Part III Lecture Notes on The Standard Model

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Part I
PCT - Conventions and Results

1 Dirac Equation and γ-Matrices

The γ-matrices are defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I \ , \tag{1.1}$$

where

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ , \tag{1.2}$$

and if $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ then it is usual to require for the hermitian conjugate matrices

$$\gamma^{0\dagger} = \gamma^0 \ , \quad \text{and} \quad \boldsymbol{\gamma}^\dagger = -\boldsymbol{\gamma} \ . \tag{1.3}$$

This condition ensures that the Dirac Hamiltonian $H_D = -i\boldsymbol{\alpha}.\boldsymbol{\nabla} + \beta m$, where $\beta = \gamma^0, \boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, is hermitian and (1.3) can alternatively be written as

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu \ . \tag{1.4}$$

The matrix γ_5 is defined by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ , \qquad (\gamma_5)^2 = I \ , \quad \gamma_5\gamma^\mu = -\gamma^\mu\gamma_5 \ , \tag{1.5}$$

and with (1.3) $\gamma_5{}^\dagger = \gamma_5$. The irreducible representations of (1.1) are given by 4×4 matrices and the representation is unique in the sense that if γ^μ, γ'^μ both satisfy (1.1) then $\gamma'^\mu = S\gamma^\mu S^{-1}$ for some S and if $\gamma^\mu = S\gamma^\mu S^{-1}$ then $S = \lambda I$.

The Dirac equation for a spinor field ψ is

$$(i\gamma^\mu \partial_\mu - m)\, \psi(x) = 0 \ , \tag{1.6}$$

and with $\overline{\psi} = \psi^\dagger \gamma^0$

$$\overline{\psi}(x) \left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m\right) = 0 \ . \tag{1.7}$$

The expansion of the quantum field which is a solution of eq.(1.6) has the form

$$\psi(x) = \sum_{p,\lambda} \left[a(p,\lambda)u(p,\lambda)e^{-ip.x} + b(p,\lambda)^\dagger v(p,\lambda)e^{ip.x}\right] \ , \tag{1.8}$$

if $u(p,\lambda)$ and $v(p,\lambda)$ are 4 component spinors satisfying $(\gamma.p - m)u(p,\lambda) = 0$ and $(\gamma.p + m)v(p,\lambda) = 0$, with $\lambda = \pm\frac{1}{2}$ a spin label and $p^\mu = (E, \mathbf{p})$ where $E = \sqrt{\mathbf{p}^2 + m^2}$ so that $p^2 = m^2$. Thus $u(p,\lambda)e^{-ip.x}$ and $v(p,\lambda)e^{ip.x}$ are positive and negative energy solutions of the Dirac equation eq.(1.6), being eigenfunctions of H_D with eigenvalues E and $-E$. In the summation

$$\sum_p \quad \text{means} \quad \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E} \ .$$

We will also use the notation $\delta_{pp'}$ for $(2\pi)^3 2E\delta^3(\mathbf{p}-\mathbf{p}')$ so that

$$\sum_{p'} \delta_{pp'} f(p') = f(p) \ . \tag{1.9}$$

The standard Bjorken-Drell conventions for γ-matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} , \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \tag{1.10}$$

although physical results are of course independent of any particular representation. In the conventions of eq.(1.10) the spinor $u(p,\lambda)$ can be written as

$$u(p,\lambda) = \sqrt{E+m} \begin{pmatrix} \chi_\lambda \\ \frac{\boldsymbol{\sigma}.\mathbf{p}}{E+m}\chi_\lambda \end{pmatrix} \tag{1.11}$$

for χ_λ a two component spinor and we may also take $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The spinor $v(p,\lambda)$ is associated with the anti-particles and will be discussed later when we consider charge conjugation.

2 Parity Inversion

2.1 Boson Fields

The operation of parity inversion, P, is associated with the spatial map $x \to x_P$ where if $x^\mu = (x_0, \mathbf{x})$ then $x_P^\mu = (x_0, -\mathbf{x})$. If $\phi(x)$ is a classical scalar field or the wave function of an associated spinless particle then the operation of parity on ϕ is defined by the transformation

$$\phi(x) \longrightarrow \eta_P \phi(x_P) \ , \tag{1.12}$$

where η_P is the *intrinsic parity* of the field or particle. Since repeating the parity operation leaves x unchanged we would expect to have $P^2 = 1$. For a classical real field ϕ this means that $\eta_P = \pm 1$. However in a quantum theory, since a complex wave function is arbitrary up to a complex phase, we need only require that $|\eta_P| = 1$.

In the case of a quantum field theory P is represented by a unitary operator $\hat{P}$ acting on the Fock space of particle states. For a quantum boson field $\phi(x)$ the parity transformation becomes

$$\hat{P}\phi(x)\hat{P}^{-1} = \eta_P \phi(x_P) \ . \tag{1.13}$$

In terms of momentum modes a scalar field ϕ, representing a spinless charged particle, has the expansion

$$\phi(x) = \sum_p \left[a(p)e^{-ip.x} + b(p)^\dagger e^{ip.x} \right] \ , \tag{1.14}$$

and eq.(1.13) becomes

$$\sum_p \left[\hat{P}a(p)\hat{P}^{-1}e^{-ip.x} + \hat{P}b(p)^\dagger \hat{P}^{-1} e^{ip.x} \right] = \sum_p \left[\eta_P a(p) e^{-ip.x_P} + \eta_P b(p)^\dagger e^{ip.x_P} \right] \ . \tag{1.15}$$

Now $p.x_P = p_P.x$ where $p_P^\mu = (E, -\mathbf{p})$ and, taking into account the invariance of the p-summation under parity, we can write the right side of eq.(1.15) as

$$\sum_p \left[\eta_P a(p_P) e^{-ip.x} + \eta_P b(p_P)^\dagger e^{ip.x} \right] \ .$$

Equating this with the left side of eq.(1.15) we conclude that

$$\hat{P}a(p)\hat{P}^{-1} = \eta_P a(p_P) \ , \qquad \hat{P}b(p)^\dagger \hat{P}^{-1} = \eta_P b(p_P)^\dagger \ . \tag{1.16}$$

If we assume also that the vacuum is parity invariant, that is $\hat{P}|0\rangle = |0\rangle$, the effect of parity on a momentum state is therefore

$$\hat{P}|p\rangle = \hat{P}a(p)^\dagger|0\rangle = \hat{P}a(p)^\dagger \hat{P}^{-1}|0\rangle = \eta_P^* a(p_P)^\dagger|0\rangle = \eta_P^*|p_P\rangle \ , \tag{1.17}$$

that is $\hat{P}$ reflects the spatial momentum and multiplies the state by the intrinsic parity of the boson. If the field ϕ is hermitian, so that $a(p) = b(p)$, then eq.(1.13) requires η_P to be real and hence $\eta_P = \pm 1$.

In a general theory $\hat{P}^2 \neq 1$, so that η_P need not be ± 1, but it should be expressible in terms of other conserved quantities in the theory. For example $\hat{P}^2 = e^{2i\alpha Q}$, where Q is the electric charge, but then we may redefine $\hat{P}e^{-i\alpha Q} \to \hat{P}$ so that then $\hat{P}^2 = 1$ and therefore $\eta_P = \pm 1$.

We normally expect parity to commute with internal symmetry transformations so the above results remain true for fields with internal symmetry indices. The isovector pion field π_α, $\alpha = 1, 2, 3$ for example obeys $\hat{P}\pi_\alpha(x)\hat{P}^{-1} = -\pi_\alpha(x_P)$ because the pion has negative intrinsic parity.

