varphi$.

For the Gaussian S_*^G in (6.1) a basis of quasi-local scaling operators $\mathcal{O}^G{}_{n,p}$, with scale dimension $\Delta_{n,p} = \Delta^F{}_{n,p}$ given by (2.12), is given in terms of $\Phi_{n,p}$, constructed from φ and its derivatives at the same point and given in part in (2.13), (2.15), by inverting (6.14)

$$\mathcal{O}^G{}_{n,p}[\varphi] = \mathcal{N}_{\hat{\mathcal{G}}}\Phi_{n,p}[\hat z\,\mathcal{G}_\delta\cdot\mathcal{Z}\cdot\varphi]\,, \qquad \mathcal{N}_{\hat{\mathcal{G}}}\Phi_{n,p}[\varphi] = e^{-\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\hat{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi}}\,\Phi_{n,p}[\varphi]\,. \tag{6.17}$$

In general $\exp(-\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\hat{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi})$ does not have a well defined action on functionals of φ for positive $\hat{\mathcal{G}}$ but in (6.17) there is no problem since $\Phi_{n,p}[\varphi]$ are polynomial in φ and the expansion of the exponential truncates. In this case determining quasi-local $\mathcal{O}^G{}_{n,p}$ is equivalent to determining local $\Phi_{n,p}$ so long as $\delta=\delta_0$. In this case

$$\hat z^2\frac{\mathrm{d}}{\mathrm{d}\hat z}\,S_*^G = -\tfrac{1}{2}\,1\cdot\mathcal{O}^G{}_{2,2}\,. \tag{6.18}$$

7 Perturbative Expansion

Perturbative calculations around the Gaussian or free theory is the time honoured method of deriving results for quantum field theories for weak coupling. For perturbations of a free CFT it is necessary that there should be a small parameter such that the perturbed theory remains conformal. For theories involving just scalar fields this is provided by the ε-expansion which depends on the presence of non trivial RG fixed points for dimensions d below the critical dimension above which mean field theory is valid. For a single scalar field and taking $d=4-\varepsilon$ there is just the Wilson Fisher fixed point. Results for scaling dimensions or critical exponents were derived in the 1970's using Wilsonian exact RG equations. To first order in ε extensive results using conformal symmetry were obtained in [15] and much more recently the ε-expansion was discussed in a CFT framework in [16]. Here we describe how the ε-expansion can be rederived using the functional respresentation of the conformal generators discussed here. The final results are not new but we emphasise the role of conformal symmetry and restrict to the theory at a CFT fixed point.

To obtain a perturbative expansion it is convenient to recast the equations for S_* (3.12) with (3.10) by expanding about the Gaussian solution (6.1) in the form

$$S_*[\varphi] = S_*{}^G[\varphi] + \mathring{S}[Y \cdot \varphi] \,. \tag{7.1}$$

Assuming Y satisfies

$$\begin{aligned} D^{(\delta)}\, Y + Y \overleftarrow{D}^{(d-\delta)} &= Y \cdot G \cdot \mathcal{Z} \,, \\ K^{(\delta)}{}_\mu\, Y + Y \overleftarrow{K}^{(d-\delta)}{}_\mu &= Y \cdot F_\mu \cdot \mathcal{Z} \,, \end{aligned} \tag{7.2}$$

then (3.12) becomes

$$\mathring{E}^{(\delta)}{}_{\mathring{S}} = \mathring{C} \,, \qquad \mathring{E}^{(\delta)}{}_{\mathring{S}\mu}[\varphi] = \mathring{C}_\mu \,, \tag{7.3}$$

with $\mathring{E}^{(\delta)}{}_{\mathring{S}}$, $\mathring{E}^{(\delta)}{}_{\mathring{S}\mu}$ as in (3.10) with $G \to \mathring{G}$, $F_\mu \to \mathring{F}_\mu$, $S \to \mathring{S}$ where

$$\mathring{G} = Y \cdot G \cdot Y^T \,, \qquad \mathring{F}_\mu = Y \cdot F_\mu \cdot Y^T \,, \tag{7.4}$$

and $\mathring{C}$, $\mathring{C}_\mu$ independent of φ and (7.2) ensures that $\mathring{G}$, $\mathring{F}_\mu$ satisfy (3.11). (3.14). Also requiring $L_{\mu\nu}\, Y + Y \overleftarrow{L}_{\mu\nu} = 0$, $\partial_\mu\, Y + Y \overleftarrow{\partial}_\mu = 0$ ensures the corresponding equations to (3.4), (3.6). To solve (7.2) as well as the other constraints we may take

$$Y = \lambda\, \mathcal{G}_\delta \cdot \mathcal{Z} \,, \tag{7.5}$$

for arbitrary λ which gives using (6.2)

$$\mathring{G} = \lambda^2\, \mathcal{G}_\delta \cdot \mathcal{Z} \cdot G \cdot \mathcal{Z} \cdot \mathcal{G}_\delta \,, \qquad \mathring{F}_\mu = \lambda^2\, \mathcal{G}_\delta \cdot \mathcal{Z} \cdot F_\mu \cdot \mathcal{Z} \cdot \mathcal{G}_\delta \,. \tag{7.6}$$

Under variations in G, $\mathrm{d}_G \mathring{S}[\varphi]$ is given by (3.80) with $\mathrm{d}\mathcal{H} \to \mathrm{d}\mathring{\mathcal{H}} = Y \cdot \mathrm{d}\mathcal{H} \cdot Y^T$.

Writing

$$\mathcal{O}[\varphi; x] = \mathring{\mathcal{O}}[Y \cdot \varphi; x] \,, \tag{7.7}$$

the eigenvalue equations for conformal primary operators (2.4) have an identical form for $\mathring{\mathcal{O}}[\varphi; x]$

$$\mathring{\mathcal{D}}\, \mathring{\mathcal{O}}[\varphi; x] = D^{(\Delta)} \mathring{\mathcal{O}}[\varphi; x] \,, \qquad \mathring{\mathcal{K}}_\mu\, \mathring{\mathcal{O}}[\varphi; x] = K^{(\Delta)}{}_\mu \mathring{\mathcal{O}}[\varphi; x] \,, \tag{7.8}$$

where $\mathring{\mathcal{D}}$, $\mathring{\mathcal{K}}_\mu$ which are identical in form to (3.1), (3.2) for $S_* \to \mathring{S}$ and $G \to \mathring{G}$, $F_\mu \to \mathring{F}_\mu$.

