Symmetry and Particle Physics, 1

1. For the tensor product space $\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}$ then the total angular momentum is $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, which can be written in terms of components $J_\pm$, J_3. Let $\mathcal{U}_M$ be the subspace for which J_3 has the eigenvalue M. Determine the dimension of $\mathcal{U}_M$. Show that if $M \geq |j_1 - j_2|$ there is a one-dimensional subspace of $\mathcal{U}_M$ which is orthogonal to $J_-\mathcal{U}_{M+1}$. Hence show that there is a single normalised state, unique up to an overall phase factor, $|\phi\rangle \in \mathcal{U}_M$ such that $J_+|\phi\rangle = 0$ if $M \geq |j_1 - j_2|$ and that we may identify $|JJ\rangle = |\phi\rangle$ for $J = M$. What happens if $M < |j_1 - j_2|$?

2. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in three dimensional Euclidean space, show that

$$T_{ij} = u_i \, v_j = \tilde{T}_{ij} + \tfrac{1}{2}\epsilon_{ijk} V_k + \tfrac{1}{3}\delta_{ij} S$$

separates the components of T_{ij} into subsets of $5, 3, 1$ that transform amongst themselves under $SO(3)$ rotations, where

$$\tilde{T}_{ij} = \tfrac{1}{2}(u_i v_j + u_j v_i) - \tfrac{1}{3}\delta_{ij} u_k v_k \quad , \quad V_k = (\mathbf{u} \times \mathbf{v})_k \quad , \quad S = \mathbf{u}\cdot\mathbf{v} \, .$$

Explain the relation to the results that, if $\mathcal{V}_j$ is the vector space for angular momentum j, then $\mathcal{V}_1 \otimes \mathcal{V}_1 \simeq \mathcal{V}_2 \oplus \mathcal{V}_1 \oplus \mathcal{V}_0$.

3. Three 3×3 matrices T_i are defined by $(T_i)_{jk} = -i\epsilon_{ijk}$. Prove the results ($a = |\mathbf{a}|$),

(i) $\quad [T_i \, , T_j] = i\epsilon_{ijk} T_k \, ,$

(ii) $\quad (\mathbf{a}\cdot\mathbf{T})^3 = a^2 \mathbf{a}\cdot\mathbf{T} \, ,$

(iii) $\quad \exp(-i\mathbf{a}\cdot\mathbf{T}) = 1 - i\mathbf{a}\cdot\mathbf{T} \, \frac{\sin a}{a} + (\mathbf{a}\cdot\mathbf{T})^2 \, \frac{\cos a - 1}{a^2} \, .$

What are the possible eigenvalues of $\mathbf{n}\cdot\mathbf{T}$ if $\mathbf{n}$ is a unit vector?

4. Assuming that the Pauli matrices σ_i are traceless and satisfy $\mathrm{tr}(\sigma_i \sigma_j) = 2\delta_{ij}$ show that for any 2×2 matrix C then $C = \sigma_i \, \frac{1}{2}\mathrm{tr}(\sigma_i C) + 1 \, \frac{1}{2}\mathrm{tr}(C)$.

If $A \in SU(2)$ explain why

$$A \sigma_j A^{-1} = \sigma_i R_{ij} \, ,$$

defines an orthogonal matrix $[R_{ij}]$. Obtain an explicit formula for R_{ij} in terms of A and show that

$$R_{ii} = (\mathrm{tr}(A))^2 - 1 \, .$$

Obtain also $R_{ij}\sigma_i\sigma_j = 2\,\mathrm{tr}(A)\,A - 1$. Hence show that for any orthogonal matrix $[R_{ij}]$ there are two possible associated $SU(2)$ matrices A.

If $[R_{ij}]$ describes a rotation about the z-axis through an angle θ show that $R_{ii} = 2\cos\theta + 1$ and obtain the associated $SU(2)$ matrices A.