2.2 Dirac Field

In determining the parity transformation properties of the Dirac spinor wave-function we require that the transformed wave-function must also satisfy the Dirac equation (1.6). It is not enough simply to invert the spatial coordinates $\mathbf{x}$ of the field $\psi(x)$. Instead we have under parity

$$\psi(x) \longrightarrow \eta_P \psi^P(x) \ , \quad \psi^P(x) = \gamma^0 \psi(x_P) \ , \tag{1.18}$$

where the matrix γ^0 is introduced in order to satisfy the requirement that $\psi^P(x)$ should satisfy the Dirac equation. To show this we have from (1.6) letting $\mathbf{x} \to -\mathbf{x}$, since $\gamma^\mu \partial_\mu = \gamma^0 \partial_t + \boldsymbol{\gamma}\cdot\boldsymbol{\nabla}$,

$$\left(i\gamma^0 \partial_t - i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} - m\right) \psi(x_P) = 0 \; . \tag{1.19}$$

Now since $\gamma^0(\gamma^0, \boldsymbol{\gamma})\gamma^0 = (\gamma^0, -\boldsymbol{\gamma})$ and $(\gamma^0)^2 = I$, or $(\gamma^0)^{-1} = \gamma^0$, it is straightforward to see that, with the definition of ψ^P in (1.18),

$$(i\gamma^\mu \partial_\mu - m)\, \psi^P(x) = 0 \; , \tag{1.20}$$

as required.

The Dirac quantum field is therefore assumed to transform under parity as

$$\hat{P}\psi(x)\hat{P}^{-1} = \eta_P \gamma^0 \psi(x_P) \; , \qquad \hat{P}\overline{\psi}(x)\hat{P}^{-1} = \eta_P^* \overline{\psi}(x_P)\gamma^0 \; . \tag{1.21}$$

It follows that under P we have

$$\begin{array}{lcll} \overline{\psi}(x)\psi(x) & \to & \overline{\psi}(x_P)\psi(x_P) & \text{scalar} \; , \\ \overline{\psi}(x)\gamma_5\psi(x) & \to & -\overline{\psi}(x_P)\gamma_5\psi(x_P) & \text{pseudoscalar} \; , \\ \overline{\psi}(x)\gamma^0\psi(x) & \to & \overline{\psi}(x_P)\gamma^0\psi(x_P) & \text{charge density} \; , \\ \overline{\psi}(x)\boldsymbol{\gamma}\psi(x) & \to & -\overline{\psi}(x_P)\boldsymbol{\gamma}\psi(x_P) & \text{current density} \; . \end{array} \tag{1.22}$$

The transformation properties of other bi-linears in ψ and $\overline{\psi}$ can be worked out in a similar fashion.

2.3 Transformation of States Under P

Under the parity transformation the positive energy Dirac wave-function of momentum p transforms as

$$u(p,\lambda)e^{-ip.x} \to \gamma^0 u(p,\lambda)e^{-ip.x_P} = u(p_P,\lambda)e^{-ip_P.x} \; , \tag{1.23}$$

assuming

$$\gamma^0 u(p,\lambda) = u(p_P,\lambda) \; , \tag{1.24}$$

which is in accord with eq.(1.11). That is the spatial part of the momentum has been reflected but the spin state has been left unaltered which is just what is expected from a parity transformation. By following a similar argument as for the bosonic field and using eq.(1.24) we see that

$$\hat{P}a(p,\lambda)\hat{P}^{-1} = \eta_P a(p_P,\lambda) \; , \tag{1.25}$$

and (again assuming invariance of the vacuum under parity) we obtain

$$\hat{P}|p,\lambda\rangle = \eta_P^* |p_P,\lambda\rangle \; , \tag{1.26}$$

where $|p,\lambda\rangle = a(p,\lambda)^\dagger|0\rangle$, which is what would have been expected on the basis of the wave-function analysis above. The transformation of $b(p,\lambda)$ and hence the anti-particle states is discussed later when the spinor $v(p,\lambda)$ has been defined in detail.

For a particle with arbitrary spin we may also assume that the single particle states transform under parity according to eq.(1.26). Since $\hat{P}\mathbf{J}\hat{P}^{-1} = \mathbf{J}$ the phase factor η_P must be independent of λ.

3 Charge Conjugation

3.1 Scalar Field

A scalar quantum field $\phi(x)$ has the decomposition in terms of creation and annihilation operators

$$\phi(x) = \sum_k \left[a(k)e^{-ik.x} + b(k)^\dagger e^{ik.x}\right] \ , \tag{1.27}$$

where $a(k)$ annihilates particles and $b(k)$ annihilates anti-particles of momentum k. Charge conjugation C interchanges particles and anti-particles. Acting on the basic Fock space we require a unitary transformation $\hat{C}$ such that for a general single particle state $\hat{C}|k,\text{particle}\rangle = \eta_C^*|k,\text{anti-particle}\rangle$, where η_C is a phase factor associated with the particle. This is achieved by requiring $\hat{C}|0\rangle = |0\rangle$ and $\hat{C}a(k)\hat{C}^{-1} = \eta_C b(k)$. Assuming also $\hat{C}b(k)\hat{C}^{-1} = \eta_C^* a(k)$ then

$$\hat{C}\phi(x)\hat{C}^{-1} = \eta_C \phi(x)^\dagger \ . \tag{1.28}$$

We have also

$$\hat{C}\phi(x)^\dagger\hat{C}^{-1} = \eta_C^* \phi(x) \ . \tag{1.29}$$

If ϕ is hermitian, $\phi = \phi^\dagger$, then η_C must be real and so $\eta_C = \pm 1$. For non hermitian ϕ, η_C in (1.28) is arbitrary since if $\eta_C = e^{2i\beta}$ then we may take $e^{-i\beta}\phi \to \phi$ so that now $\eta_C \to 1$.

The possibility that a $\pm$ sign may be involved in the charge conjugation properties of neutral fields is non-trivial and of physical significance. The electromagnetic 4-vector field $A_\mu(x)$ obeys

$$\hat{C}A_\mu(x)\hat{C}^{-1} = -A_\mu(x) \ , \tag{1.30}$$

which is necessary to ensure that electromagnetic interactions are invariant under C. An N photon state therefore has charge conjugation $(-1)^N$ and a π^0 can decay to two photons but not three, assuming charge conjugation is an exact symmetry of electromagnetic and strong interactions.

3.2 Dirac Field

In the same way as for the charged scalar field the charge conjugation operation on the Dirac field interchanges particles and anti-particles. The transformation therefore involves the hermitian conjugation of the field. The field after charge conjugation must however satisfy the Dirac equation. We use the following notational conventions:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} , \qquad \psi(x)^* = \begin{pmatrix} \psi_1(x)^\dagger \\ \psi_2(x)^\dagger \\ \psi_3(x)^\dagger \\ \psi_4(x)^\dagger \end{pmatrix} , \tag{1.31}$$

and

$$\psi(x)^\dagger = \Big(\psi_1(x)^\dagger,\ \psi_2(x)^\dagger,\ \psi_3(x)^\dagger,\ \psi_4(x)^\dagger\Big) \ . \tag{1.32}$$

and so $\overline{\psi}(x) = \psi(x)^\dagger\gamma^0$.

