Symmetry and Particle Physics, 3

1. Define $\sigma_\mu = (1, \boldsymbol{\sigma})$ and $\bar\sigma_\mu = (1, -\boldsymbol{\sigma})$, with both four independent 2×2 matrices. Show

$$\sigma_\mu \bar\sigma_\nu + \sigma_\nu \bar\sigma_\mu = 2 g_{\mu\nu} 1 \, , \quad (\bar\sigma_\mu)^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma_{\beta\dot\beta} \, .$$

Without assuming any explicit expression for the Pauli matrices $\boldsymbol{\sigma}$ show that we must then have $\frac{1}{2}\mathrm{tr}(\sigma_\mu x^\mu \, \bar\sigma_\nu x^\nu) = \det(\sigma_\mu x^\mu) = g_{\mu\nu} x^\mu x^\nu$.

For a matrix $A = [A_{\alpha\dot\alpha}]$ explain why $A = \sigma_\mu \, \frac{1}{2}\mathrm{tr}(\bar\sigma^\mu A)$. For $B = [B_\alpha{}^\beta]$ verify that

$$B = 1 \, \tfrac{1}{2}\mathrm{tr}\big(B\big) - \sigma_{[\mu}\bar\sigma_{\nu]} \, \tfrac{1}{8}\mathrm{tr}\big(\sigma^{[\mu}\bar\sigma^{\nu]}B\big) \, .$$

2. A Lie group has group elements $g(a)$ depending on group parameters a^r, with $g(0) = e$, the identity, and under group multiplication $g(a)g(b) = g(\varphi(a,b))$ for some $\varphi^r(a,b)$. Let $g(a)^{-1} = g(\bar a)$ where $\varphi(\bar a, a) = 0$. Why must $\varphi^r(a,0) = a^r$, $\varphi^r(0,b) = b^r$? Assume $\varphi^r(a,b)$ is expanded near the origin according to

$$\varphi^a(a,b) = a^a + b^a + c^a{}_{bc} a^b b^c + \mathrm{O}(a^2 b, ab^2) \, .$$

Use this to find $\bar a(a)$ for a small. Let $g(d) = g(a)^{-1} g(b)^{-1} g(a) g(b)$ and show that for a, b small $d^a = f^a_{\ bc} a^b b^c$ where $f^a_{\ bc} = c^a_{\ bc} - c^a_{\ cb}$. Using an expansion to one higher order show that the associativity condition $\varphi(\varphi(a,b),c) = \varphi(a, \varphi(b,c))$ leads to the Jacobi identity.

Assume the Lie group has generators T_a satisfying $[T_a, T_b] = f^a_{\ bc} T_c$. For an element of the Lie algebra $a^a T_a$ there is an associated group element given by $g(a) = \exp(a^a T_a)$. Use the Baker-Campbell-Hausdorff formula for $\exp C = \exp A \exp B$ in the form $C = A + B + \frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] + [B,[B,A]]) + \ldots$ to obtain $\varphi(a,b)$ for a, b small and verify that this is compatible with the general expansion of φ.

3. Using the same notation as the previous question, where $c^r = \varphi^r(a,b)$, obtain

$$\frac{\partial c^r}{\partial b^s} = \lambda_s{}^a(b) \, \mu_a{}^r(c) \, , \quad \mu_a{}^r(c) = \frac{\partial}{\partial b^a} \varphi^r(c,b) \bigg|_{b=0} \, , \quad \mu_a{}^r(c) \lambda_r{}^b(c) = \delta_a{}^b \, .$$

Show that the equation for the structure constants $f^a_{\ bc}$ may also be expressed as

$$\frac{\partial}{\partial b^r}\lambda_s{}^a(b) - \frac{\partial}{\partial b^s}\lambda_r{}^a(b) = -f^a_{\ bc}\lambda_r{}^b(b)\lambda_s{}^c(b) \, .$$

For those familiar with differential forms.
Defining the differential form $\omega^a(b) = \mathrm{d}b^r \lambda_r{}^a(b)$ verify that the above equation is equivalent, if d is the exterior derivative, to $\mathrm{d}\omega^a(b) = -\frac{1}{2} f^a_{\ bc} \, \omega^b(b) \wedge \omega^c(b)$. Show that the Jacobi identity for $f^a_{\ bc}$ is entailed by $\mathrm{d}^2\omega^a(b) = 0$. If g are matrices forming a representation, $g(c) = g(a)g(b)$, show that $g(b)^{-1}\mathrm{d}g(b) = \omega^a(b) t_a$ where $\{t_a\}$ are matrix generators satisfying $[t_a, t_b] = f^c_{\ ab} t_c$.

