Symmetry and Particle Physics, 4

1. The generators of the Lie algebra of $SU(3)$ are $E_{i\pm}$ for $i = 1, 2, 3$ and H_1, H_2 where $[H_1, H_2] = 0$, $[E_{1+}, E_{2+}] = E_{3+}$, $[E_{1+}, E_{3+}] = [E_{2+}, E_{3+}] = 0$ and $E_{i-} = E_{i+}{}^{\dagger}$. $E_{i\pm}, H_i$ obey, for each i, the $SU(2)$ algebra $[E_+, E_-] = H$, $[H, E_\pm] = \pm 2E_\pm$ if $H_3 = H_1 + H_2$. Also we have $[H_1, E_{2\pm}] = \mp E_{2\pm}$, $[H_2, E_{1\pm}] = \mp E_{1\pm}$.

A state $|\psi\rangle$ has weight $[r_1, r_2]$ if $H_1|\psi\rangle = r_1|\psi\rangle$, $H_2|\psi\rangle = r_2|\psi\rangle$. $|n_1, n_2\rangle_{\rm hw}$ is a highest weight state with weight $[n_1, n_2]$ satisfying $E_{i+}|n_1, n_2\rangle_{\rm hw} = 0$. Let

$$|\psi_i^{(l,k)}\rangle = (E_{3-})^i (E_{1-})^{l-i} (E_{2-})^{k-i} |n_1, n_2\rangle_{\rm hw} \, .$$

Show that these all have weight $[n_1 + k - 2l, n_2 - 2k + l]$. If $l \geq k$ explain why, in order to construct a basis, it is necessary to restrict $i = 0, \ldots, k$ except if $k > n_2$ when $i = k - n_2, \ldots, k$.

For $SU(2)$ generators $E_\pm, H$ obtain

$$[E_+, (E_-)^n] = n(E_-)^{n-1}(H - n + 1) \, .$$

From the $SU(3)$ commutators $[E_{3+}, E_{1-}] = -E_{2+}$, $[E_{3+}, E_{2-}] = E_{1+}$, $[E_{2+}, E_{3-}] = E_{1-}$ and $[E_{2+}, E_{1-}] = [E_{1+}, E_{2-}] = 0$ obtain also

$$\begin{aligned} [E_{3+}, (E_{1-})^{l-i}] &= -(l-i)(E_{1-})^{l-i-1} E_{2+} \, , & [E_{3+}, (E_{2-})^{k-i}] &= (k-i)(E_{2-})^{k-i-1} E_{1+} \, , \\ [E_{2+}, (E_{3-})^i] &= i \, E_{1-} (E_{3-})^{i-1} \, , & [E_{2+} (E_{1-})^{l-i}] &= 0 \, . \end{aligned}$$

Hence obtain

$$\begin{aligned} E_{3+}|\psi_i^{(l,k)}\rangle &= i(n_1 + n_2 - k - l + i + 1)|\psi_{i-1}^{(l-1,k-1)}\rangle - (l-i)(k-i)(n_2 - k + i + 1)|\psi_i^{(l-1,k-1)}\rangle \, , \\ E_{2+}|\psi_i^{(l,k)}\rangle &= E_{1-}\big(i\,|\psi_{i-1}^{(l-1,k-1)}\rangle + (k-i)(n_2 - k + i + 1)|\psi_i^{(l-1,k-1)}\rangle\big) \, . \end{aligned}$$

Use this to show that $\{|\psi_i^{(l,k)}\rangle, i = 0, \ldots k\}$ are linearly independent, *i.e.* there is no non trivial solution of $\sum_{i=0}^k c_i |\psi_i^{(l,k)}\rangle = 0$, if $\{|\psi_i^{(l-1,k-1)}\rangle, i = 0, \ldots k-1\}$ are linearly independent, so long as $k \leq n_2$.