Under charge conjugation we therefore assume

$$\psi(x) \longrightarrow \eta_C\psi^C(x) \ , \quad \psi^C(x) = C\overline{\psi}(x)^t \ , \tag{1.33}$$

with t denoting transpose. The matrix C is then chosen to ensure $\psi^C(x)$ satisfies the Dirac equation. From the transpose of (1.7) we have

$$\Big(-i(\gamma^t)^\mu\partial_\mu - m\Big)\,\overline{\psi}(x)^t = 0 \ , \tag{1.34}$$

and so

$$\Big(-iC(\gamma^\mu)^tC^{-1}\partial_\mu - m\Big)\,\psi^C(x) = 0 \ . \tag{1.35}$$

Assuming C satisfies

$$C(\gamma^\mu)^tC^{-1} = -\gamma^\mu \ . \tag{1.36}$$

then from (1.35)

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi^C(x) = 0 \ , \tag{1.37}$$

as required. The existence of a matrix C such that (1.36) holds is guaranteed since $-(\gamma^\mu)^t$ also obey the essential definition of gamma matrices in eq.(1.1). From (1.36) we can further straightforwardly obtain

$$C[\gamma^\mu,\gamma^\nu]^tC^{-1} = -[\gamma^\mu,\gamma^\nu] \ , \quad C{\gamma_5}^tC^{-1} = \gamma_5 \ , \quad C(\gamma^\mu\gamma_5)^tC^{-1} = \gamma^\mu\gamma_5 \ . \tag{1.38}$$

Taking the transpose of (1.36) and then eliminating $(\gamma^\mu)^t$ gives $C^{-1t}\gamma^\mu C^t = C^{-1}\gamma^\mu C$ or $\gamma^\mu C^tC^{-1} = C^tC^{-1}\gamma^\mu$ which requires $C^tC^{-1} \propto I$. Hence we must have $C^t = \pm C$. The sign is further determined since $\Gamma^A = (I, \gamma^\mu, \frac{1}{2}[\gamma^\mu,\gamma^\nu], \gamma^\mu\gamma_5, \gamma_5)$ form a basis for 4×4 matrices and in order for Γ^AC to give 10 symmetric and 6 antisymmetric matrices we must then require

$$C^t = -C \ . \tag{1.39}$$

By taking the hermitian conjugate of (1.36) and using the hermeticity conditions (1.3) we find $C^{-1\dagger}(\gamma^\mu)^t C^\dagger = -\gamma^\mu$ and then using $(\gamma^\mu)^t = -C^{-1}\gamma^\mu C$ from (1.36) we may find $(CC^\dagger)^{-1}\gamma^\mu CC^\dagger = \gamma^\mu$ or $CC^\dagger$ commutes with γ^μ. This requires that $CC^\dagger$ is proportional to the identity and, with a choice of normalisation, we therefore have

$$C^\dagger = C^{-1} \ . \tag{1.40}$$

In the Bjorken and Drell representation of (1.10) we may find an explicit form for C by taking

$$C = i\gamma^0\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \ , \tag{1.41}$$

since for the standard Pauli matrices $i\sigma_2 \boldsymbol{\sigma}^t i\sigma_2 = \boldsymbol{\sigma}$ and $(i\sigma_2)^2 = -I$. For this particular representation it is easy to see that (1.39) and (1.40) are satisfied.

Charge conjugation on the quantum Dirac field is then given by

$$\begin{aligned} \hat{C}\psi(x)\hat{C}^{-1} &= \eta_C C\overline{\psi}(x)^t \equiv \eta_C \psi^C(x) \ , \\ \hat{C}\overline{\psi}(x)\hat{C}^{-1} &= \eta_C{}^* \overline{\psi}(x)^* C^\dagger \gamma^0 = -\eta_C{}^* \psi(x)^t C^{-1} \ , \end{aligned} \tag{1.42}$$

applying (1.40). From (1.42), with the assumed definition of ψ^C, we have

$$\hat{C}\psi^C(x)\hat{C}^{-1} = -\eta_C{}^* C(C^{-1})^t \psi(x) = \eta_C{}^* \psi(x) \ , \tag{1.43}$$

using (1.39).

3.3 Transformation of Bi-Linears Under C

A particularly important operator is the electric current $j_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$. If ψ is a quantum operator it is convenient to amend the definition as follow:

$$\overline{\psi}(x)\gamma_\mu\psi(x) = (\gamma_\mu)_{\alpha\beta}\overline{\psi}_\alpha(x)\psi_\beta(x) \to \tfrac{1}{2}(\gamma_\mu)_{\alpha\beta}[\overline{\psi}_\alpha(x), \psi_\beta(x)] \ , \tag{1.44}$$

where the simple product of operators has been replaced by the anti-symmetrized product. This definition ensures that j_μ has well defined properties under charge conjugation and may also eliminate spurious infinities. We have then

$$\begin{aligned} \hat{C}j_\mu(x)\hat{C}^{-1} &= \tfrac{1}{2}(\gamma_\mu)_{\alpha\beta}[\hat{C}\overline{\psi}_\alpha(x)\hat{C}^{-1}, \hat{C}\psi_\beta(x)\hat{C}^{-1}] \\ &= -\tfrac{1}{2}(\gamma_\mu)_{\alpha\beta}[(\psi(x)^t C^{-1})_\alpha, (C\overline{\psi}(x)^t)_\beta] \\ &= -\tfrac{1}{2}(C^{-1}\gamma_\mu C)_{\alpha'\beta'}[\psi_{\alpha'}(x), \overline{\psi}_{\beta'}(x)] \ . \end{aligned} \tag{1.45}$$

But $C^{-1}\gamma_\mu C = -\gamma_\mu{}^t$ so therefore

$$\begin{aligned} \hat{C}j_\mu(x)\hat{C}^{-1} &= \tfrac{1}{2}(\gamma_\mu)_{\beta'\alpha'}[\psi_{\alpha'}(x), \overline{\psi}_{\beta'}(x)] \\ &= -j_\mu(x) \ . \end{aligned} \tag{1.46}$$

The effect of C on other bi-linear expressions in $\overline{\psi}$ and ψ may be worked out in a similar way.

Note that, if as indicated above, we assume that the photon is negative under charge conjugation then the electromagnetic interaction $j^\mu(x)A_\mu(x)$ is invariant under charge conjugation. It therefore causes transitions only between states with the same charge conjugation eigenvalues.

3.4 Negative Energy Solutions

In a previous section we left the detailed form for the negative energy solutions, or equivalently for the spinor $v(p,\lambda)$ in eq.(1.8), of the Dirac equation unresolved. We can use the charge conjugation properties of the Dirac field to fill in this gap. Applying (1.42) to eq.(1.8)

$$\hat{C}\psi(x)\hat{C}^{-1} = \eta_C \sum_{p,\lambda} \left[b(p,\lambda)C\overline{v}(p,\lambda)^t e^{-ip.x} + a(p,\lambda)^\dagger C\overline{u}(p,\lambda)^t e^{ip.x}\right] \ . \tag{1.47}$$

However the effect of charge conjugation is just to interchange particles and anti-particles, leaving momentum and spin unchanged. Therefore we must require

$$\hat{C}a(p,\lambda)\hat{C}^{-1} = \eta_C b(p,\lambda) \ , \quad \hat{C}b(p,\lambda)^\dagger\hat{C}^{-1} = \eta_C a(p,\lambda)^\dagger \ , \tag{1.48}$$

and hence

$$\hat{C}\psi(x)\hat{C}^{-1} = \eta_C\psi^C(x) \ , \quad \psi^C(x) = \sum_{p,\lambda} \left[b(p,\lambda)u(p,\lambda)e^{-ip.x} + a(p,\lambda)^\dagger v(p,\lambda)e^{ip.x}\right] \ . \tag{1.49}$$