For any $V[\varphi]$ such that

$$\mathcal{D}^F V[\varphi] = \mathcal{K}^F{}_\mu V[\varphi] = 0 \,, \tag{7.9}$$

with $\mathcal{D}^F$, $\mathcal{K}^F{}_\mu$ the free field generators given in (2.7), (7.3) may be solved iteratively giving a perturbation expansion for $\mathring{S}$ of the form

$$\begin{aligned} \mathring{S}[\varphi] = {} & \mathring{V}[\varphi] - \tfrac{1}{2}\, \mathring{V}[\varphi] \big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi} \cdot \mathring{\mathcal{H}} \cdot \frac{\delta}{\delta\varphi}} - 1 \big)\, \mathring{V}[\varphi] \\ & + \tfrac{1}{2}\, \mathring{V}[\varphi] \big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi} \cdot \mathring{\mathcal{H}} \cdot \frac{\delta}{\delta\varphi}} - 1 \big)\, \mathring{V}[\varphi] \big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi} \cdot \mathring{\mathcal{H}} \cdot \frac{\delta}{\delta\varphi}} - 1 \big)\, \mathring{V}[\varphi] \\ & + \tfrac{1}{6}\, \mathrm{tr}\Big(\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi} \cdot \mathring{\mathcal{H}} \cdot \frac{\delta}{\delta\varphi}} - 1 \big)\, \mathring{V}[\varphi] \big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi} \cdot \mathring{\mathcal{H}} \cdot \frac{\delta}{\delta\varphi}} - 1 \big)\, \mathring{V}[\varphi] \big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi} \cdot \mathring{\mathcal{H}} \cdot \frac{\delta}{\delta\varphi}} - 1 \big)\, \mathring{V}[\varphi] \Big) \\ & + \mathrm{O}(V^4) \,, \end{aligned} \tag{7.10}$$

for

$$\mathring{V}[\varphi] = \mathcal{N}_{\mathring{\mathcal{G}}} V[\varphi] = e^{-\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathring{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi}} V[\varphi]\,. \tag{7.11}$$

Here $\mathring{\mathcal{H}}, \mathring{\mathcal{G}}$ are defined as solutions of

$$\begin{aligned} D^{(\delta)}\,\mathring{\mathcal{H}} + \mathring{\mathcal{H}}\,\overleftarrow{D}^{(\delta)} &= \mathring{G}\,, \qquad & K^{(\delta)}{}_\mu\,\mathring{\mathcal{H}} + \mathring{\mathcal{H}}\,\overleftarrow{K}^{(\delta)}{}_\mu &= \mathring{F}_\mu\,, \\ D^{(\delta)}\,\mathring{\mathcal{G}} + \mathring{\mathcal{G}}\,\overleftarrow{D}^{(\delta)} &= \mathring{G}\,, \qquad & K^{(\delta)}{}_\mu\,\mathring{\mathcal{G}} + \mathring{\mathcal{G}}\,\overleftarrow{K}^{(\delta)}{}_\mu &= \mathring{F}_\mu\,. \end{aligned} \tag{7.12}$$

Although the equations are identical the solutions are different since $\mathring{\mathcal{H}}(s)$ is required to fall off for large s, to ensure $\mathring{S}$ is quasi-local, while $\mathring{\mathcal{G}}(s)$ is regular as $s \to 0$, which is necessary for $\mathring{V}[\varphi]$ to be well defined. It is useful to note that

$$\begin{aligned} \Big(D^{(\delta)}\varphi\cdot\frac{\delta}{\delta\varphi} - \frac{1}{2}\,\frac{\delta}{\delta\varphi}\cdot\mathring{G}\cdot\frac{\delta}{\delta\varphi}\Big)\mathring{V}[\phi] &= e^{-\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathring{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi}}\big(\mathcal{D}^F V[\phi]\big)\,, \\ \Big(K^{(\delta)}{}_\mu\varphi\cdot\frac{\delta}{\delta\varphi} - \frac{1}{2}\,\frac{\delta}{\delta\varphi}\cdot\mathring{F}_\mu\cdot\frac{\delta}{\delta\varphi}\Big)\mathring{V}[\phi] &= e^{-\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathring{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi}}\big(\mathcal{K}^F{}_\mu V[\phi]\big)\,. \end{aligned} \tag{7.13}$$

Expanding the exponentials in (7.10) the functional derivatives $\frac{\overleftarrow{\delta}}{\delta\varphi}, \frac{\delta}{\delta\varphi}$ act to the left, right just on the adjacent $\mathring{V}[\varphi]$ and of course the trace is invariant under cyclic permutation. This generates the usual perturbative expansion for $-\mathring{S}$ in terms of connected graphs with vertices determined by $V[\varphi]$. The result given in (7.10) and (7.11) corresponds to summing over all lines linking different vertices with propagator $-\mathring{\mathcal{H}}$ and the same vertex with propagator $-\mathring{\mathcal{G}}$.

Using (6.2) and (7.4) solutions of (7.12) with the relevant boundary conditions are given by

$$\mathring{\mathcal{H}} = \lambda^2\big(\mathcal{G}_\delta\cdot\mathcal{Z}\cdot\mathcal{G}_\delta - \mathcal{G}_\delta/z\big)\,, \qquad \mathring{\mathcal{G}} = \lambda^2\big(\mathcal{G}_\delta\cdot\mathcal{Z}\cdot\mathcal{G}_\delta - \mathcal{G}_\delta/\hat{z}\big)\,. \tag{7.14}$$

Defining, as in (4.6),

$$e^{\mathring{T}[\varphi]} = e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathring{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi}}\, e^{-\mathring{S}[\varphi]}\,, \tag{7.15}$$

then $\mathring{T}$ has an expansion[3]

$$\begin{aligned} \mathring{T}[\varphi] = {} & -V[\varphi] + \tfrac{1}{2}\,V[\varphi]\,\big(e^{\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{G}}_\delta\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,V[\varphi] \\ & - \tfrac{1}{2}\,V[\varphi]\,\big(e^{\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{G}}_\delta\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,V[\varphi]\,\big(e^{\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{G}}_\delta\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,V[\varphi] \\ & - \tfrac{1}{6}\,\mathrm{tr}\Big(\big(e^{\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{G}}_\delta\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,V[\varphi]\,\big(e^{\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{G}}_\delta\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,V[\varphi]\,\big(e^{\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{G}}_\delta\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,V[\varphi]\Big) \\ & + \mathrm{O}(V^4)\,, \end{aligned} \tag{7.16}$$

[3] Note that

$$\begin{aligned} e^{-\mathring{S}[\varphi]} = {} & 1 - \mathring{V}[\varphi] + \tfrac{1}{2}\,\mathring{V}[\varphi]\,e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}}\,\mathring{V}[\varphi] \\ & - \tfrac{1}{6}\,\mathrm{tr}\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}}\,\mathring{V}[\varphi]\,e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}}\,\mathring{V}[\varphi]\,e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}}\,\mathring{V}[\varphi]\big) + \mathrm{O}(V^4)\,, \end{aligned}$$

and if d is a differential operator obeying the usual Leibnitz rules then $e^{\frac{1}{2}\mathrm{d}^2}(f\,g) = \hat{f}\,e^{\overleftarrow{\mathrm{d}}\,\mathrm{d}}\,\hat{g}$ and also $e^{\frac{1}{2}\mathrm{d}^2}(f\,g\,h) = \mathrm{tr}\big(e^{\overleftarrow{\mathrm{d}}\,\mathrm{d}}\,\hat{f}\,e^{\overleftarrow{\mathrm{d}}\,\mathrm{d}}\,\hat{g}\,e^{\overleftarrow{\mathrm{d}}\,\mathrm{d}}\,\hat{h}\big)$ for $\hat{f} = e^{\frac{1}{2}\mathrm{d}^2}f$ and similarly for $\hat{g}, \hat{h}$.

for

$$\mathring{\mathcal{G}}_\delta = \mathring{\mathcal{G}} - \mathring{\mathcal{H}} = \mathring{k}\,\mathcal{G}_\delta\,, \qquad \mathring{k} = \frac{\lambda^2 k}{z\hat{z}}\,. \tag{7.17}$$

It is easy to verify that $\mathring{T}$ satisfies the conformal identities (4.7) assuming (7.9). Of course, since $\mathring{\mathcal{G}}_\delta(s) \propto s^{-\delta}$, $T[\varphi]$ is a very non local functional.

