Operator Product Expansion in Quantum Field Theory

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Introduction

The operator product expansion provides an algebraic structure in quantum field theory. In a sense it supercedes or rather transcends the equal time commutation relations which provide the traditional starting point for the canonical quantisation of any quantum field theory. The essential idea is that for any two local operator quantum fields at spacetime points x_1, x_2 their product may be expressed in terms of a series of other local quantum fields at a point x, which may be identified with x_1 or x_2, times c-number coefficient functions which depend on $x_1 - x_2$. The set of operators which may appear depends on the particular quantum field theory and must of course be in accord with any requirements of conserved quantum numbers. The coefficient functions depend on $x_1 - x_2$ in a fashion which depends on the dimensions of the various operators involved, at least up to renormalisation group corrections. The most singular contributions are those for the operators appearing in the OPE with lowest scale dimension. From a phemenonological point of view only the first few terms in the operator product expansion are of relevance. However theoretically, especially for conformal field theories, it is desirable to know the full expansion to all orders in powers of $x_1 - x_2$ in such a way that the operator product may be replaced by the full expansion in appropriate correlation functions. We first discuss the OPE for free theories and then the interacting case.

Free Field Theory

The operator product expansion is most straightforward in free field theory when it almost reduces to a Taylor series expansion. For a simple free massless scalar field $\phi(x)$ then in four dimensions we may write

$$\phi(x)\phi(0) = \frac{C}{x^2} + :\phi(x)\phi(0):\,, \tag{1}$$

where $: \, :$ denotes normal ordering (moving all annihilation operators to the right of creation operators) and C is just a normalisation numerical constant (for canonical normalisation $C = 1/4\pi^2$). The $1/x^2$ term proportional to the identity operator reflects the leading singular behaviour at short distances of $\phi(x)\phi(0)$, the power being determined by ϕ having dimension 1. For the normal ordered term we may expand in terms of an infinite set of local operators by using the Taylor expansion

$$:\phi(x)\phi(0): = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^{\mu_1} \dots x^{\mu_n} \; :\partial_{\mu_1} \dots \partial_{\mu_n}\phi(0)\,\phi(0):\,, \tag{2}$$

where the operator appearing in the n'th term has dimension $n+2$. Manifestly at short distances only the leading terms are relevant. The equation (1) also provides a point splitting definition of the local composite operator $:\phi^2(0):$ in terms of limit of $\phi(x)\phi(0)$ as $x \to 0$ after subtraction of the singular C/x^2 term.

The OPE can be easily generalised to composite operators defined by normal ordering. For $:\phi^2:$ we have, by applying Wick's theorem,

$$:\phi^2(x)::\phi^2(0): = \frac{2C^2}{x^4} + \frac{4C}{x^2}:\phi(x)\phi(0): + :\phi^2(x)\,\phi^2(0):\,, \tag{3}$$

where Taylor series expansion may be applied to both $:\phi(x)\phi(0):$ and also $:\phi^2(x)\,\phi^2(0):$ to give an infinite sequence of local operators of increasing dimensions.

The expansion in terms of local operators may be reordered. For instance from (1) we may write, using $\partial^2\phi = 0$,

$$\begin{aligned}\phi(x)\phi(0) = \frac{C}{x^2} + \Big(1 + \tfrac{1}{2}x^\mu\partial_\mu + \tfrac{1}{4}x^\mu x^\nu\partial_\mu\partial_\nu + \tfrac{1}{16}x^2\partial^2\Big) :\phi^2(0): \\ - \tfrac{1}{2}x^\mu x^\nu\, T_{\mu\nu} + \mathrm{O}(x^3)\,, \end{aligned} \tag{4}$$

where

$$T_{\mu\nu} =: \partial_\mu\phi\partial_\nu\phi : - \tfrac{1}{4}\eta_{\mu\nu} : \partial\phi\cdot\partial\phi :\,, \tag{5}$$

is the energy momentum tensor. In (4), and also in a similar context subsequently, we define $\partial :\phi^2(0): = \partial_y :\phi^2(y):|_{y=0}$. The expansion (4) provides a point splitting definition of $T_{\mu\nu}$ and also demonstrates that many operators appearing in the operator product expansion are expressible in terms of overall derivatives of lower dimension operators. We may also note that without further input there is an ambiguity in the definition of $T_{\mu\nu}$ of the form

$$T_{\mu\nu} \sim T_{\mu\nu} + a(\partial_\mu\partial_\nu - \tfrac{1}{4}\eta_{\mu\nu}\partial^2) :\phi^2: \,. \tag{6}$$

In a conformal theory however we require $a = -\frac{1}{6}$.