5. Show that

$$\mathrm{tr}_{\mathcal{V}_j}\left(e^{-i\theta J_3}\right) = \chi_j(\theta) = \frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,,$$

and that the characters $\chi_j(\theta)$ satisfy the orthogonality conditions

$$\int_0^{2\pi} \mathrm{d}\theta\ \sin^2\tfrac{1}{2}\theta\,\chi_j(\theta)\chi_{j'}(\theta) = \pi\,\delta_{jj'}\,.$$

6. Let $C_{i_1\dots i_l}$ be a symmetrical traceless tensor of rank l. Let $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ be a three-dimensional unit vector giving a point on the unit sphere. Define a tangential derivative such that $\nabla_i \hat{x}_j = \delta_{ij} - \hat{x}_i\hat{x}_j$. For the spherical harmonic $Y_l(\hat{\mathbf{x}}) = C_{i_1\dots i_l}\hat{x}_{i_1}\dots\hat{x}_{i_l}$ show that

$$\nabla^2 Y_l(\hat{\mathbf{x}}) = -l(l+1)\,Y_l(\hat{\mathbf{x}})\,.$$

Show also $\partial^2 Y_l(\mathbf{x}) = 0$ is equivalent to the traceless condition on $C_{i_1\dots i_l}$. Verify that $\partial^2\big(x_i Y_l(\mathbf{x}) - \frac{1}{2l+1}\,\mathbf{x}^2\partial_i Y_l(\mathbf{x})\big) = 0$ so that this defines a symmetric traceless tensor of rank $l+1$.

7. The pion states $|\pi^\pm\rangle$, $|\pi^0\rangle$ have $I=1$ and are defined so that

$$I_\pm|\pi^0\rangle = \sqrt{2}\,|\pi^\pm\rangle\,.$$

Show that

$$e^{-i\pi I_2}|\pi^\pm\rangle = |\pi^\mp\rangle\,, \qquad e^{-i\pi I_2}|\pi^0\rangle = -|\pi^0\rangle\,.$$

The charge conjugation operator C acts on a particle state to give the corresponding anti-particle state, up to an overall phase. Why must $CI_3C^{-1} = -I_3$? Why cannot we have $C(I_1,I_2,I_3)C^{-1} = (I_1,I_2,-I_3)$? Assume $C(I_1,I_2,I_3)C^{-1} = (-I_1,I_2,-I_3)$. Show that

$$C|\pi^0\rangle = |\pi^0\rangle \quad \Rightarrow \quad C|\pi^\pm\rangle = -|\pi^\mp\rangle\,.$$

Define $G = Ce^{-i\pi I_2}$ and hence show

$$G|\pi^\pm\rangle = -|\pi^\pm\rangle\,, \qquad G|\pi^0\rangle = -|\pi^0\rangle\,.$$

Assume that strong interactions are invariant under charge conjugation and $SU(2)_I$. Show that in any strong interaction process $\pi\pi \to n\pi$ then n must be even. $\rho^\pm,\rho^0$ and ω are spin one mesons with $I=1$ and $I=0$ respectively, and $C|\rho^0\rangle = -|\rho^0\rangle$, $C|\omega\rangle = -|\omega\rangle$. What multi-pion decays are allowed for $\rho^\pm,\rho^0$ and ω ($m_\rho \approx m_\omega \approx 750MeV$, $m_\pi \approx 140MeV$)?

8. Explain the experimental results

$$2\frac{d\sigma}{d\Omega}(np\to\pi^0 d) = \frac{d\sigma}{d\Omega}(pp\to\pi^+ d)\,, \qquad \frac{d\sigma}{d\Omega}(pd\to\pi^+ H_3) = 2\frac{d\sigma}{d\Omega}(pd\to\pi^0 He_3)\,.$$

d is the deuteron, while H_3, He_3 are the tritium, helium-3 nuclei.

9. Show that the six possible charge combinations for the decays $\Delta \to \pi + N$ separate into pairs whose amplitudes A are equal because of symmetry under $I_3 \to -I_3$ for all particles. Show that the decay widths, proportional to $|A|^2$, satisfy

$$2\Gamma_{\Delta^{++}\to\pi^+p} = 3\Gamma_{\Delta^+\to\pi^0 p} = 6\Gamma_{\Delta^0\to\pi^- p}\,.$$