A similar formal perturbative expansion gives solutions of (7.8) for $\Delta = \Delta^F{}_{n,p}$ starting from solutions of (2.12),

$$\begin{aligned}
\mathring{\mathcal{O}}_{n,p}[\varphi] = {}& \mathring{\Phi}_{n,p}[\varphi] - \mathring{V}[\varphi]\,\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,\mathring{\Phi}_{n,p}[\varphi] \\
& + \mathring{V}[\varphi]\,\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,\mathring{V}[\varphi]\,\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,\mathring{\Phi}_{n,p}[\varphi] \\
& + \tfrac{1}{2}\,\mathring{V}[\varphi]\,\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,\mathring{\Phi}_{n,p}[\varphi]\,\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,\mathring{V}[\varphi] \\
& + \tfrac{1}{2}\,\mathrm{tr}\Big(\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,\mathring{V}[\varphi]\,\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,\mathring{V}[\varphi]\,\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,\mathring{\Phi}_{n,p}[\varphi]\Big) \\
& + \mathrm{O}(V^3)\,.
\end{aligned} \tag{7.18}$$

However this discussion is of course over simplified. The Gaussian solution dictates $\delta = \delta_0$. In the perturbative expansion we then take

$$\delta = \delta_0 + \tfrac{1}{2}\,\eta = \tfrac{1}{2}(d - 2 + \eta)\,, \tag{7.19}$$

so that η plays the role of an anomalous dimension. Assuming now Y satisfies (7.2) for $\delta = \delta_0$ then (7.3) is modified to

$$\begin{aligned}
\mathring{E}^{(\delta_0)}{}_{\mathring{S}} - \mathring{C} = {}& -\tfrac{1}{2}\eta\,\varphi\cdot\frac{\delta}{\delta\varphi}\,\mathring{S}[\varphi] - \tfrac{1}{2}\eta\,\frac{1}{\lambda^2}\,\partial^2\varphi\cdot\mathcal{Z}^{-1}\cdot\partial^2\varphi\,, \\
\mathring{E}^{(\delta_0)}{}_{\mathring{S}\mu}[\varphi] - \mathring{C}_\mu = {}& -\eta\,\varphi\cdot\mathring{X}_\mu\cdot\frac{\delta}{\delta\varphi}\,\mathring{S}[\varphi] - \tfrac{1}{2}\eta\,\frac{1}{\lambda^2}\,\partial^2\varphi\cdot\big(X_\mu\cdot\mathcal{Z}^{-1} + \mathcal{Z}^{-1}\cdot X_\mu\big)\cdot\partial^2\varphi\,,
\end{aligned} \tag{7.20}$$

for

$$\mathring{X}_\mu = Y\cdot X_\mu\cdot Y^{-1}\,. \tag{7.21}$$

From (7.4) and (3.7) $\mathring{F}_\mu = \mathring{X}_\mu\cdot\mathring{G} + \mathring{G}\cdot\mathring{X}_\mu{}^T$. We also have from (7.8)

$$\begin{aligned}
\big(\mathring{\mathcal{D}} - D^{(\Delta_{n,p})}\big)\mathring{\mathcal{O}}_{n,p}[\varphi;x] = {}& -\tfrac{1}{2}\eta\,\varphi\cdot\frac{\delta}{\delta\varphi}\,\mathring{\mathcal{O}}_{n,p}[\varphi;x]\,, \\
\big(\mathring{\mathcal{K}}_\mu - K^{(\Delta_{n,p})}{}_\mu\big)\mathring{\mathcal{O}}_{n,p}[\varphi;x] = {}& -\eta\,\varphi\cdot\mathring{X}_\mu\cdot\frac{\delta}{\delta\varphi}\,\mathring{\mathcal{O}}_{n,p}[\varphi;x]\,,
\end{aligned} \tag{7.22}$$

with $\delta = \delta_0$ in $\mathring{\mathcal{D}}, \mathring{\mathcal{K}}_\mu$.

It is also essential to adapt the discussion above to take account that the perturbation is slightly relevant rather than exactly marginal as in (7.9) which was assumed in obtaining the solution (7.10). In $d = 4 - \varepsilon$ dimensions we take

$$V[\varphi] = g\,1\cdot\Phi_{4,0} + \tfrac{1}{2}\eta\,\partial^2\varphi\cdot\mathcal{K}\cdot\partial^2\varphi\,, \qquad \Phi_{4,0} = \varphi^4\,, \tag{7.23}$$

so that, from (3.41), (3.42), with $\delta = \delta_0$,

$$\begin{aligned}\mathcal{D}^F V[\varphi] &= -\varepsilon\, g\, 1 \cdot \Phi_{4,0} - \tfrac{1}{2}\eta\, \partial^2\varphi \cdot \big(D^{(\delta_0)}\, \kappa + \kappa\, \overleftarrow{D}^{(\delta_0)} \big) \cdot \partial^2 \varphi\,, \\ \mathcal{K}^F{}_\mu V[\varphi] &= -2\varepsilon\, g\, 1 \cdot X_\mu \cdot \Phi_{4,0} - \tfrac{1}{2}\eta\, \partial^2\varphi \cdot \big(K^{(\delta_0)}{}_\mu\, \kappa + \kappa\, \overleftarrow{K}^{(\delta_0)}{}_\mu \big) \cdot \partial^2 \varphi\,. \end{aligned} \tag{7.24}$$

The resulting additional terms in the equations for $\mathring{S}$ are then cancelled against contributions resulting from the need to regularise short distance singularities arising from the singular behaviour of products of $\mathring{\mathcal{H}}(s)$ as $s \to 0$, which would otherwise generate poles in ε in (7.10). The cancellation determines g, η as an expansion in positive powers of ε, and also κ, so as to solve (7.20) order by order in ε where, to leading order, $g = \mathrm{O}(\varepsilon)$, $\eta = \mathrm{O}(g^2) = \mathrm{O}(\varepsilon^2)$. Similarly in (7.22) the corresponding contributions generate corrections to the scale dimension so that $\Delta_{n,p} = \Delta^F{}_{n,p} + \mathrm{O}(\varepsilon)$.