2. A Lie algebra has a Cartan subalgebra $\underline{H} = (H_1, \ldots, H_r)$ and the remaining generators are $E_{\underline{\alpha}}$, corresponding to roots $\underline{\alpha}$, where $[\underline{H}, E_{\underline{\alpha}}] = \underline{\alpha}\, E_{\underline{\alpha}}$. Assume $[E_{\underline{\alpha}}, E_{-\underline{\alpha}}] = H_{\underline{\alpha}} = 2\underline{\alpha} \cdot \underline{H}/\underline{\alpha}^2$. For a root $\underline{\beta}$, $E_{\underline{\beta}}$ satisfies

$$[E_{\underline{\alpha}}, E_{\underline{\beta}}] = 0 \, , \quad [H_{\underline{\alpha}}, E_{\underline{\beta}}] = n \, E_{\underline{\beta}} \, , \qquad \underbrace{[E_{-\underline{\alpha}}, [\ldots, [E_{-\underline{\alpha}}}_{r}, E_{\underline{\beta}}] \ldots]] = E_{\underline{\beta} - r\underline{\alpha}} \, .$$

Show that

$$[E_{\underline{\alpha}}, E_{\underline{\beta} - r\,\underline{\alpha}}] = r(n - r + 1)\, E_{\underline{\beta} - (r-1)\underline{\alpha}} \, .$$

For n an integer show that we may assume $E_{\underline{\beta} - (n+1)\underline{\alpha}} = 0$.

3. A Lie algebra has simple roots $\boldsymbol{\alpha}_1, \ldots \boldsymbol{\alpha}_r$. The fundamental weights satisfy

$$\frac{2\mathbf{w}_i \cdot \boldsymbol{\alpha}_j}{\boldsymbol{\alpha}_j \cdot \boldsymbol{\alpha}_j} = \delta_{ij} \, .$$

Show that $\boldsymbol{\alpha}_i = \sum_j K_{ij} \mathbf{w}_j$ where $[K_{ij}]$ is the Cartan matrix.

A rank two Lie algebra has simple roots $\boldsymbol{\alpha}_1 = (1, 0)$ and $\boldsymbol{\alpha}_2 = (-1, 1)$. What is the Cartan matrix? Assuming any other positive roots are equal in length to either one of the simple roots, show that $\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_4 = 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ are the other positive roots. Draw the root diagram, and show that the dimension of the Lie algebra is ten.

Construct the fundamental weights $\mathbf{w}_1$, $\mathbf{w}_2$. How is the highest weight of the representation whose weights coincide with the roots of the Lie algebra related to the fundamental weights?

4. The Lie algebra of $U(n)$ may be represented by a basis consisting first of the $n^2 - n$ off diagonal matrices $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ for $i \neq j$ and also the n diagonal matrices $(h_i)_{kl} = \delta_{ik}\delta_{kl}$, no sum on k, where $i, j, k, l = 1, \dots n$. For $SU(n)$ it is necessary to restrict to traceless matrices given by $h_i - h_j$ for some i, j. The $n-1$ independent $h_i - h_j$ correspond to the Cartan subalgebra. Show that

$$[h_i, E_{jk}] = (\delta_{ij} - \delta_{ik})E_{jk}\,, \qquad [E_{ij}, E_{ji}] = h_i - h_j\,, \quad \text{no summation convention}\,.$$

Let $\mathbf{e}_i$ be orthogonal n-dimensional unit vectors, $(\mathbf{e}_i)_j = \delta_{ij}$. Show that E_{ij} is associated with the root vector $\mathbf{e}_i - \mathbf{e}_j$ while E_{ji} corresponds to the root vector $\mathbf{e}_j - \mathbf{e}_i$. Hence show that there are $n(n-1)$ root vectors belonging to the $n-1$ dimensional hyperplane orthogonal to $\sum_i \mathbf{e}_i$. Verify that we may take as simple roots

$$\boldsymbol{\alpha}_1 = \mathbf{e}_1 - \mathbf{e}_2\,, \quad \boldsymbol{\alpha}_2 = \mathbf{e}_2 - \mathbf{e}_3\,, \ \dots, \quad \boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}\,, \ \dots, \quad \boldsymbol{\alpha}_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n\,,$$

by showing that all roots may be expressed in terms of the $\boldsymbol{\alpha}_i$ with either positive or negative integer coefficients. Determine the Cartan matrix and write down the corresponding Dynkin diagram.

5. The Lie algebra for $SO(n)$ is given by real antisymmetric $n \times n$ matrices. Show that the dimension is $\frac{1}{2}n(n-1)$. A basis for the Lie algebra is given by matrices $L_{ij} = -L_{ji}$, $i, j = 1, \dots n$, where $(L_{ij})_{mn} = -\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}$. Show that

$$[L_{ij}, L_{kl}] = \delta_{ik}L_{jl} - \delta_{il}L_{jk} - \delta_{jk}L_{il} + \delta_{jl}L_{ik}\,.$$

For $n = 2r$ or $n = 2r+1$ verify that a maximal set of commuting hermitian matrices is given by

$$iL_{12}\,, \qquad iL_{34}\,, \qquad \dots\,, \qquad iL_{2r-1\,2r}\,,$$

so that the rank is r in both cases.