Comparing eq.(1.47) and eq.(1.49) we see that the required definition for $v(p,\lambda)$ is such that

$$v(p,\lambda) = C\overline{u}(p,\lambda)^t \ , \tag{1.50}$$

which then implies $u(p,\lambda) = C\overline{v}(p,\lambda)^t$. In the Bjorken and Drell representation from eq.(1.11) and eq.(1.41),

$$v(p,\lambda) = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \sqrt{E+m} \begin{pmatrix} \chi_\lambda^* \\ -\frac{\boldsymbol{\sigma}^t.\mathbf{p}}{E+m}\chi_\lambda^* \end{pmatrix} \ . \tag{1.51}$$

Using the fact that the Pauli matrices satisfy $i\sigma_2\boldsymbol{\sigma}^t i\sigma_2 = \boldsymbol{\sigma}$ we see that

$$v(p,\lambda) = \sqrt{E+m} \begin{pmatrix} \frac{\boldsymbol{\sigma}.\mathbf{p}}{E+m}\chi_\lambda^c \\ \chi_\lambda^c \end{pmatrix} \ , \tag{1.52}$$

where $\chi_\lambda^c = i\sigma_2\chi_\lambda^*$. This completes the construction of the negative energy solutions of the Dirac equation.

If $a(p,\lambda)$ and $b(p,\lambda)$ are different, as for a usual quantised Dirac field, then η_C may be eliminated by redefining the phases of $a(p,\lambda)$ and/or $b(p,\lambda)$. A 'real' or Majorana fermion field is one where $a(p,\lambda) = b(p,\lambda)$ so that

$$\psi(x) = \psi^C(x) \ . \tag{1.53}$$

3.5 Charge Conjugation and Parity

If the parity of a Dirac field is η_P then we have shown in eq.(1.21) that

$$\hat{P}\psi(x)\hat{P}^{-1} = \eta_P \gamma^0 \psi(x_P) \ . \tag{1.54}$$

The u spinor has the property (1.24) which leads to eq.(1.25). With the definition for the v spinor in eq.(1.50) we may now obtain

$$\begin{aligned} \gamma^0 v(p,\lambda) &= -C(\overline{u}(p,\lambda)\gamma^0)^t \\ &= -C\overline{u}(p_P,\lambda)^t \\ &= -v(p_P,\lambda) \ . \end{aligned} \tag{1.55}$$

Hence the parity transformation properties of the Dirac field in eq.(1.21) now require

$$\hat{P}b(p,\lambda)^\dagger \hat{P}^{-1} = -\eta_P b(p_P,\lambda)^\dagger \ . \tag{1.56}$$

Assuming $\eta_P = \pm 1$ we see that the parity of a spin $\frac{1}{2}$ anti particle is opposite to that of the associated particle. Thus if we choose positive parity for the electron then the positron has negative parity. Similarly quarks have positive parity and anti-quarks have negative parity.

Equivalently using

$$\hat{P}\overline{\psi}(x)\hat{P}^{-1} = \eta_P^* \, \overline{\psi}(x_P)\gamma^0 \ , \tag{1.57}$$

we find for conjugate field $\psi^C(x) = C\overline{\psi}(x)^t$

$$\begin{aligned} \hat{P}\psi^C(x)\hat{P}^{-1} &= C\hat{P}\overline{\psi}(x)^t\hat{P}^{-1} \ , \\ &= C(\eta_P^*\overline{\psi}(x_P)\gamma^0)^t \ , \\ &= -\eta_P^*\gamma^0 C\overline{\psi}(x_P)^t \ , \\ &= -\eta_P^*\gamma^0\psi^C(x_P) \ , \end{aligned} \tag{1.58}$$

since $C(\gamma^0)^t = -\gamma^0 C$. This is opposite to eq.(1.21) if η_P is real. If $\psi(x) = \psi^C(x)$, as for a Majorana field, then we must take $\eta_P = -\eta_P^*$ or $\eta_P = \pm i$.

4 Time Reversal

4.1 Classical Theory

Newton's equation of motion for a particle of mass m subject to a force $\mathbf{F}(\mathbf{x})$ is

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \ . \tag{1.59}$$

It is easy to check that if $\mathbf{x}(t)$ is a solution then so is $\mathbf{x}^T(t) = \mathbf{x}(-t)$. This is what is meant by time reversibility in the classical case, namely that the backwards running motion is just as good a solution as the original motion. If there is a velocity dependent force due to the presence of a magnetic field $\mathbf{B}$ then the equation of motion

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + q\dot{\mathbf{x}} \times \mathbf{B}(\mathbf{x}) \ , \tag{1.60}$$

is only time reversal invariant if we also reverse the magnetic field. That is both $\mathbf{x}^T(t)|_{\mathbf{B}} = \mathbf{x}(-t)|_{-\mathbf{B}}$ are solutions of this equation of motion. This is an indication of how it is necessary to consider the effect of time reversal on all the fields in a problem in order to exhibit invariance. For electric and magnetic fields in general we must require $\mathbf{E}(t,\mathbf{x}) \to \mathbf{E}(-t,\mathbf{x})$, $\mathbf{B}(t,\mathbf{x}) \to -\mathbf{B}(-t,\mathbf{x})$.

4.2 Quantum Mechanics

The wave function for a non-relativistic particle satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\psi(t,\mathbf{x}) \ . \tag{1.61}$$

A solution corresponding to time-reversed evolution should involve $\psi(-t,\mathbf{x})$. However this time-reversed wave-function does not satisfy the Schrödinger equation. To obtain such a solution it is necessary to combine time-reversal with complex conjugation so that the action of time reversal on a wave function is

$$\psi(t,\mathbf{x}) \longrightarrow \eta_T \psi^T(t,\mathbf{x}) \ , \quad \psi^T(t,\mathbf{x}) = \psi(-t,\mathbf{x})^* \ , \tag{1.62}$$

for $|\eta_T| = 1$. It is easy to see that ψ^T does satisfy the Schrödinger equation. The complete time-reversal transformation therefore is **anti-linear**. Thus if ψ is replaced by $\alpha\psi$ then under time reversal

$$\alpha\psi(t,\mathbf{x}) \longrightarrow \alpha^* \eta_T \psi^T(t,\mathbf{x}) \ . \tag{1.63}$$

Note the effect of anti-linearity on scalar products:

$$\begin{aligned}
(\psi(t),\phi(t)) \longrightarrow (\psi^T(t),\phi^T(t)) &= \int \mathrm{d}^3\mathbf{x}\, \psi^T(t,\mathbf{x})^* \phi^T(t,\mathbf{x}) \\
&= \int \mathrm{d}^3\mathbf{x}\, \phi(-t,\mathbf{x})^* \psi(-t,\mathbf{x}) \\
&= (\phi(-t),\psi(-t)) \\
&= (\psi(-t),\phi(-t))^* \ .
\end{aligned} \tag{1.64}$$

Time reversal therefore complex conjugates scalar products. Probabilities which depend only on the modulus of scalar products are unaffected by the anti-linearity of the transformation.