To lowest order the short distance singularities for $d = 4 - \varepsilon$ are reflected by simple poles in ε and arise from

$$\mathcal{G}_{\delta_0}{}^n \sim \frac{1}{\varepsilon}\, \sigma_n\, (\partial^2)^{n-2} I\,, \qquad \sigma_n = \frac{1}{(16\pi^2)^{n-1}}\, \frac{2}{(n-1)!^2}\,, \quad n \ge 2\,. \tag{7.25}$$

Although $\mathring{\mathcal{H}}(s)$ falls off rapidly for large s for small s it is proportional to $\mathcal{G}_{\delta_0}(s)$ it may be expressed in the same form as (3.28) for $k \to \mathring{k}$, $\mathcal{G} \to \mathring{\mathcal{G}}$ with $\mathring{\mathcal{G}}(s)$ having a similar expansion to (3.29). Regularised products of $\mathring{\mathcal{H}}(s)$ may then be defined by

$$\mathcal{R}\big(-\mathring{\mathcal{H}}\big)^n = \big(-\mathring{\mathcal{H}}\big)^n - \frac{1}{\varepsilon} \sum_{r=2}^{n} \binom{n}{r} \mathring{k}^r \sigma_r (-\mathring{\mathcal{G}})^{n-r}\, (\partial^2)^{r-2} I\,, \quad n = 2, 3, \dots\,, \tag{7.26}$$

which cancels all poles in ε due to short distance singularities. Using

$$\begin{aligned} D^{(r\delta_0)} (\partial^2)^{r-2} I + (\partial^2)^{r-2} I \overleftarrow{D}^{(r\delta_0)} &= -\varepsilon\, (r-1) (\partial^2)^{r-2} I\,, \\ K^{(r\delta_0)}{}_\mu\, (\partial^2)^{r-2} I + (\partial^2)^{r-2} I \overleftarrow{K}^{(r\delta_0)}{}_\mu &= -\varepsilon\, (r-1) \big(X_\mu \cdot (\partial^2)^{r-2} I + (\partial^2)^{r-2} I \cdot X_\mu \big)\,, \end{aligned} \tag{7.27}$$

and (7.12)

$$\begin{aligned} & D^{(n\delta_0)} \mathcal{R}\big(-\mathring{\mathcal{H}}\big)^n + \mathcal{R}\big(-\mathring{\mathcal{H}}\big)^n \overleftarrow{D}^{(n\delta_0)} + n\, \mathcal{R}\big(-\mathring{\mathcal{H}}\big)^{n-1} \mathring{G} \\ &\quad = \sum_{r=2}^{n} (r-1) \binom{n}{r} \mathring{k}^r \sigma_r (-\mathring{\mathcal{G}})^{n-r}\, (\partial^2)^{r-2} I = \sum_{r=0}^{n-2} \rho_{n,r}\, (\partial^2)^r I\,, \\ & K^{(n\delta_0)}{}_\mu\, \mathcal{R}\big(-\mathring{\mathcal{H}}\big)^n + \mathcal{R}\big(-\mathring{\mathcal{H}}\big)^n \overleftarrow{K}^{(n\delta_0)}{}_\mu + n\, \mathcal{R}\big(-\mathring{\mathcal{H}}\big)^{n-1} \mathring{F}_\mu \\ &\quad = \sum_{r=2}^{n} (r-1) \binom{n}{r} \mathring{k}^r \sigma_r (-\mathring{\mathcal{G}})^{n-r}\, \big(X_\mu \cdot (\partial^2)^{r-2} I + (\partial^2)^{r-2} I \cdot X_\mu \big) \\ &\qquad\qquad = \sum_{r=0}^{n-2} \rho_{n,r}\, \big(X_\mu \cdot (\partial^2)^r I + (\partial^2)^r I \cdot X_\mu \big)\,. \end{aligned} \tag{7.28}$$

Using the expansion (3.29) for $\mathcal{G} \to \mathring{\mathcal{G}}$, $G \to \mathring{G}$ the coefficients $\rho_{n,r}$ are determined from $s^p (\partial^2)^q \delta^d \propto (\partial^2)^{q-p} \delta^d$ for $p \le q$, $s^p (\partial^2)^q \delta^d = 0$ for $p > q$. Hence

$$\begin{aligned} \rho_{n,n-2} &= \mathring{k}^n\, (n-1) \sigma_n\,, \qquad\qquad \rho_{n,n-3} = -\mathring{k}^{n-1} n(n-2)\, \sigma_{n-1}\, \mathring{\mathcal{G}}(0)\,, \\ \rho_{n,n-4} &= \mathring{k}^{n-2} \tfrac{1}{2} n(n-1)(n-3)\, \sigma_{n-2}\, \mathring{\mathcal{G}}(0)^2 \\ &\qquad - \mathring{k}^{n-1} 2n(n-2)(n-3)(2(n-4)+d)\, \sigma_{n-1}\, \mathring{\mathcal{G}}'(0)\,. \end{aligned} \tag{7.29}$$

To lowest order we extend (7.23) by following (7.10) to

$$\mathring{S}[\varphi] = g\,1\cdot\mathring{\Phi}_{4,0} + \tfrac{1}{2}\,\eta\,\partial^2\varphi\cdot\kappa\cdot\partial^2\varphi - \tfrac{1}{2}\,g^2\,1\cdot\mathring{\Phi}_{4,0}\,\mathcal{R}\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\,1\cdot\mathring{\Phi}_{4,0} + \mathrm{O}(V^3) + \text{const.}\,, \tag{7.30}$$

with

$$\mathring{\Phi}_{4,0} = e^{-\frac{1}{2}\mathring{\mathcal{G}}(0)\frac{\partial^2}{\partial\varphi^2}}\varphi^4 = \varphi^4 - 6\,\mathring{\mathcal{G}}(0)\,\varphi^2 + 3\,\mathring{\mathcal{G}}(0)^2\,. \tag{7.31}$$

In (7.30) $\mathcal{R}$ is necessary, and is defined according to (7.26), after the action of the functional derivatives on $1\cdot\mathring{\Phi}_{4,0}$ since this is a purely local functional of φ. Only the first four terms in the expansion of the exponential are now relevant. Using (7.28) for $n=2,3$ with (7.30)

$$\begin{aligned}
\mathring{E}^{(\delta_0)}{}_{\mathring{S}} = {} & -\varepsilon\,g\,1\cdot\mathring{\Phi}_{4,0} - \tfrac{1}{2}\eta\,\partial^2\varphi\cdot\big(D^{(\delta_0)}\,\kappa + \kappa\,\overleftarrow{D}^{(\delta_0)}\big)\cdot\partial^2\varphi \\
& + \tfrac{1}{2}\,g^2\big(\tfrac{1}{2}\,\rho_{2,0}\,\mathring{\Phi}_{4,0}{}''\cdot\mathring{\Phi}_{4,0}{}'' + \tfrac{1}{6}\,\rho_{3,0}\,\Phi_{4,0}{}'''\cdot\Phi_{4,0}{}''' + \tfrac{1}{6}\,\rho_{3,1}\,\Phi_{4,0}{}'''\cdot\partial^2\Phi_{4,0}{}'''\big) \\
& + \varepsilon\,g^2\,1\cdot\mathring{\Phi}_{4,0}\,\mathcal{R}\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)1\cdot\mathring{\Phi}_{4,0} + \mathrm{O}(V^3) + \text{const.}\,, \\
\mathring{E}^{(\delta_0)}{}_{\mathring{S}\,\mu} = {} & -2\varepsilon\,g\,1\cdot X_\mu\cdot\mathring{\Phi}_{4,0} - \tfrac{1}{2}\eta\,\partial^2\varphi\cdot\big(K^{(\delta_0)}{}_\mu\,\kappa + \kappa\,\overleftarrow{K}^{(\delta_0)}{}_\mu\big)\cdot\partial^2\varphi \\
& + g^2\big(\tfrac{1}{2}\,\rho_{2,0}\,\mathring{\Phi}_{4,0}{}''\cdot X_\mu\cdot\mathring{\Phi}_{4,0}{}'' + \tfrac{1}{6}\,\rho_{3,0}\,\Phi_{4,0}{}'''\cdot X_\mu\cdot\Phi_{4,0}{}''' \\
& \qquad + \tfrac{1}{6}\,\rho_{3,1}\,\Phi_{4,0}{}'''\cdot X_\mu\cdot\partial^2\Phi_{4,0}{}'''\big) \\
& + 2\varepsilon\,g^2\,1\cdot X_\mu\cdot\mathring{\Phi}_{4,0}\,\mathcal{R}\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)1\cdot\mathring{\Phi}_{4,0} + \mathrm{O}(V^3) + \text{const.}\,.
\end{aligned} \tag{7.32}$$