4. An $SU(2)$ matrix may be represented by

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \, , \qquad |\alpha|^2 + |\beta|^2 = 1 \, .$$

Let $\alpha = |\alpha| \, e^{i\lambda}$ where $|\alpha|^2 = 1 - |\beta|^2$ so that A is regarded as depending on the real parameter λ and the complex parameter β. Determine $\mathrm{d}\lambda$, $\mathrm{d}\beta$, $\mathrm{d}\bar\beta$ in terms of infinitesimal ϕ, θ, $\bar\theta$ where

$$\begin{pmatrix} \alpha + \mathrm{d}\alpha & \beta + \mathrm{d}\beta \\ -\bar\beta - \mathrm{d}\bar\beta & \bar\alpha + \mathrm{d}\bar\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \begin{pmatrix} 1 + i\phi & \theta \\ -\bar\theta & 1 - i\phi \end{pmatrix} \, .$$

Hence show that the invariant measure for $SU(2)$ with these parameters is $\mathrm{d}\lambda \, \mathrm{d}^2\beta$. What are the ranges of λ and β? Calculate $\int \mathrm{d}\lambda \, \mathrm{d}^2\beta$. (For $\mathrm{d}\beta = \mathrm{d}\beta_1 + i\mathrm{d}\beta_2$, $\mathrm{d}^2\beta = \mathrm{d}\beta_1 \mathrm{d}\beta_2$.)

5. Consider two $SU(2)$ algebras, $I_\pm, I_3$ and $U_\pm, U_3$, where $[I_3, I_\pm] = \pm I_\pm$, $[I_+, I_-] = 2I_3$ with $I_3{}^\dagger = I_3$, $I_+{}^\dagger = I_-$, and also similarly for $I \to U$. Assume

$$[I_-, U_+] = 0\,, \quad [I_3, U_+] = -\tfrac{1}{2}U_+\,, \qquad [U_-, I_+] = 0 \qquad [U_3, I_+] = -\tfrac{1}{2}I_+\,.$$

Define $V_+ = [I_+, U_+]$ and $V_- = V_+{}^\dagger$. Explain why $[I_+, V_+] = [U_+, V_+] = 0$. Evaluate $[V_+, V_-]$ and show that $V_\pm$, $V_3 = I_3 + U_3$ form a $SU(2)$ algebra. Let $U_3 = -\frac{1}{2}I_3 + \frac{3}{2}Y$. Work out the various commutators involving Y.

Explain how the **I**-spin and **U**-spin operators with the above relations generate a $SU(3)$ algebra.

6. For a simple Lie algebra $\mathcal{L}$, with elements X_i such that $[X_i, X_j] = i f_{ijk} X_k$ where f_{ijk} is totally antisymmetric, let T_i be matrices forming a representation R of $\mathcal{L}$ and assume $T_i T_i = C_R 1$. Define

$$\langle X_i, X_j \rangle = \mathrm{tr}(T_i T_j)\frac{\dim \mathcal{L}}{C_R \dim R}\,.$$

(a) Evaluate $\langle J_3, J_3 \rangle$ in the j-th irreducible representation of $SU(2)$ and show that the result is independent of j.

(b) For $SU(3)$ show that the representation given by $T_i = \frac{1}{2}\lambda_i$ gives the same value for $\langle X_i, X_j \rangle$ as does the adjoint representation $(T_i^{\mathrm{ad}})_{jk} = -i f_{ijk}$.