4.3 Specification of an Anti-Linear Operator

The complete effect of a linear operator can be determined by specifying its action on a basis set of the vector space of physical states and then extending its application by exploiting the linearity of the map. Similarly the complete effect of an anti-linear map can be determined by specifying its effect on a basis and extending the result using its anti-linearity. For example according to previous definitions above time-reversal acting on the momentum state wave-function $e^{i\mathbf{p}.\mathbf{x}/\hbar}$ is to produce the complex conjugate wave-function $e^{-i\mathbf{p}.\mathbf{x}/\hbar}$. In Dirac notation in this basis $\hat{T}$ is defined by

$$\hat{T}|\mathbf{p}\rangle = |-\mathbf{p}\rangle \ . \tag{1.65}$$

A general state $|\psi\rangle$ can be represented as

$$|\psi\rangle = \sum_{\mathbf{p}} \tilde{\psi}(\mathbf{p})|\mathbf{p}\rangle \ . \tag{1.66}$$

The effect of time-reversal is then

$$\begin{aligned} \hat{T}|\psi\rangle &= \hat{T}\sum_{\mathbf{p}} \tilde{\psi}(\mathbf{p})|\mathbf{p}\rangle = \sum_{\mathbf{p}} \tilde{\psi}^*(\mathbf{p})\hat{T}|\mathbf{p}\rangle = \sum_{\mathbf{p}} \tilde{\psi}^*(\mathbf{p})|-\mathbf{p}\rangle = \sum_{\mathbf{p}} \tilde{\psi}^*(-\mathbf{p})|\mathbf{p}\rangle \\ &= |\psi^T\rangle \ . \end{aligned} \tag{1.67}$$

If the state $|\phi\rangle = \sum_{\mathbf{p}} \tilde{\phi}(\mathbf{p})|\mathbf{p}\rangle$ is defined similarly then the scalar product for these two states $\langle\phi|\psi\rangle = \sum_{\mathbf{p}} \tilde{\phi}^*(\mathbf{p})\tilde{\psi}(\mathbf{p})$ so that

$$\langle\phi|\psi\rangle = \langle\phi^T|\psi^T\rangle^* \ . \tag{1.68}$$

This results holds generally for the action of the anti-unitary operator $\hat{T}$ and it is easy to check that this result is essential in accord to be consistent with the required anti-linear properties of $\hat{T}$, when if $|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle$ then $|\psi^T\rangle = {a_1}^*|{\psi_1}^T\rangle + {a_2}^*|{\psi_2}^T\rangle$.

4.4 Time-Reversal for Relativistic Wave-Functions

In line with non-relativistic quantum mechanics the effect of time-reversal on the complex wave-function $\phi(x)$ of a relativistic scalar particle is

$$\phi(x) \longrightarrow \eta_T\phi^T(x) \ , \quad \phi^T(x) = \phi(x_T)^* \ , \tag{1.69}$$

where $x_T^\mu = (-t, \mathbf{x})$. Using the anti-linear properties of the transformation it is easy to see, by modifying ϕ by a suitable complex phase, that we may take $\eta_T = 1$ but we keep η_T for later convenience. The transformation under time reversal for a state of definite momentum, $\phi(p,x) = e^{-ip.x}$, is therefore given by

$$\phi^T(p,x) = \left\{e^{-ip.x_T}\right\}^* = e^{-ip_T.x} = \phi(p_T,x) \ , \tag{1.70}$$

where $p_T^\mu = (E, -\mathbf{p})$.

Dirac wave-functions are dealt with in a similar way but with an additional linear transformation in order to ensure that the result satisfies the Dirac equation,

$$\psi(x) \longrightarrow \eta_T \psi^T(x) \ , \quad \psi^T(x) = B\psi(x_T)^* \ . \tag{1.71}$$

Hence from the complex conjugate of the Dirac equation for $\psi(x)$ and taking $x^0 \to -x^0$ we find

$$\left(i\gamma^{0*}\partial_t - i\boldsymbol{\gamma}^*\cdot\boldsymbol{\nabla} - m\right)\psi(x_T)^* = 0 \ . \tag{1.72}$$

If we assume that the matrix B is defined by the requirement

$$B(\gamma^{0*}, -\boldsymbol{\gamma}^*)B^{-1} = (\gamma^0, \boldsymbol{\gamma}) \ , \tag{1.73}$$

then, with the definition of ψ^T in eq.(1.71),

$$B\left(i\gamma^{0*}\partial_t - i\boldsymbol{\gamma}^*\cdot\boldsymbol{\nabla} - m\right)\psi(x_T)^* = (i\gamma^\mu\partial_\mu - m)\,\psi^T(x) \tag{1.74}$$

so that ψ^T satisfies the Dirac equation if ψ does. To satisfy eq.(1.73) it is sufficient, since from the hermeticity condition eq.(1.3) $(\gamma^{0*}, -\boldsymbol{\gamma}^*) = (\gamma^{0t}, \boldsymbol{\gamma}^t)$, to take

$$B = \gamma_5 C \ , \tag{1.75}$$

where we use the basic property of C in eq.(1.36) and also the result that γ_5 anti-commutes with γ^μ from (1.5). With the result (1.75) for B we have also

$$B^t = -C\gamma_5{}^t = -\gamma_5 C = -B \ , \quad B^\dagger = C^\dagger\gamma_5 = B^{-1} \quad \Rightarrow \quad B^* = -B^{-1} \ , \tag{1.76}$$

using the hermeticity of γ_5 and eqs.(1.39,1.38,1.40).

Acting on spinors $u(p,\lambda)$ time reversal not only reflects the spatial part of the momentum but also reverses the spin label since the action on the angular momentum operator is given by $\hat{T}\mathbf{J}\hat{T}^{-1} = -\mathbf{J}$. Thus the time reversed spinor which is given by eq.(1.71) has the form

$$Bu(p_P, \lambda)^* = u(p, -\lambda)\eta_\lambda \ , \tag{1.77}$$

where $\eta_\lambda = \pm 1$ depends on the conventions chosen for relating different spin components. This result may be regarded as giving a definition for $u(p, -\frac{1}{2})$ in terms of $u(p, \frac{1}{2})$. Taking the complex conjugate of (1.77) and using (1.76) we find

$$u(p_P, \lambda) = -Bu(p, -\lambda)^*\eta_\lambda \ , \tag{1.78}$$

so we must require $\eta_{-\lambda}\eta_\lambda = -1$.

From the definition eq.(1.50) for $v(p,\lambda)$ and using eqs.(1.75,1.76,1.77), so that $B^{-1*} = -C\gamma_5{}^t$, then with $C^* = -C^{-1}, \gamma_5{}^\dagger = \gamma_5$ and $C\gamma_5{}^t C^{-1} = \gamma_5$ we also find similar result for the v spinor as for the u spinor in (1.77),

$$\begin{aligned} v(p,\lambda)^* &= C^*\overline{u}(p,\lambda)^\dagger = -C^{-1}\gamma^0 u(p,\lambda) \\ &= -C^{-1}\gamma^0 B^{-1*}u(p_P,-\lambda)^*\eta_\lambda \\ &= -C^{-1}\gamma^0 B^{-1*}\gamma^{0\,t}\overline{u}(p_P,-\lambda)^t\eta_\lambda \\ &= C^{-1}\gamma^0 C\gamma_5{}^t\gamma^{0\,t}C^{-1}v(p_P,-\lambda)\eta_\lambda \\ &= -C^{-1}\gamma^0\gamma_5\gamma^0 v(p_P,-\lambda)\eta_\lambda \\ &= B^{-1}v(p_P,-\lambda)\eta_\lambda\ . \end{aligned} \tag{1.79}$$

Explicitly in the Bjorken and Drell representation eq.(1.75) gives, as a result of eq.(1.10) and eq.(1.41),

$$B = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}\ . \tag{1.80}$$

With the form given by eq.(1.11) for the spinor $u(p,\lambda)$ the effect of time-reversal is

$$Bu(p,\lambda)^* = \sqrt{E+m}\begin{pmatrix} i\sigma_2\chi_\lambda^* \\ -\frac{\boldsymbol{\sigma}.\mathbf{p}}{E+m}i\sigma_2\chi_\lambda^* \end{pmatrix}\ , \tag{1.81}$$

since $i\sigma_2\boldsymbol{\sigma}^*i\sigma_2 = \boldsymbol{\sigma}$. In the standard representation χ_λ are real and it is easy to see that $i\sigma_2\chi_\lambda = \chi_{-\lambda}(-1)^{\frac{1}{2}+\lambda}$. Hence we see that eq.(1.77) is satisfied with $\eta_\lambda = (-1)^{\frac{1}{2}+\lambda}$.