For cancellation of the non derivative terms to $\mathrm{O}(\varepsilon^2)$ we must take

$$\varepsilon\,g\,\mathring{\Phi}_{4,0} = \tfrac{1}{2}\,\big(\tfrac{1}{2}\,\rho_{2,0}\,(\mathring{\Phi}_{4,0}{}'')^2 + \tfrac{1}{6}\,\rho_{3,0}\,(\Phi_{4,0}{}''')^2\big) + \text{const.} \quad\Rightarrow\quad 36\,\rho_{2,0}\,g = \varepsilon\,, \tag{7.33}$$

where, since $\rho_{3,0} = -3\mathring{\mathcal{G}}(0)\,\rho_{2,0}$, this also ensures cancellation of $\mathring{\mathcal{G}}(0)\varphi^2$ terms.

For the derivative terms

$$\eta\,\big(D^{(\delta_0)}\,\kappa + \kappa\,\overleftarrow{D}^{(\delta_0)}\big) + 96\,\rho_{3,1}\,g^2\,\mathcal{G}_{\delta_0} = \eta\,\frac{1}{\lambda^2}\,\mathcal{Z}^{-1}\,. \tag{7.34}$$

$$\eta = 96\,\frac{\mathring{k}\,\rho_{3,1}}{\rho_{2,0}{}^2}\,(\rho_{2,0}\,g)^2 = 24\,(\rho_{2,0}\,g)^2 = \frac{\varepsilon^2}{54}\,. \tag{7.35}$$

To determine corrections to the scale dimensions $\Delta_{n,0}$ we extend (7.18) for the non derivative local operators $\mathring{\mathcal{O}}_{n,0}[\varphi;x]$ at this order to

$$\mathring{\mathcal{O}}_{n,0} = \mathring{\Phi}_{n,0} + \tau\,\mathring{\Phi}_{n-2,2} - g\,1\cdot\mathring{\Phi}_{4,0}\,\mathcal{R}\big(e^{-\frac{\overleftarrow{\delta}}{\delta\varphi}\cdot\mathring{\mathcal{H}}\cdot\frac{\delta}{\delta\varphi}} - 1\big)\mathring{\Phi}_{n,0} + \mathrm{O}(g^2)\,, \tag{7.36}$$

where, with definitions in (2.13),

$$\mathring{\Phi}_{n,0} = e^{-\frac{1}{2}\mathring{\mathcal{G}}(0)\frac{\partial^2}{\partial\varphi^2}}\varphi^n\,, \qquad \mathring{\Phi}_{n,2} = e^{-\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\mathring{\mathcal{G}}\cdot\frac{\delta}{\delta\varphi}}\Phi_{n,2} = -\partial^2\varphi\,\mathring{\Phi}_{n-1,0} + 2d\,\mathring{\mathcal{G}}'(0)\,\mathring{\Phi}_{n-2,0}\,. \tag{7.37}$$

With these expressions and using (7.29)

$$\begin{aligned}
& \big(\mathring{\mathcal{D}} - D^{(\Delta^F{}_{n,0})}\big)\mathring{\mathcal{O}}_{n,0} - \varepsilon\,\tau\,\mathring{\Phi}_{n-2,2} \\
& = g\big(\tfrac{1}{2}\,\rho_{2,0}\,\mathring{\Phi}_{4,0}{}''\,\mathring{\Phi}_{n,0}{}'' + \tfrac{1}{6}\,\rho_{3,0}\,\Phi_{4,0}{}'''\,\mathring{\Phi}_{n,0}{}''' + \tfrac{1}{24}\,\rho_{4,0}\,\Phi_{4,0}{}''''\mathring{\Phi}_{n,0}{}'''' + \tfrac{1}{6}\,\rho_{3,1}\,\partial^2\Phi_{4,0}{}'''\,\mathring{\Phi}_{n,0}{}'''\big) \\
& = \tfrac{1}{2}\,\rho_{2,0}\,g\,e^{-\frac{1}{2}\mathring{\mathcal{G}}(0)\frac{\partial^2}{\partial\varphi^2}}\big(\Phi_{4,0}{}''\,\Phi_{n,0}{}''\big) + 4\,\rho_{3,1}\,g\big(\partial^2\varphi\,\mathring{\Phi}_{n,0}{}''' - 2d\;\mathring{\mathcal{G}}'(0)\,\mathring{\Phi}''''_{n,0}\big) \\
& = 6n(n-1)\,\rho_{2,0}\,g\,\mathring{\Phi}_{n,0} - 4n(n-1)(n-2)\,\rho_{3,1}\,g\,\mathring{\Phi}_{n-2,2}\,.
\end{aligned} \tag{7.38}$$

Hence we must have to first order in ε

$$\Delta^1{}_{n,0} = 6n(n-1)\,\rho_{2,0}\,g = \tfrac{1}{6}n(n-1)\,\varepsilon\,, \tag{7.39}$$

and also τ is determined to cancel the $\mathring{\Phi}_{n-2,2}$ terms.

7.1 Higher Order Corrections

At the next order it is necessary to add the contributions to $\mathring{S}$ corresponding to the second and third lines in (7.10) where, after the action of the functional derivatives, it is crucial as before to introduce regularised products of $\mathring{\mathcal{H}}$'s to avoid poles in ε. For contributions proportional to φ^4, not involving derivatives, it is sufficient to use just

$$\begin{aligned} \mathcal{R}\big(\mathring{\mathcal{H}}(s_{xy})^2\,\mathring{\mathcal{H}}(s_{xz})\,\mathring{\mathcal{H}}(s_{yz})\big) &= \mathcal{R}\big(\mathring{\mathcal{H}}(s_{xy})^2\big)\,\mathring{\mathcal{H}}(s_{xz})\,\mathring{\mathcal{H}}(s_{yz}) + \frac{1}{2\varepsilon^2}(1-\tfrac{1}{2}\varepsilon)\,\rho_{2,0}{}^2\,\delta_{xy}\,\delta_{xz}\,, \\ \mathcal{R}\big(\mathring{\mathcal{H}}(s_{xy})^2\big) &= \mathring{\mathcal{H}}(s_{xy})^2 - \frac{1}{\varepsilon}\,\rho_{2,0}\,\delta_{xy}\,, \end{aligned} \tag{7.40}$$

along with $\mathcal{R}\big(\mathring{\mathcal{H}}(s_{xy})^p\,\mathring{\mathcal{H}}(s_{yz})^q\big) = \mathcal{R}\big(\mathring{\mathcal{H}}(s_{xy})^p\big)\,\mathcal{R}\big(\mathring{\mathcal{H}}(s_{yz})^q\big)$. In (7.40) and subsequently we use the notation $s_{xy} = (x-y)^2$, $\delta_{xy} = \delta^d(x-y)$, etc. The necessary subtractions in (7.40) may be derived from an analysis of the short distance ε poles present in the products $\mathcal{G}_{\delta_0}(s_{xy})^2\,\mathcal{G}_{\delta_0}(s_{xz})\,\mathcal{G}_{\delta_0}(s_{yz})$.[4]