7. Let $\{T^i{}_j\}$ be $n \times n$ matrices such that $T^i{}_j$ has a 1 in the i'th row and j'th column and is zero otherwise. Show that they satisfy the Lie algebra

$$[T^i{}_j, T^k{}_l] = \delta^k{}_j T^i{}_l - \delta^i{}_l T^k{}_j\,.$$

Define $X = T^i{}_j X^j{}_i$ with arbitrary components $X^j{}_i$. Determine the adjoint matrix $(X^{\mathrm{ad}})^n{}_m{}^k{}_l$ by

$$[X, T^k{}_l] = T^m{}_n (X^{\mathrm{ad}})^n{}_m{}^k{}_l\,,$$

and show that

$$\kappa(X,Y) = \mathrm{tr}\big(X^{\mathrm{ad}} Y^{\mathrm{ad}}\big) = 2\big(n \textstyle\sum_{i,j} X^j{}_i Y^i{}_j - \sum_i X^i{}_i \sum_j Y^j{}_j\big)\,.$$

Show that $1 + \epsilon X \in U(n)$ for infinitesimal ϵ if $(X^j{}_i)^* = -X^i{}_j$. Hence show that in this case

$$\kappa(X,X) = -2n \textstyle\sum_{i,j} |\hat{X}^j{}_i|^2\,, \qquad \hat{X}^j{}_i = X^j{}_i - \frac{1}{n}\,\delta^j{}_i \sum_k X^k{}_k\,,$$

and therefore $\kappa(X,X) = 0 \;\Leftrightarrow\; X^{\mathrm{ad}} = 0$. What restrictions must be made for $SU(N)$ and verify that in this case the generators satisfy $\kappa(X,X) < 0$ so the group is semi-simple.

8. For a group with a Lie algebra with a basis $\{T_a\}$ such that $[T_a, T_b] = f^c{}_{ab} T_c$ let $g_{ab} = \langle T_a, T_b \rangle$ where $\langle\ ,\ \rangle$ is an invariant symmetric bilinear form so that $\langle [X,Y], Z \rangle = -\langle Y, [X,Z] \rangle$. Show that $f_{abc} = g_{ad} f^d{}_{bc}$ is totally antisymmetric. If D_μ is an appropriate covariant derivative involving a gauge field A_μ^a verify

$$\partial_\mu \langle X(x), Y(x) \rangle = \langle D_\mu X(x), Y(x) \rangle + \langle X(x), D_\mu Y(x) \rangle\,.$$

Let $T^\mu{}_\nu = \langle F^{\mu\sigma}, F_{\nu\sigma} \rangle - \frac{1}{4}\delta^\mu{}_\nu \langle F^{\sigma\rho}, F_{\sigma\rho} \rangle$. Show, using the Bianchi identity, $\partial_\mu T^\mu{}_\nu = \langle D_\mu F^{\mu\sigma}, F_{\nu\sigma} \rangle$. For a variation δA_μ^a obtain also $\delta\, \frac{1}{4}\epsilon^{\mu\nu\sigma\rho} \langle F_{\mu\nu}, F_{\sigma\rho} \rangle = \partial_\mu \epsilon^{\mu\nu\sigma\rho} \langle \delta A_\nu, F_{\sigma\rho} \rangle$. By letting $A_\mu \to t A_\mu$ and differentiating with respect to t and then integrating show that

$$\tfrac{1}{4}\epsilon^{\mu\nu\sigma\rho} \langle F_{\mu\nu}, F_{\sigma\rho} \rangle = \partial_\mu \epsilon^{\mu\nu\sigma\rho} \big\langle A_\nu, \partial_\sigma A_\rho + \tfrac{1}{3}[A_\sigma, A_\rho] \big\rangle\,.$$

9. With notation as in the previous question define a three dimensional Lagrangian

$$\mathcal{L} = \epsilon^{\mu\nu\rho}\big(g_{ab} A_\mu^a \partial_\nu A_\rho^b + \tfrac{1}{3} f_{abc} A_\mu^a A_\nu^b A_\rho^c\big)\,.$$

For a a gauge transformation $\delta A_\mu^a = -\partial_\mu \lambda^a - f^a{}_{bc} A_\mu^b \lambda^c$ show that $\delta \mathcal{L} = -\partial_\mu\big(\epsilon^{\mu\nu\rho} g_{ab} \lambda^a \partial_\nu A_\rho^b\big)$ so that $\int \mathrm{d}^3 x\, \mathcal{L}$ is invariant.