4.5 Time-Reversal and Quantum Fields

In order to achieve the same time-reversal properties on the particle states for a scalar quantum field as is suggested by the previous discussion of wave functions, we demand $\hat{T}|0\rangle = |0\rangle$ and

$$\hat{T}a(p)\hat{T}^{-1} = \eta_T a(p_P)\ , \tag{1.82}$$

which gives for single particle states

$$\hat{T}|p\rangle = \hat{T}a(p)^\dagger|0\rangle = \hat{T}a(p)^\dagger\hat{T}^{-1}\hat{T}|0\rangle = \eta_T^* a(p_P)^\dagger|0\rangle = \eta_T^*|p_P\rangle\ . \tag{1.83}$$

Again the phase η_T has no absolute significance since if $\eta_T = e^{2i\gamma}$ we may remove it from (1.82) by letting $e^{i\gamma}a(p) \to a(p)$, taking account of the anti-linear properties of $\hat{T}$.

The operation of time-reversal on quantum fields can then be found in terms of its action on the creation and annihilation operators. For a scalar field

$$\phi(x) = \sum_p \left[a(p)e^{-ip.x} + b(p)^\dagger e^{ip.x}\right] \tag{1.84}$$

remembering that $\hat{T}$ is *anti-linear*

$$\begin{aligned}\hat{T}\phi(x)\hat{T}^{-1} &= \sum_p \left[\hat{T}a(p)\hat{T}^{-1}e^{ip.x} + \hat{T}b(p)^\dagger\hat{T}^{-1}e^{-ip.x}\right] \quad , &(1.85)\\ &= \eta_T\sum_p \left[a(p_P)e^{ip.x} + b(p_P)^\dagger e^{-ip.x}\right] \; , &(1.86)\end{aligned}$$

where we also require

$$\hat{T}b(p)^\dagger\hat{T}^{-1} = \eta_T b(p_P)^\dagger \; . \tag{1.87}$$

Using the invariance of the range of the p-summation under $p \to p_P$ and $p_P.x = -p.x_T$ it follows that for a scalar field

$$\hat{T}\phi(x)\hat{T}^{-1} = \eta_T\phi(x_T) \; . \tag{1.88}$$

Analogous reasoning holds for the Dirac spinor field so that

$$\hat{T}\psi(x)\hat{T}^{-1} = \eta_T B^{-1}\psi(x_T) \; . \tag{1.89}$$

We have also

$$\hat{T}\overline{\psi}(x)\hat{T}^{-1} = \eta_T^*\overline{\psi}(x_T)B \; . \tag{1.90}$$

Using eqs.(1.77,1.79) it is easy to see that

$$\hat{T}a(p,\lambda)\hat{T}^{-1} = \eta_T(-1)^{\frac{1}{2}+\lambda}a(p_P,-\lambda) \; , \quad \hat{T}b(p,\lambda)\hat{T}^{-1} = \eta_T^*(-1)^{\frac{1}{2}+\lambda}b(p_P,-\lambda) \; . \tag{1.91}$$

Some important examples of the transformation of bi-linears under T are the scalar

$$\hat{T}\overline{\psi}(x)\psi(x)\hat{T}^{-1} = \overline{\psi}(x_T)BB^{-1}\psi(x_T) = \overline{\psi}(x_T)\psi(x_T) \; , \tag{1.92}$$

and the electric current

$$\hat{T}j^\mu(x)\hat{T}^{-1} = \overline{\psi}(x_T)B(\gamma^\mu)^*B^{-1}\psi(x_T) \tag{1.93}$$

so that

$$\hat{T}j^0(x)\hat{T}^{-1} = j^0(x_T) \; , \tag{1.94}$$

and

$$\hat{T}\mathbf{j}(x)\hat{T}^{-1} = -\mathbf{j}(x_T) \; . \tag{1.95}$$

Time-reversal therefore leaves the charge density unchanged but reverses the flow of the current.

4.6 Transformation of States Under T

For a general single particle state of spin S the transformation properties under time-reversal is given by

$$\hat{T}|p,\lambda\rangle = \eta_T^*(-1)^{S+\lambda}|p_P,-\lambda\rangle \ . \tag{1.96}$$

The dependence of the phase factor on λ is dictated by compatibility with standard conventions relating spin states. For the particle at rest $\bar{p}^\mu = (m,\mathbf{0})$ we require $J_\pm|\bar{p},\lambda\rangle = N_{S,\pm\lambda}|\bar{p},\lambda\pm 1\rangle$, with $N_{S,\lambda}$ real, and under time reversal $\hat{T}J_\pm\hat{T}^{-1} = -J_\mp$. Clearly it follows from eq.(1.96) that

$$\hat{T}^2|p,\lambda\rangle = \eta_T(-1)^{S+\lambda}\hat{T}|p_P,-\lambda\rangle = (-1)^{2S}|p,\lambda\rangle \,, \tag{1.97}$$

and hence in general $\hat{T}^2 = (-1)^F$, where F measures the fermion number.

5 Applications of P, C and T

As an example consider the interaction Lagrangian of QED, namely

$$\mathcal{L}_I(x) = -e\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \ . \tag{1.98}$$

The associated interaction Hamiltonian is

$$V(t) = -\int \mathrm{d}^3\mathbf{x}\,\mathcal{L}_I(x) \ , \tag{1.99}$$

and the S-matrix relating in states to out states is

$$S = \mathcal{T}\exp\left\{-i\int_{-\infty}^{\infty}\mathrm{d}t\,V(t)\right\} \ . \tag{1.100}$$

Taking into account the transformation properties of all the fields we see that

$$\begin{aligned}
\hat{P}\mathcal{L}_I(x)\hat{P}^{-1} &= \mathcal{L}_I(x_P) \ , && (1.101)\\
\hat{C}\mathcal{L}_I(x)\hat{C}^{-1} &= \mathcal{L}_I(x) \ , && (1.102)\\
\hat{T}\mathcal{L}_I(x)\hat{T}^{-1} &= \mathcal{L}_I(x_T) \ . && (1.103)
\end{aligned}$$

These results have the implications

$$\begin{aligned}
\hat{P}V(t)\hat{P}^{-1} &= V(t) \ , && (1.104)\\
\hat{C}V(t)\hat{C}^{-1} &= V(t) \ , && (1.105)\\
\hat{T}V(t)\hat{T}^{-1} &= V(-t) \ . && (1.106)
\end{aligned}$$

For the S-matrix we find

$$\begin{aligned}
\hat{P}S\hat{P}^{-1} &= S \ , && (1.107)\\
\hat{C}S\hat{C}^{-1} &= S \ . && (1.108)
\end{aligned}$$

The S-matrix therefore commutes with $\hat{P}$ and $\hat{C}$. This in turn implies that the parity and charge conjugation properties of the initial and final states in QED are the same.