Define

$$E_{\epsilon\eta} = L_{13} + i\epsilon L_{23} + i\eta(L_{14} + i\epsilon L_{24})\,, \quad \epsilon, \eta = \pm 1\,,$$

and verify the commutators

$$[iL_{12}, E_{\epsilon\eta}] = \epsilon E_{\epsilon\eta}\,, \qquad [iL_{34}, E_{\epsilon\eta}] = \eta E_{\epsilon\eta}\,, \qquad [iL_{2i-1\,2i}, E_{\epsilon\eta}] = 0\,, \ i = 3, \dots r\,,$$

so that $E_{\epsilon\eta}$ corresponds to a root vector $(\epsilon, \eta, 0, \dots, 0)$. Using the notation of the previous question where $\mathbf{e}_i$ are orthogonal unit vectors in a r-dimensional space show that $\pm\mathbf{e}_i \pm \mathbf{e}_j$ for all $i, j = 1, \dots r$, $i \neq j$ give in general $2r(r-1)$ root vectors.

For $n = 2r$ choose as simple roots

$$\boldsymbol{\alpha}_1 = \mathbf{e}_1 - \mathbf{e}_2\,, \quad \boldsymbol{\alpha}_2 = \mathbf{e}_2 - \mathbf{e}_3\,, \ \dots, \quad \boldsymbol{\alpha}_{r-1} = \mathbf{e}_{r-1} - \mathbf{e}_r\,, \quad \boldsymbol{\alpha}_r = \mathbf{e}_{r-1} + \mathbf{e}_r\,.$$

Show that $\mathbf{e}_i - \mathbf{e}_j$, for $i < j$, and $\mathbf{e}_i + \mathbf{e}_j$ may be expressed as linear combinations of these simple roots with positive or zero integer coefficients. Show also that the other roots are given by negative linear combinations. Work out the Cartan matrix and determine the Dynkin diagram.

For $n = 2r+1$ verify that

$$[iL_{12}, E_{\pm 1}] = \pm E_{\pm 1}\,, \quad [iL_{2i-1\,2i}, E_{\pm 1}] = 0\,, \ i = 2, \dots r\,, \qquad E_{\pm 1} = L_{1\,2r+1} \pm iL_{2\,2r+1}\,,$$

corresponding to roots $(\pm 1, 0, \dots, 0)$. Hence show that there are $2r$ additional roots in this case $\pm\mathbf{e}_i$, $i = 1, \dots, r$. In a similar fashion as previously show that in this case we may take as simple roots

$$\boldsymbol{\alpha}_1 = \mathbf{e}_1 - \mathbf{e}_2\,, \quad \boldsymbol{\alpha}_2 = \mathbf{e}_2 - \mathbf{e}_3\,, \ \dots, \quad \boldsymbol{\alpha}_{r-1} = \mathbf{e}_{r-1} - \mathbf{e}_r\,, \quad \boldsymbol{\alpha}_r = \mathbf{e}_r\,.$$

Hence obtain the Cartan matrix and determine the Dynkin diagram.

6. For the Lie algebra of G_2 the simple roots are $\boldsymbol{\alpha}_1 = (1, 0)$ and $\boldsymbol{\alpha}_2 = \frac{1}{2}(-3, \sqrt{3})$. Determine the fundamental weights $\mathbf{w}_1$ and $\mathbf{w}_2$. Let $|q_1, q_2\rangle$ be a state corresponding to the weight $q_1\mathbf{w}_1 + q_2\mathbf{w}_2$. Assuming $E_{i\pm}$, H_i are the $SU(2)$ generators associated with the roots $\boldsymbol{\alpha}_i$ construct a basis for the representation space starting from a highest weight vector (*i*) $|1, 0\rangle$ and (*ii*) $|0, 1\rangle$ by the successive action of E_{1-} and E_{2-} on the highest weight state. Show that the dimensions of the space are respectively 7 and 14 (in the second case there are two independent states with $q_1 = q_2 = 0$). Construct the weight diagram and in the 14 dimensional case show that it coincides with the root diagram.