Interacting Theories

The operator product expansion becomes an essential tool in the context of interacting quantum field theories. For renormalisable quantum field theories various results can be proved to all orders in the standard perturbative expansion and are naturally assumed to be properties of the complete theory. In interacting theories we may no longer use normal ordering to define composite operators which in general have anomalous dimensions. The coefficient functions appearing in the operator product expansion also gain perturbative corrections but these are constrained by renormalisation group Callan-Symanzik equations.

Again if we consider the simplest case of a massless scalar theory as above but now with a renormalised coupling constant g the leading terms in the expansion of $\phi(x)\phi(0)$ are of the form (here we assume a Z_2 symmetry under $\phi \to -\phi$, otherwise the operator ϕ would be expected to appear in the operator product expansion)

$$\phi(x)\phi(0) = \frac{C(g,\mu^2x^2)}{x^2} + D(g,\mu^2x^2)\,\phi^2(0) + \dots\,, \tag{7}$$

where μ is an arbitrary renormalisation scale. This arbitrariness is reflected in the RG equation

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_\phi(g)\Big)C(g,\mu^2x^2) = 0\,. \tag{8}$$

At a fixed point $\beta(g_*) = 0$ this equation may be solved with an arbitrary choice of normalisation to give $C(g_*,\mu^2x^2) = (\mu^2x^2)^{-\gamma_\phi(g_*)}$ which corresponds to the fields ϕ having a modified scale dimension $1 + \gamma_\phi(g_*)$. In a similar fashion the coefficient $D(g,\mu^2x^2)$ in (7) satisfies

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_\phi(g) - \gamma_{\phi^2}(g)\Big)D(g,\mu^2x^2) = 0\,, \tag{9}$$

where it is necessary to introduce a new anomalous dimension function $\gamma_{\phi^2}(g)$ related to the composite operator ϕ^2. Although it is natural to label the operator as ϕ^2 its definition in terms of the elementary field ϕ is essentially only as given in terms of the OPE (9). At a fixed point again $D(g_*,\mu^2x^2) = k(\mu^2x^2)^{-\gamma_\phi(g_*)+\frac{1}{2}\gamma_{\phi^2}(g_*)}$ where the coefficient k is determined by the scale of the three point function $\langle\phi(x)\phi(y)\phi^2(0)\rangle$. In asymptotically free theories the RG equations show that at short distances the coefficient functions tend to those of free field theory but with calculable logarithmic corrections. More generally for a set of operators $\{O_i\}$ the OPE has the form

$$O_i(x)O_j(0) \sim \frac{1}{(x^2)^p}\sum_k C_{ijk}(g,\mu^2x^2)O_k(0)\,, \tag{10}$$

where p is determined by the free scale dimensions of the O_i and

$$\begin{aligned}&\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\Big)C_{ijk}(g,\mu^2x^2) \qquad (11)\\ &= \sum_n \Big(\gamma_{kn}(g)C_{ijn}(g,\mu^2x^2) - \gamma_{in}(g)C_{njk}(g,\mu^2x^2) - \gamma_{jn}(g)C_{ink}(g,\mu^2x^2)\Big)\,,\end{aligned}$$

with $\gamma_{in}(g)$ the anomalous dimension matrix arising from the mixing of composite operators.

An important aspect of the operator product expansion is that the coefficient functions may be calculated perturbatively, essentially by applying the OPE in some suitable correlation function. Essentially the OPE provides a factorisation between short distance UV singularities and non perturbative effects. In a Feynman graph the short distances in an operator product correspond to the large momentum behaviour and power counting theorems allow a factorisation up to calculable logarithmic corrections. A detailed analysis depends on the detailed technicalities of the proofs of renormalisation to all orders of perturbation theory.