With the regularised expressions in (7.40) we may straightforwardly obtain

$$\begin{aligned} &\big(D_x{}^{(3\delta_0)} + D_y{}^{(3\delta_0)} + D_z{}^{(2\delta_0)}\big)\mathcal{R}\big(\mathring{\mathcal{H}}(s_{xy})^2\,\mathring{\mathcal{H}}(s_{xz})\,\mathring{\mathcal{H}}(s_{yz})\big) \\ &- 2\,\mathring{G}(s_{xy})\,\mathring{\mathcal{H}}(s_{xy})\,\mathring{\mathcal{H}}(s_{xz})\,\mathring{\mathcal{H}}(s_{yz}) - \mathcal{R}\big(\mathring{\mathcal{H}}(s_{xy})^2\big)\big(\mathring{G}(s_{xz})\,\mathring{\mathcal{H}}(s_{yz}) + \mathring{\mathcal{H}}(s_{xz})\,\mathring{G}(s_{yz})\big) \\ &\quad = \rho_{2,0}\,\delta_{xy}\,\mathcal{R}\big(\mathring{\mathcal{H}}(s_{xz})^2\big) + \tfrac{1}{2}\,\rho_{2,0}{}^2\,\delta_{xy}\,\delta_{xz}\,, \end{aligned} \tag{7.41}$$

which extends

$$\big(D_x{}^{(2\delta_0)} + D_y{}^{(2\delta_0)}\big)\mathcal{R}\big(\mathring{\mathcal{H}}(s_{xy})^2\big) - 2\,\mathring{G}(s_{xy})\,\mathring{\mathcal{H}}(s_{xy}) = \rho_{2,0}\,\delta_{xy}\,,$$

used previously. Hence requiring the $\mathrm{O}(\varphi^4)$ terms in $\mathring{E}^{(\delta_0)}{}_{\mathring{S}}$ to satisfy (7.20) now gives to this order

$$-\varepsilon + 36\,\rho_{2,0}\,g - 6\,(12\,\rho_{2,0}\,g)^2 = -2\,\eta \quad\Rightarrow\quad 36\,\rho_{2,0}\,g = \varepsilon + \tfrac{17}{27}\,\varepsilon^2\,. \tag{7.42}$$

[4] The relevant result is derived in appendix C of [17].

A Alternative Representations of Conformal Generators

The form for the generator of scale transformations (3.1) and the corresponding consistency relation (3.10a) are essentially of the form at a fixed point of the exact RG equations due to Wilson and later Polchinski. There are various related equations. These are equivalent by allowing transformations of the form $\tilde{\varphi}(p) \to e^{f(p^2)}\tilde{\varphi}(p)$ for some $f(p^2)$ analytic near $p^2 = 0$. Such changes can be absorbed in a change of the cut off function G. There is also a corresponding change in the $D^{(\delta)}\varphi$ term in (3.1), (3.10a). This can be removed by writing $S_*[\varphi] = \mathcal{S}_*[\varphi] + \frac{1}{2}\,\varphi\cdot\mathcal{C}\cdot\varphi$ and choosing $\mathcal{C}$ appropriately. In this case (3.10a) becomes a similar equation for $\mathcal{S}_*$ but in general has an extra term quadratic in φ. Although the equations given in the text have the virtue of simplicity we describe here some generalisations.

Such modifications may be obtained by adding an extra term to the functional differential expressions for the conformal generators $\mathcal{D}, \mathcal{K}$ given by (3.1) and (3.2) of the form

$$\hat{\mathcal{D}} = \mathcal{D} + \varphi \cdot g \cdot \frac{\delta}{\delta\varphi}\,, \qquad \hat{\mathcal{K}}_\mu = \mathcal{K}_\mu + \varphi \cdot f_\mu \cdot \frac{\delta}{\delta\varphi}\,. \tag{A.1}$$

where g, f_μ are not required to be symmetric. Commutation relations with $\mathcal{P}_\mu, \mathcal{M}_{\mu\nu}$ dictate

$$L_{\mu\nu}\, f_\sigma + f_\sigma \overleftarrow{L}_{\mu\nu} = -\delta_{\mu\sigma}\, f_\nu + \delta_{\nu\sigma}\, f_\mu\,, \qquad \partial_\mu\, f_\nu + f_\nu \overleftarrow{\partial}_\mu = 2\delta_{\mu\nu}\, g\,. \tag{A.2}$$

Closure of the conformal algebra for $\big[\hat{\mathcal{D}}, \hat{\mathcal{K}}_\mu\big]$ now requires instead of just (3.11)

$$D^{(d-\delta)} f_\mu + f_\mu \overleftarrow{D}^{(\delta)} - K^{(d-\delta)}{}_\mu\, g - g\, \overleftarrow{K}^{(\delta)}{}_\mu - g \cdot f_\mu + f_\mu \cdot g = f_\mu\,, \tag{A.3a}$$

$$D^{(\delta)} F_\mu + F_\mu \overleftarrow{D}^{(\delta)} - K^{(\delta)}{}_\mu\, G - G\, \overleftarrow{K}^{(\delta)}{}_\mu + g^T \cdot F_\mu + F_\mu \cdot g - f_\mu{}^T \cdot G - G \cdot f_\mu = F_\mu\,. \tag{A.3b}$$

Imposing $\big[\hat{\mathcal{K}}_\mu, \hat{\mathcal{K}}_\nu\big] = 0$ also gives more general equations

$$K^{(d-\delta)}{}_\mu\, f_\nu + f_\nu \overleftarrow{K}^{(\delta)}{}_\mu - K^{(d-\delta)}{}_\nu\, f_\mu - f_\mu \overleftarrow{K}^{(\delta)}{}_\nu - f_\mu \cdot f_\nu + f_\nu \cdot f_\mu = 0\,, \tag{A.4a}$$

$$K^{(\delta)}{}_\mu\, F_\nu + F_\nu \overleftarrow{K}^{(\delta)}{}_\mu - K^{(\delta)}{}_\nu\, F_\mu - F_\mu \overleftarrow{K}^{(\delta)}{}_\nu + f_\mu{}^T \cdot F_\nu + F_\nu \cdot f_\mu - f_\nu{}^T \cdot F_\mu - F_\mu \cdot f_\mu = 0\,. \tag{A.4b}$$

