Symmetry and Particle Physics, 2

1. Let $T^{\alpha_1 \dots \alpha_{2j}}$ be a symmetric $SU(2)$ tensor for $j = \frac{1}{2}, 1, \frac{3}{2}, \dots$. Show that the action of the spin operator is given by

$$\mathbf{S}^{\alpha_1 \dots \alpha_{2j}}{}_{\beta_1 \dots \beta_{2j}} T^{\beta_1 \dots \beta_{2j}} = \sum_{i=1}^{2j} \tfrac{1}{2} (\boldsymbol{\sigma})^{\alpha_i}{}_{\beta} T^{\alpha_1 \dots \alpha_{i-1} \beta \alpha_{i+1} \dots \alpha_{2j}} \ ,$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. Define for $m = -j, -j+1, \dots, j$

$$T^{(jm)} = \big[(j+m)!(j-m)!\big]^{-\frac{1}{2}} T^{\overbrace{1 \dots 1}^{j+m}\,\overbrace{2 \dots 2}^{j-m}} .$$

Calculate $S_\pm T^{(jm)}$ and $S_3 T^{(jm)}$. Show that $\bar{T}_{\alpha_1 \dots \alpha_{2j}} T^{\alpha_1 \dots \alpha_{2j}} = (2j)! \sum_m T^{(jm)*} T^{(jm)}$ for $\bar{T}_{\alpha_1 \dots \alpha_{2j}}$ the conjugate tensor.

2. A field $\phi(x)$ transforms under the action of a Poincaré transformation (Λ, a) such that $U[\Lambda, a]\phi(x)U[\Lambda, a]^{-1} = \phi(\Lambda x + a)$. For an infinitesimal transformation, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and correspondingly $U[\Lambda, a] = 1 - i\frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} - i a^\mu P_\mu$ show that

$$[M_{\mu\nu}, \phi(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x) \, , \qquad [P_\mu, \phi(x)] = i\partial_\mu \phi(x) \, .$$

Verify that $M_{\mu\nu} \to i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ and $P_\mu \to -i\partial_\mu$ satisfy the algebra for $[M_{\mu\nu}, M_{\sigma\rho}]$ and $[M_{\mu\nu}, P_\sigma]$ expected for the Poincaré group.

3. Show how $B(\theta, \mathbf{n}) \in Sl(2, \mathbb{C})$ where

$$B(\theta, \mathbf{n}) = \cosh \tfrac{1}{2}\theta + \boldsymbol{\sigma} \cdot \mathbf{n} \, \sinh \tfrac{1}{2}\theta \, , \qquad \mathbf{n}^2 = 1 \, ,$$

corresponds to a Lorentz boost with velocity $\mathbf{v} = \tanh\theta \, \mathbf{n}$. Show that

$$\big(1 + \tfrac{1}{2}\boldsymbol{\sigma} \cdot \delta\mathbf{v}\big) B(\theta, \mathbf{n}) = B(\theta', \mathbf{n}') R \, ,$$

where, to first order in $\delta\mathbf{v}$,

$$\theta' = \theta + \delta\mathbf{v} \cdot \mathbf{n} \, , \quad \mathbf{n}' = \mathbf{n} + \coth\theta (\delta\mathbf{v} - \mathbf{n} \, \mathbf{n} \cdot \delta\mathbf{v}) \, ,$$

and R is an infinitesimal rotation given by

$$R = 1 + \tanh \tfrac{1}{2}\theta \ \tfrac{1}{2} i \, (\delta\mathbf{v} \times \mathbf{n}) \cdot \boldsymbol{\sigma} = 1 + \tfrac{\gamma}{\gamma+1} \ \tfrac{1}{2} i \, (\delta\mathbf{v} \times \mathbf{v}) \cdot \boldsymbol{\sigma} \, , \qquad \gamma = (1 - \mathbf{v}^2)^{-\frac{1}{2}} \, .$$

Show that we must have $\mathbf{v}' = \mathbf{v} + \delta\mathbf{v} - \mathbf{v} \, \mathbf{v} \cdot \delta\mathbf{v}$. [Note $\boldsymbol{\sigma} \cdot \mathbf{a} \, \boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \, 1 + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b})$.]