The coefficient functions in the OPE should be independent of any infra red or non perturbative long distance effects (such as confinement in QCD). However the operators which appear in the OPE, such as ϕ^2 above, may have non zero expectation values which are absent to all orders in perturbation theory.

Perturbative Example

The general considerations can be illustrated by considering a scalar field theory to lowest order in a perturbative expansion. We consider a four dimensional theory with a single scalar field and a potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{24}g\phi^4$. Using dimensional regularisation m^2, as well as g, is treated as a coupling with an associated β-function $\gamma_{\phi^2}(g)m^2$. With a mass term the operator ϕ^2 mixes with the identity operator so that

$$\begin{aligned} &\big(\mathcal{D} + \gamma_{\phi^2}(g)\big)\langle\phi^2(0)\rangle = -\gamma_{\phi^2 I}(g)m^2\,, \\ &\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_{\phi^2}(g)m^2\frac{\partial}{\partial m^2}\,, \end{aligned} \tag{12}$$

where $\gamma_{\phi^2 I}$ reflects the mixing. At one loop order we have

$$\beta(g) = \frac{3g^2}{16\pi^2}\,, \qquad \gamma_{\phi^2}(g) = \frac{g}{16\pi^2}\,, \qquad \gamma_{\phi^2 I}(g) = \frac{1}{8\pi^2}\,, \tag{13}$$

and we may also set $\gamma_\phi(g) = 0$. In this case in the operator product expansion (7) the coefficient C also depends on m^2x^2 and the RG equations (8) and (9) are now modified to include the effects of mixing

$$\begin{aligned}&\mathcal{D}C(g,m^2x^2,\mu^2x^2) = m^2x^2\gamma_{\phi^2 I}(g)\, D(g,\mu^2x^2)\,,\\ &\big(\mathcal{D}-\gamma_{\phi^2}(g)\big)C(g,\mu^2x^2) = 0\,.\end{aligned} \tag{14}$$

From lowest order perturbation theory with (13), and using (14) to include all orders in $g\ln\mu^2x^2$, we have in this approximation

$$\begin{aligned}&C(g,m^2x^2,\mu^2x^2)\\ &= \frac{1}{4\pi^2} - \frac{2m^2x^2}{g}\Big(1+\frac{3g}{32\pi^2}\ln\mu^2x^2\Big)^{\frac{2}{3}}\bigg(\Big(1+\frac{3g}{32\pi^2}\ln\mu^2x^2\Big)^{-\frac{1}{3}}-1\bigg)\,,\\ &D(g,\mu^2x^2) = \Big(1+\frac{3g}{32\pi^2}\ln\mu^2x^2\Big)^{\frac{1}{3}}\,.\end{aligned} \tag{15}$$

The operator product expansion then reproduces the small x behaviour of the two point function $\langle\phi(x)\phi(0)\rangle$ at one loop, expanding C, D to first order in g, if we take

$$\langle\phi^2(0)\rangle = -\frac{m^2}{8\pi^2}\ln\frac{\mu}{m} + \mathrm{O}(g)\,, \tag{16}$$

which is in accord with (12). If $m^2<0$ the symmetry $\phi\leftrightarrow-\phi$ is broken and it is necessary to shift the field $\phi = v+f$, with $v^2=-6m^2/g$ and the field f has a mass m_f with $m_f^2=-2m^2$. The operator product expansion (7) with the same coefficient functions as in (15) remains valid. The two point function $\langle\phi(x)\phi(0)\rangle$, which includes a non perturbative term v^2, is again reproduced for small x at one loop now if

$$\langle\phi^2(0)\rangle = -\frac{6m^2}{g} - \frac{m^2}{2\pi^2}\ln\frac{\mu}{m_f} + \mathrm{O}(g)\,, \tag{17}$$

but in this case it is necessary to expand $D(g,\mu^2x^2)$ to $\mathrm{O}(g^2)$ as a consequence of the leading $1/g$ term in (17). Note that both (16) and (17) contain the non perturbative dependance on $\ln m$ and $\ln m_f$ which is present in the two point function.