There are also additional contributions arising from (A.1) so that in (3.12) it is necessary to take $E_{S_*}[\varphi] \to \hat{E}_{S_*}[\varphi]$ and $E_{S_*\mu}[\varphi] \to \hat{E}_{S_*\mu}[\varphi]$ where

$$\hat{E}_{S_*}[\varphi] = E_{S_*}[\varphi] + \varphi \cdot g \cdot \frac{\delta}{\delta\varphi}\, S_*[\varphi]\,, \quad \hat{E}_{S_*\mu}[\varphi] = E_{S_*\mu}[\varphi] + \varphi \cdot f_\mu \cdot \frac{\delta}{\delta\varphi}\, S_*[\varphi]\,, \tag{A.5}$$

A particular solution for g, f_μ is obtained by taking

$$g = -\mathcal{G}^{-1} \cdot G\,, \qquad f_\mu = -\mathcal{G}^{-1} \cdot F_\mu\,, \tag{A.6}$$

and using (5.7). With the choice (A.6) (A.2) follows from (3.4), (3.6) and the additional g, f_μ terms in (A.3b) and (A.4b) vanish. It is also easy to verify (A.3a), (A.4a). The solutions for G, F_μ given by (3.5), (3.7) then remain valid while for $\mathcal{G}^{-1}$ it is sufficient just to solve (4.3). If we let

$$S_*[\varphi] = \mathcal{S}_*[\varphi] + \tfrac{1}{2}\,\varphi \cdot \mathcal{G}^{-1} \cdot \varphi\,, \tag{A.7}$$

then $\hat{\mathcal{D}} = \mathcal{D}$, $\hat{\mathcal{K}}_\mu = \mathcal{K}_\mu$ if in the expressions (3.1) and (3.2) we let $S_* \to \mathcal{S}_*$. Furthermore using (A.6)

$$\hat{E}_{S_*}[\varphi] = E_{\mathcal{S}_*}[\varphi] - \tfrac{1}{2}\,\mathrm{tr}\big(G \cdot \mathcal{G}^{-1}\big)\,, \qquad \hat{E}_{S_*\mu}[\varphi] = E_{\mathcal{S}_*\mu}[\varphi] - \tfrac{1}{2}\,\mathrm{tr}\big(F_\mu \cdot \mathcal{G}^{-1}\big)\,, \tag{A.8}$$

so that the conditions on $\mathcal{S}_*$ are the same as those on S_* without the additional terms in (A.1).

As a variant we consider restricting $\delta = \delta_0$ as in (2.14) in which case the solution of (4.3) is simple in terms of the Fourier transforms $\tilde{\mathcal{G}}(p^2), \tilde{G}(p^2)$ since

$$\tilde{\mathcal{G}}(p^2) = \frac{K(p^2)}{p^2}\,, \qquad \tilde{G}(p^2) = -2K'(p^2)\,. \tag{A.9}$$

We may also relax (A.6) by now taking

$$g = -\mathcal{G}^{-1} \cdot G + \tfrac{1}{2}\eta\, I\,, \qquad f_\mu = -\mathcal{G}^{-1} \cdot F_\mu + \eta\, X_\mu\,, \tag{A.10}$$

for I, X_μ defined in (2.2), and η an arbitrary parameter. The additional terms in (A.10) involving η satisfy (A.2) and also (A.3a),(A.3b) and (A.4a),(A.4b) still hold as a consequence of $X_\mu \cdot G + G \cdot X_\mu = F_\mu$, $X_\mu \cdot F_\nu + F_\nu \cdot X_\mu = X_\nu \cdot F_\mu + F_\mu \cdot X_\nu$ and $X_\mu \cdot X_\nu = X_\nu \cdot X_\mu$. Assuming (A.10) and $S_*, \mathcal{S}_*$ related as in (A.7) then $\hat{\mathcal{D}} = \mathcal{D}$, $\hat{\mathcal{K}}_\mu = \mathcal{K}_\mu$ if now in (3.1) and (3.2) we let $S_* \to \mathcal{S}_*$ and assume (7.19). Instead of (A.8) there are now additional terms quadratic in φ,

$$\begin{aligned} \hat{E}_{S_*}[\varphi] &= E_{\mathcal{S}_*}[\varphi] + \tfrac{1}{2}\eta\, \varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \tfrac{1}{2}\,\mathrm{tr}\big(G \cdot \mathcal{G}^{-1}\big)\,, \\ \hat{E}_{S_*\mu}[\varphi] &= E_{\mathcal{S}_*\mu}[\varphi] + \tfrac{1}{2}\eta\, \varphi \cdot (X_\mu \cdot \mathcal{G}^{-1} + \mathcal{G}^{-1} \cdot X_\mu) \cdot \varphi - \tfrac{1}{2}\,\mathrm{tr}\big(F_\mu \cdot \mathcal{G}^{-1}\big)\,. \end{aligned} \tag{A.11}$$

If $\eta = 0$ a solution is $\mathcal{S}_* = 0$.

Although the functional forms are somewhat different the functional representation of the conformal group generators provided by (A.1) with (A.10) is equivalent to that given in section 3. To demonstrate this it is sufficient in the above to rescale the fields where

$$\hat{\varphi} = \mathcal{G}_{\delta_0} \cdot \mathcal{G}^{-1} \cdot \varphi\,, \qquad \frac{\delta}{\delta \hat{\varphi}} = \mathcal{G}_{\delta_0}{}^{-1} \cdot \mathcal{G} \cdot \frac{\delta}{\delta \varphi}\,, \tag{A.12}$$

and correspondingly replace G, F_μ by $\hat{G}, \hat{F}_\mu$ where

$$\hat{G} = \mathcal{G}_{\delta_0} \cdot \mathcal{G}^{-1} \cdot G \cdot \mathcal{G}^{-1} \cdot \mathcal{G}_{\delta_0}\,, \quad \hat{F}_\mu = \mathcal{G}_{\delta_0} \cdot \mathcal{G}^{-1} \cdot F_\mu \cdot \mathcal{G}^{-1} \cdot \mathcal{G}_{\delta_0}\,, \tag{A.13}$$

satisfying (3.4) and (3.6), so that, with (A.9), $\tilde{\hat{G}}(p^2) = -2K'(p^2)/K(p^2)^2$. Applying (A.12) and using $D^{(\delta)}\hat{\varphi} \cdot \frac{\delta}{\delta\hat{\varphi}} = (D^{(\delta)}\varphi - G \cdot \mathcal{G}^{-1} \cdot \varphi) \cdot \frac{\delta}{\delta\varphi}$, $K^{(\delta)}{}_\mu \hat{\varphi} \cdot \frac{\delta}{\delta\hat{\varphi}} = (K^{(\delta)}{}_\mu \varphi - F_\mu \cdot \mathcal{G}^{-1} \cdot \varphi) \cdot \frac{\delta}{\delta\varphi}$ ensures

$$\begin{aligned} E_{S_*}[\hat{\varphi}]\big|_{G \to \hat{G}} &= E_{\mathcal{S}_*}[\varphi] + \tfrac{1}{2}\eta\, \varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \tfrac{1}{2}\,\mathrm{tr}\big(G \cdot \mathcal{G}^{-1}\big)\,, \\ E_{S_*\mu}[\hat{\varphi}]\big|_{F_\mu \to \hat{F}_\mu} &= E_{\mathcal{S}_*\mu}[\varphi] + \tfrac{1}{2}\eta\, \varphi \cdot (X_\mu \cdot \mathcal{G}^{-1} + \mathcal{G}^{-1} \cdot X_\mu) \cdot \varphi - \tfrac{1}{2}\,\mathrm{tr}\big(F_\mu \cdot \mathcal{G}^{-1}\big)\,, \end{aligned} \tag{A.14}$$

for

$$S_*[\hat{\varphi}] = \mathcal{S}_*[\varphi] + \tfrac{1}{2}\, \varphi \cdot \mathcal{G}^{-1} \cdot \varphi\,. \tag{A.15}$$

The associated eigenvalue equations determining exponents and local scaling operators are also equivalent since $\mathcal{D}, \mathcal{K}_\mu$ are identical with the expressions (3.1) and (3.2) for S_* given by (A.15) and also for $G \to \hat{G}$.