Time-reversal is a little more complicated. Expanding the time-ordered exponential in eq.(1.100) we have

$$S = \sum_n (-i)^n \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n \, V(t_1)V(t_2)\ldots V(t_n) \ . \tag{1.109}$$

We have then

$$S_T = \hat{T} S \hat{T}^{-1} = \sum_n i^n \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n \, V(-t_1)V(-t_2)\ldots V(-t_n) \ . \tag{1.110}$$

If now we set $\tau_i = -t_{n-i+1}$ we find after a consideration of ranges of integration

$$S_T = \sum_n i^n \int_{-\infty}^{\infty} \mathrm{d}\tau_1 \int_{-\infty}^{\tau_1} \mathrm{d}\tau_2 \ldots \int_{-\infty}^{\tau_{n-1}} \mathrm{d}\tau_n \, V(\tau_n)V(\tau_{n-1})\ldots V(\tau_1) \ , \tag{1.111}$$

with the consequence that

$$S_T = S^\dagger. \tag{1.112}$$

Because $\hat{T}$ is an anti-linear transformation we have for any states $|a\rangle$ and $|b\rangle$ (we use the notation $|a_T\rangle = \hat{T}|a\rangle$ etc.)

$$\langle a_T | S_T{}^\dagger | b_T \rangle = \langle a | S^\dagger | b \rangle^*, \tag{1.113}$$

that is

$$\langle b | S | a \rangle = \langle a_T | S_T{}^\dagger | b_T \rangle \ . \tag{1.114}$$

Time reversal invariance of the theory implies that $S_T{}^\dagger = S$ therefore

$$\langle b | S | a \rangle = \langle a_T | S | b_T \rangle \ . \tag{1.115}$$

In turn this implies that the probabilities, rates or cross-sections are equal for two processes related by time-reversal.

5.1 The CPT Theorem

If a theory is invariant under T, P and C separately then it is invariant under the combined (anti-linear) transformation CPT. Even if T, P and C are not separate symmetries it can be shown that for any Lorentz invariant Lagrangian $\mathcal{L}(x)$ formed products of quantum fields at the point x then it is also invariant under CPT. This is a version of the CPT Theorem. In general the CPT Theorem implies that any Lorentz invariant local quantum field theory will be invariant under the combined transformation. The consequence is that if we set $\Theta = \hat{C}\hat{P}\hat{T}$ then

$$\Theta S \Theta^{-1} = S^\dagger \ . \tag{1.116}$$

Invariance under CPT is sufficient to ensure that particles and anti-particles that are unstable have the same lifetime. Note also it is a consequence of the theorem that if a theory is time-reversal invariant then it is certainly invariant under CP, or if it is invariant under any two of the transformations it will be invariant under the third.

To verify how invariance of $\mathcal{L}$ follows we consider first the CPT transformation on a scalar field ϕ where our previous results give

$$\Theta\phi(x)\Theta^{-1} = \eta_T\eta_P\eta_C\phi(-x)^\dagger \ . \tag{1.117}$$

We now choose phases so that $\eta_T\eta_P\eta_C = 1$. For an arbitrary bosonic quantum tensor field $\phi_{\mu_1,\ldots\mu_n}$ we generalise this to

$$\Theta\phi_{\mu_1,\ldots\mu_n}(x)\Theta^{-1} = (-1)^n\phi_{\mu_1,\ldots\mu_n}(-x)^\dagger \ . \tag{1.118}$$

Note that this result is consistent with forming new fields by taking derivatives, e.g. $\phi_\mu = \partial_\mu\phi$. Acting on Dirac fields it also follows from the previous results for the action of P, C, T separately that, with a similar choice of phases,

$$\Theta\psi(x)\Theta^{-1} = -\theta\overline{\psi}(-x)^t \ , \qquad \Theta\overline{\psi}(x)\Theta^{-1} = \psi(-x)^t\theta^{-1} \ , \tag{1.119}$$

where the matrix $\theta = -B^{-1}\gamma^0 C$. From eqs.(1.21,1.42,1.89) and the result for B in eq.(1.75) we find $\theta = {\gamma_5}^t\gamma^{0t}$. Hence eq.(1.119) can be rewritten as

$$\Theta\psi(x)\Theta^{-1} = -(\psi(-x)^\dagger\gamma_5)^t \ , \quad \Theta\overline{\psi}(x)\Theta^{-1} = (\gamma_5\overline{\psi}(-x)^\dagger)^t \ . \tag{1.120}$$

With this result we may find for the transformation of an arbitrary bi-linear formed from Dirac fields ψ_1, ψ_2,

$$\begin{aligned} \Theta\overline{\psi}_1(x)M\psi_2(x)\Theta^{-1} &= -(\gamma_5\overline{\psi}_1(-x)^\dagger)^t M^*(\psi_2(-x)^\dagger\gamma_5)^t \\ &= \psi_2(-x)^\dagger\gamma_5 M^\dagger\gamma_5\overline{\psi}_1(-x)^\dagger \\ &= \Big[\overline{\psi}_1(-x)\gamma_5 M\gamma_5\psi_2(-x)\Big]^\dagger \ . \end{aligned} \tag{1.121}$$

It is then straightforward to see that an n rank tensor field constructed from Dirac fields by choosing an appropriate form for the matrix M has the same transformation properties as in eq.(1.118) (note that from eq.(1.5) $\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_n}\gamma_5 = (-1)^n\gamma_{\mu_1}\ldots\gamma_{\mu_n}$).

Thus a Lorentz invariant Lagrangian which is a sum of monomials in the fields, including possible bi-linears in fermion fields, which are Lorentz scalars formed from contracting Lorentz indices satisfies under a CPT transformation, assuming the boson fields and fermion bi-linears transform according to eqs.(1.118,1.121),

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(-x)^* = \mathcal{L}(-x) \ . \tag{1.122}$$

In particular the interaction Lagrangian formed from the operator fields satisfies

$$\Theta\mathcal{L}_I(x)\Theta^{-1} = \mathcal{L}_I(-x) \ , \tag{1.123}$$

so that the S-matrix operator given by eqs.(1.99,1.100) obeys eq.(1.116). This result holds even if $\hat{C}, \hat{P}$ and $\hat{T}$ are not separately defined in the quantum theory, such as when the corresponding discrete symmetries are broken and the required transformed states may not exist, since the operator Θ may always be defined by its action in eqs.(1.118,1.120). On particle states CPT transforms particles into anti-particles with opposite spin since $\Theta a(p,\lambda)\Theta^{-1} = (-1)^{\frac{1}{2}+\lambda} b(p,-\lambda)$.

Part II
Broken Symmetries in Field Theory

1 Symmetry in Quantum Theory

In theoretical physics as a whole and in quantum mechanics in particular the idea of symmetries which combine to form a symmetry group is of crucial importance. In a quantum theory the action of symmetry transformations conventionally correspond to unitary (or perhaps anti-unitary) operators acting on the space of states for the given theory.