Conformal Field Theories

When the β-function vanishes and a quantum field theory enjoys conformal invariance the operator product expansion is a potentially convergent expansion. It is natural to restrict to conformal quasi-primary operators which do not mix with lower scale dimensions under conformal transformations. If we consider for instance two scalar operators ϕ with scale dimension Δ_ϕ then the operator product expansion has the generic form

$$\phi(x)\phi(0) = \frac{1}{x^{2\Delta_\phi}} + \sum_I C_{\phi\phi O^I} \frac{1}{(x^2)^{\frac{1}{2}(2\Delta_\phi - \Delta^I + \ell)}} C^{(\ell)}_{\Delta^I}(x, \partial)^{\mu_1,\dots,\mu_\ell} O^I_{\mu_1,\dots,\mu_\ell}(0)\,, \tag{18}$$

where there is a sum over quasi-primary operators $O^I_{\mu_1,\dots,\mu_\ell}$ with scale dimension Δ^I and spin ℓ, so they are symmetric traceless tensors of rank ℓ. In the first term in (18) the coefficient is chosen to be 1 by a choice of normalisation. The coefficients $C_{\phi\phi O^I}$, with a standard normalisation for O^I, are then determined by the coefficients of the corresponding three point functions involving $\phi\phi$ and O^I. In (18) $C^{(\ell)}_{\Delta^I}$ are differential operators which sum up the contributions of all derivatives or descendants of the quasi-primary operator O^I. They can be explicitly given in terms of an integral representation, for any space-time dimension, where the scale is fixed by requiring for the leading term $C^{(\ell)}_{\Delta^I}(x,0)^{\mu_1,\dots,\mu_\ell} = x^{\mu_1}\dots x^{\mu_\ell} -$ traces. The spectrum of operators which appear is obviously a property of the particular conformal field theory.

Ward Identities

If the theory has a symmetry with corresponding conserved currents then there are Ward identities which constrain the operator product expansion of fields with the conserved current. For a current $J_{\mu a}$ then we have in d-dimensions the singular contribution in the operator product expansion is given by

$$J_{\mu a}(x)O(0) \sim -\frac{1}{S_d}\frac{x_\mu}{(x^2)^{\frac{1}{2}d}} t_a O(0)\,, \tag{19}$$

where t_a are a set of matrix generators corresponding to the symmetry acting on the fields O and S_d is the volume of the unit $(d-1)$-dimensional sphere, $S_4 = 2\pi^2$. For a conserved current there are no anomalous dimensions and the coefficient in (19), which depends on the normalisation for the current $J_{\mu a}$, is chosen so that $[Q_a, O(0)] = -t_a O(0)$ with Q_a the charge formed from $J_{\mu a}$. For the energy momentum tensor the operator there is an analogous result. We consider the simpler case of a conformal theory when the energy momentum tensor is both conserved and traceless and

$$T_{\mu\nu}(x)\, O(0) \sim A_{\mu\nu}(x) O(0) + B_{\mu\nu\lambda}(x) \partial^\lambda O(0) + \dots\,, \tag{20}$$

where $A_{\mu\nu}(x) = \mathrm{O}(x^{-d})$ and $B_{\mu\nu\lambda}(x) = \mathrm{O}(x^{-d+1})$. As a distribution $A_{\mu\nu}(x)$ is ambiguous up to terms proportional to $\delta^d(x)$. If Δ is the scale dimension of O and $s_{\mu\nu}$ are the Lorentz spin generators acting on O the Ward identities then give

$$\begin{aligned} &\partial^\mu A_{\mu\nu}(x) = \Big(\frac{\Delta}{d}\, \eta_{\nu\lambda} + C_{\nu\lambda} + \tfrac{1}{2} s_{\nu\lambda} \Big) \partial^\lambda \delta^d(x)\,, \\ &\eta^{\mu\nu} A_{\mu\nu}(x) = \eta^{\mu\nu} C_{\mu\nu} \delta^d(x)\,, \qquad \partial^\mu B_{\mu\nu\lambda}(x) = - \eta_{\nu\lambda} \delta^d(x)\,, \end{aligned} \tag{21}$$

where $C_{\mu\nu}$ is a constant tensor reflecting the arbitrariness in $A_{\mu\nu}$, it is immaterial as far as Ward identities are concerned. We may choose