With the representation (A.1), (A.10) we may also determine exact local scaling operators $\Phi_R[\varphi]$, $\Phi_\varphi[\varphi]$ with $\Delta = d - \delta, \delta$ for δ given by (7.19). The results are identical to those given in (3.20) and (3.21) for $\varphi \to \hat{\varphi}$ and $\mathcal{H} \to \hat{\mathcal{H}}$, satisfying (3.23) for $G \to \hat{G}$ and $F_\mu \to \hat{F}_\mu = \mathcal{G}_{\delta_0} \cdot \mathcal{G}^{-1} \cdot F_\mu \cdot \mathcal{G}^{-1} \cdot \mathcal{G}_{\delta_0}$, where

$$\tilde{\hat{\mathcal{H}}}(p^2) = -(p^2)^{\frac{1}{2}\eta - 1} \int_0^{p^2} \mathrm{d}x \, x^{-\frac{1}{2}\eta} \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{K(x)} \, . \tag{A.16}$$

B Operator Product Expansion

The short distance expression (3.57) for the bilocal functional $\mathcal{E}_{12}[\varphi; x, y]$, defined by (3.55) with the asympotic condition (3.56), can be extended in a form analogous to the operator product expansion for the product $\mathcal{O}_1(x)\, \mathcal{O}_2(y)$ for $x \sim y$. For $\mathcal{O}$ a local conformal primary operator with scale dimension Δ, satisfying (2.3), (2.4),

$$\mathcal{E}_{12}[\varphi; x, y] \sim \frac{c_{12}}{s^{\Delta_1}}\, \delta_{\Delta_1 \Delta_2} + \frac{c_{12\mathcal{O}}}{s^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta)}}\, C(x - y, \partial_y)\, \mathcal{O}[\varphi; y] \, , \tag{B.1}$$

solves (3.55) if the differential operator $C(x - y, \partial_y)$, which generates the contributions to $\mathcal{E}_{12}$ of all descendants of $\mathcal{O}$, satisfies the crucial equations

$$D_x{}^{(0)} C(x - y, \partial_y) + \big[D_y{}^{(0)} \, , C(x - y, \partial_y) \big] = 0 \, , \tag{B.2}$$

and

$$\begin{aligned} \Delta (x - y)_\mu \, C(x - y, \partial_y) + \big[K_y{}^{(\Delta)}{}_\mu , C(x - y, \partial_y) \big] + K_x{}^{(0)}{}_\mu C(x - y, \partial_y) & \\ = 2 y_\nu \, C(x - y, \partial_y) S_{\mu\nu} \, , & \end{aligned} \tag{B.3}$$

with $S_{\mu\nu}$ the appropriate spin matrix acting on $\mathcal{O}$ as in (1.7). Choosing a convenient normalisation for $C(x - y, \partial_y)$, $\mathcal{E}_{12}$ then determines $c_{12\mathcal{O}}/c_{12}$ which is independent of the arbitrary normalisations of $\mathcal{O}_1, \mathcal{O}_2$.

(B.3) gives

$$(L_{x\,\mu\nu} + L_{r\,\mu\nu})\, C(x, r) = C(x, r) S_{\mu\nu} \, , \qquad (x \cdot \partial_x - r \cdot \partial_r)\, C(x, r) = 0 \, , \tag{B.4}$$

with $L_{\mu\nu}$ given by (1.6), as well as the second order equation

$$\big(- x_\nu L_{x\,\mu\nu} + x_\mu\, (r \cdot \partial_r + \Delta) - 2 (r \cdot \partial_r + \Delta)\, \partial_{r\,\mu} + r_\mu\, \partial_r{}^2 \big)\, C(x, r) = -2\, \partial_{r\,\nu} C(x, r) S_{\mu\nu} \, . \tag{B.5}$$

(B.4) is equivalent to (B.2) as well as rotational invariance. As an expansion (B.5) with (B.4) for scalar $\mathcal{O}$ gives

$$C(x, r) = 1 + \tfrac{1}{2}\, x \cdot r + \frac{\Delta + 2}{8(\Delta + 1)}\, (x \cdot r)^2 - \frac{\Delta}{16(\Delta + 1)(\Delta - \frac{1}{2} d + 1)}\, x^2 r^2 + \dots \, , \tag{B.6}$$

assuming $C(0,r)=1$.

Taking $\mathcal{O}_1 = \mathcal{O}_2 = \Phi$, $\Delta_1 = \Delta_2 = \Delta_\Phi$ the expansion (B.1) can be extended to also include the energy momentum tensor $\Theta_{\alpha\beta}$

$$\begin{aligned}\mathcal{E}_{\Phi\Phi}[\varphi;x,y] \sim \frac{c_{\Phi\Phi}}{s^{\Delta_\Phi}} + \frac{c_{\Phi\Phi\mathcal{O}}}{s^{\Delta_\Phi-\frac{1}{2}\Delta}}\, C(x-y,\partial_y)\mathcal{O}[\varphi;y] \\ + \frac{c_{\Phi\Phi\Theta}}{s^{\Delta_\Phi-\frac{1}{2}d+1}}\,(x-y)_\alpha(x-y)_\beta\,\Theta_{\alpha\beta}[\varphi;y] + \dots\,. \end{aligned} \tag{B.7}$$

The coefficient $c_{\Phi\Phi\Theta}$ in (B.7) is constrained by Ward identities [4] so that

$$c_{\Phi\Phi\Theta} = -\frac{d\Delta_\Phi}{d-1}\,\frac{c_{\Phi\Phi}}{c_{\Theta\Theta}}\,. \tag{B.8}$$

As an illustration for $\Phi = \varphi$, $\Delta_\Phi = \delta_0$, $\mathcal{O}_{\varphi\varphi}[\varphi;x,y] = \varphi(x)\varphi(y)$ then in (B.7) we should take $\mathcal{O} = \varphi^2$ with $\Delta = 2\delta_0 = d-2$ and $c_{\Phi\Phi} = 1/(d-2)$, $c_{\Theta\Theta} = d/(d-1)$. Then $c_{\varphi\varphi\mathcal{O}} = 1$, $c_{\varphi\varphi\Theta} = -\frac{1}{2}$ and it is easy to see that $\Theta_{\alpha\beta}$ is in accord with (2.16).

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