4. The group of four dimensional space-time symmetries may be generalised to conformal transformations $x \to x'$ defined by the requirement

$$\mathrm{d}x'^2 = \Omega(x)^2 \mathrm{d}x^2 \,,$$

where $\mathrm{d}x^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. For an infinitesimal transformation $x'^\mu = x^\mu + f^\mu(x)$, $\Omega(x)^2 = 1 + 2\sigma(x)$. Obtain in this case

$$\partial_\mu f_\nu + \partial_\nu f_\mu = 2\sigma\, g_{\mu\nu} \qquad \Rightarrow \qquad 4\sigma = \partial \cdot f \,.$$

Hence obtain

$$4\,\partial_\sigma \partial_\mu f_\nu = g_{\mu\nu}\,\partial_\sigma \partial \cdot f + g_{\sigma\nu}\,\partial_\mu \partial \cdot f - g_{\sigma\mu}\,\partial_\nu \partial \cdot f \,.$$

From this obtain $2\partial_\sigma \partial_\mu \partial \cdot f = -g_{\sigma\mu}\,\partial^2 \partial \cdot f$ and hence show that we must have $\partial_\sigma \partial_\mu \partial \cdot f = 0$. Why does it then follow that $f_\mu(x)$ can only be quadratic in x? Show that $f^\mu(x)$ must then have the general form

$$f^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2b \cdot x\, x^\mu \,, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \,.$$

Show also that an inversion $x'^\mu = x^\mu / x^2$ is a conformal transformation. Calculate the finite conformal transformation obtained by an inversion followed by a translation by b^μ followed by another inversion and show that it is compatible with the result for $f^\mu(x)$.

5. A four-dimensional space is defined in terms of 6-vectors $\eta^A = (\eta^\mu, \eta^+, \eta^-)$, $\mu = 0, 1, 2, 3$, subject to the relations

$$\eta \cdot \eta = g_{AB}\eta^A \eta^B = g_{\mu\nu}\eta^\mu \eta^\nu - \eta^+ \eta^- = 0 \,, \qquad \eta^A \sim C\eta^A \,.$$

Using $\eta^\pm = \eta^4 \pm \eta^5$ show that this space is invariant under transformations $\eta^A \to G^A{}_B \eta^B$ where $[G^A{}_B]$ are matrices belonging to the group $SO(4,2)$. For an infinitesimal transformation, $G^A{}_B = \delta^A{}_B + \omega^A{}_B$, show that $\omega^A{}_B$ may be decomposed in the form

$$\big[\omega^A{}_B\big] = \begin{pmatrix} \omega^\mu{}_\nu & a^\mu & b^\mu \\ 2b_\nu & -\lambda & 0 \\ 2a_\nu & 0 & \lambda \end{pmatrix} \,, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \,.$$

Suppose, for $\eta^+ \neq 0$, $\eta^A = \eta^+(x^\mu, 1, x^2)$. Using $\delta\eta^A = \omega^A{}_B \eta^B$ determine the corresponding δx^μ. What transformation corresponds to $\eta^+ \leftrightarrow \eta^-$? For four points x_i, $i = 1, 2, 3, 4$, calculate $\eta_1 \cdot \eta_2\, \eta_3 \cdot \eta_4 / (\eta_1 \cdot \eta_3\, \eta_2 \cdot \eta_4)$. Why is this a conformal invariant?

6. Consider the subgroup of the Galilean group corresponding to translations and boosts where

$$t' = t + b \,, \qquad \mathbf{x}' = \mathbf{x} + \mathbf{a} + \mathbf{v}t \,.$$

Denoting the corresponding group element by $(b, \mathbf{a}, \mathbf{v})$ work out the group multiplication law and show that $(b_2, \mathbf{a}_2, \mathbf{v}_2)^{-1}(b_1, \mathbf{a}_1, \mathbf{v}_1)^{-1}(b_2, \mathbf{a}_2, \mathbf{v}_2)(b_1, \mathbf{a}_1, \mathbf{v}_1) = (0, 0, b_1\mathbf{v}_2 - b_2\mathbf{v}_1)$. Suppose, for infinitesimal $b, \mathbf{a}, \mathbf{v}$, the associated unitary operator has the form

$$U[b, \mathbf{a}, \mathbf{v}] = 1 + ib\,H - i\mathbf{a} \cdot \mathbf{P} + i\mathbf{v} \cdot \mathbf{K} \,,$$

work out the corresponding commutators. For general $\mathbf{a}, \mathbf{v}$ require $U[0, \mathbf{a}, \mathbf{v}] = T[\mathbf{a}]U_B[\mathbf{v}]$ and assume now $T[\mathbf{a}]U_B[\mathbf{v}] = e^{im\,\mathbf{a}\cdot\mathbf{v}}U_B[\mathbf{v}]T[\mathbf{a}]$ for some positive constant m. Show that this leads to the modified commutation relation $[K_i, P_j] = im\,\delta_{ij}$.

Suppose $\mathbf{P}|\mathbf{0}\rangle = \mathbf{0}$ and define $|m\mathbf{v}\rangle = U_B[\mathbf{v}]|\mathbf{0}\rangle$. Show that is an eigenvector of $\mathbf{P}$ and that $H|\mathbf{p}\rangle = (E_0 + \frac{\mathbf{p}^2}{2m})|\mathbf{p}\rangle$.