$$\frac{\Delta}{d}\, \eta_{\nu\lambda} + C_{\nu\lambda} = 0\,. \tag{22}$$

(If desired we might also take $A'_{\mu\nu}(x) = A_{\mu\nu}(x) + \frac{1}{2} s_{\mu\nu} \delta^d(x)$ in which case $\partial^\mu A'_{\mu\nu}(x) = 0\,,\ A'_{[\mu\nu]}(x) = \frac{1}{2} s_{\mu\nu} \delta^d(x)$ but such an antisymmetric piece seems unnatural). In general there is no unique form for $A_{\mu\nu}(x)$, as a consequence of the freedom of choice for $C_{\nu\lambda}$ in (21). However for a scalar field O we must have, for $x \neq 0$,

$$\begin{aligned} A_{\mu\nu}(x) &= \frac{\Delta}{d-1} \frac{1}{S_d} \Big(\eta_{\mu\nu} - d\, \frac{x_\mu x_\nu}{x^2} \Big) \frac{1}{(x^2)^{\frac{1}{2}d}} \\ &= - \frac{\Delta}{(d-1)(d-2)} \frac{1}{S_d} \partial_\mu \partial_\nu \frac{1}{(x^2)^{\frac{1}{2}d-1}}\,, \end{aligned} \tag{23}$$

with the overall scale determined by (21).

For the operator product of the current $J_{\mu a}$ with itself there is an additional term proportional to the identity operator of the form

$$J_{\mu a}(x) J_{\nu b}(0) \sim C_J\, \delta_{ab} \Big(\eta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2} \Big) \frac{1}{x^{2(d-1)}}\,, \tag{24}$$

where the coefficient C_J, which is determines the scale of the two point function for $J_{\mu a}$, is well defined since the normalisation of the current is determined through the Ward identity. A similar result also holds for the operator product of the energy momentum tensor with itself, with an overall coefficient C_T. In general we may also write for the operator product of two scalar fields O

$$O(x)O(0) \sim C_O \, \frac{1}{x^{2\Delta}} - \frac{C_O}{C_T S_d} \, \frac{d\Delta}{d-1} \, \frac{1}{(x^2)^{\Delta-\frac{1}{2}d+1}} x^\mu x^\nu T_{\mu\nu}(0) \, , \qquad (25)$$

neglecting other contributions. The contribution of the energy momentum tensor does not therefore introduce any new coefficient.

Two Dimensions

In two dimensions the operator product expansion plays an essential role in the discussion of conformal field theories. For a Euclidean metric it is natural to use complex variables z and $\bar{z}$. The energy momentum tensor in this case reduces to a chiral field $T(z)$ and its conjugate $\bar{T}(\bar{z})$. For the operator product with a chiral field $\phi(z)$ with scale dimension Δ

$$T(z)\phi(0) \sim \frac{\Delta}{z^2} \, \phi(0) + \frac{1}{z} \, \phi'(0) \, , \qquad (26)$$

and for the operator product of T with itself

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2} \, T(0) + \frac{1}{z} \, T'(0) \, , \qquad (27)$$

Here c is the Virasoro central charge which plays a critical role in the discussion of two dimensional conformal field theories, it is given by the two point function which follows from (27), $\langle T(z)T(0)\rangle = \frac{1}{2}c \, z^{-4}$.

In simple rational conformal field theories the operators are organised into conformal blocks by the infinite dimensional extended conformal symmetry in two dimensions. This allows the full spectrum of operators and their dimensions to be determined and in consequence complete results for the operator product expansion to be found in many cases.

Further Remarks

The operator product expansion reflects the locality properties of quantum field theories and can be extended without difficulty to curved space backgrounds. For a product $\phi(x)\phi(0)$ the separation x^2 may be replaced by a biscalar at x and 0 but it is necessary to include in the operator product expansion contributions involving the background Riemann tensor as well as the operator fields present in flat space. There is also a generalisation of the operator product expansion for superfields on superspace.

At a fundamental level although the operator product expansion can be derived to all orders in perturbation theory the contribution of non perturbative effects such as instantons to the coefficients is not entirely clear. Issues of associativity have yet to be fully analysed.

There are also important applications to the phemenonological analysis of QCD when assumptions about the OPE and saturation of sum rules can lead to results for the vacuum expectation value of gauge invariant operators such as $F^{\mu\nu}F_{\mu\nu}$.

Further Reading

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