In general a transformation g acting on the states belonging to the Hilbert space for a quantum mechanical system, $|\psi\rangle \underset{g}{\to} |\psi^g\rangle$, where $|\psi^g\rangle$ is the state corresponding to the transformed physical system, is a symmetry if for all states $|\psi\rangle$, $|\phi\rangle$ transition probabilities are invariant,

$$|\langle\phi^g|\psi^g\rangle|^2 = |\langle\phi|\psi\rangle|^2 \,. \tag{2.1}$$

A theorem due to Wigner asserts that there are two possibilities, either

$$|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle \;\Rightarrow\; |\psi^g\rangle = a_1|\psi_1{}^g\rangle + a_2|\psi_2{}^g\rangle \quad \text{and} \quad \langle\phi^g|\psi^g\rangle = \langle\phi|\psi\rangle \,, \tag{2.2}$$

or

$$|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle \;\Rightarrow\; |\psi^g\rangle = a_1^*|\psi_1{}^g\rangle + a_2^*|\psi_2{}^g\rangle \quad \text{and} \quad \langle\phi^g|\psi^g\rangle = \langle\phi|\psi\rangle^* \,, \tag{2.3}$$

In case (2.2) there is a unitary operator $U(g)$ such that for all $|\phi\rangle$

$$U(g)|\psi\rangle = |\psi^g\rangle \,, \tag{2.4}$$

while for (2.3) the corresponding operator is anti-unitary (if the symmetry transformations are continuously connected to the identity then $U(g)$ can only be unitary). If the symmetry transformations belong to a symmetry group G then for any two symmetry transformations $g_1, g_2 \in G$ we may define their product $g_1 g_2 \in G$ and it is natural to suppose

$$U(g_1)U(g_2) = U(g_1 g_2) \,, \tag{2.5}$$

Actually in quantum mechanics states are only defined up to a complex phase of modulus 1. This leads to the freedom of introducing complex phase factors on the right hand sides of (2.5). In many cases such complications can be avoided with the assumption of standard phase conventions for a suitable basis of states in the

Hilbert space for the theory. The assumption of the symmetry being invariant under time evolution of course means that the Hamiltonian itself is invariant

$$U(g)^\dagger H U(g) = H\,. \tag{2.6}$$

In this cases the states with a given energy must form a representation space for G

$$U(g)|r\rangle = \sum_s |s\rangle D_{sr}(g)\,, \tag{2.7}$$

where r labels the states and $D(g)$ defines a finite dimensional representation of the group G. Thus the space of states of given energy may be classified in terms of the representations of the group G. As well as the usual rotational group in particle physics the isospin group $SU(2)_I$ and its extension $SU(3)_F$ are well known and although they do not define exact symmetries of the Hamiltonian they classify particle states of nearly degenerate masses. To the extent that the symmetry is exact the vacuum must be invariant, or form a trivial singlet representation of the group,

$$U(g)|0\rangle = |0\rangle\,. \tag{2.8}$$

1.1 Spontaneous Symmetry Breakdown, Discrete Symmetries

Although the above is the conventional way in which symmetries are realised in quantum theory it is not the only possibility. The crucial assumption is that contained in (2.8), namely that the vacuum state is invariant. In classical physics there are many instances when the ground state does not respect the basic symmetry of the Lagrangian or Hamiltonian. To illustrate this in field theory we consider the simplest case of a Lagrangian density for a single scalar field ϕ,

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)\,, \tag{2.9}$$

which is invariant under the Z_2 symmetry (Z_2 is virtually the simplest possible group with only two elements $\{1,-1\}$),

$$\phi \leftrightarrow -\phi\,. \tag{2.10}$$

The assumption of symmetry under (2.10) requires

$$V(\phi) = V(-\phi)\,, \tag{2.11}$$

and as a typical field theory example we may take

$$V(\phi) = \tfrac{1}{2}\,m^2\phi^2 + \tfrac{1}{4!}\,g\phi^4\,, \qquad g > 0\,, \tag{2.12}$$

with the condition on the coupling g necessary to ensure that $V(\phi)$ is bounded below. For $m^2 > 0$ the conventional picture is realised at least classically since the minimum of V occurs at $\phi = 0$ which is invariant under (2.10). In the quantum theory we would expect that the symmetry should be realised by a unitary operator U such that $U^2 = 1$, the identity. The assumption of symmetry, from (2.6) and (2.8), in this case,

$$U^\dagger H U = H \,, \qquad U|0\rangle = |0\rangle \,, \tag{2.13}$$

implies that the energy eigenstates can be written $|\psi_\pm\rangle$ where

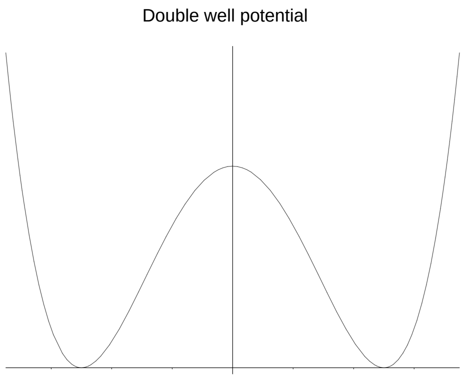
$$U|\psi_\pm\rangle = \pm|\psi_\pm\rangle \,, \tag{2.14}$$

since there are just two possible representations of this very simple group Z_2. The states $|\phi_+\rangle$ and $|\psi_-\rangle$ are respectively created by the application of even and odd numbers of field operators ϕ in the quantum field theory to the vacuum state $|0\rangle$.

However a very different picture emerges if $m^2 < 0$. In this case by addition of a constant we might rewrite $V(\phi)$ in the form

$$V(\phi) = \tfrac{1}{4!}\, g(\phi^2 - v^2)^2 \,, \tag{2.15}$$

which has the form the form of a double well, shown below



In the ground state of minimum energy there are two possibilities classically, $\phi = \pm v$, and in the quantum field theory there are expected to be two vacua $|0_\pm\rangle$ such that

$$\langle 0_\pm|\phi(x)|0_\pm\rangle = \pm v_R \,, \tag{2.16}$$

with v_R some renormalised value, including quantum corrections, of the constant v. For the two vacua it is possible to construct two independent Hilbert spaces of states $\mathcal{H}_\pm$ by the application of field operators to $|0_\pm\rangle$. These two Hilbert spaces have no overlap, all states in $\mathcal{H}_+$ are entirely distinct from those in $\mathcal{H}_-$, but they define two equivalent quantum field theories. Although there is an exact one to one mapping between states in the two spaces there is no unitary operator acting on the states which realises this as a physical symmetry. To set up a perturbative expansion for this theory it is necessary to shift the field

$$\phi = v + f \,, \tag{2.17}$$

so that the Lagrangian density defined by (2.9) and (2.15) becomes

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu f \partial_\mu f - \tfrac{1}{6}g(v^2 f^2 + v f^3 + \tfrac{1}{4}f^4) \,, \tag{2.18}$$

so that f is a massive field with cubic and quartic interactions. In perturbation theory then to lowest order $\langle 0|f|0\rangle = 0$ but there are corrections which will make this non zero resulting from Feynman diagrams with one external line.

It is important to recognise that in $\mathcal{H}_+$ there may be states $|\psi\rangle$ which are essentially identical to $|0_-\rangle$ for some finite region $\mathcal{V}$, i.e.

$$\langle\psi|\phi(0,\mathbf{x})|\psi\rangle = -v_R \ \text{ for } \ \mathbf{x}\in\mathcal{V}\,, \quad \langle\psi|\phi(0,\mathbf{x})|\psi\rangle \to v_R \ \text{ as } \ |\mathbf{x}|\to\infty\,. \tag{2.19}$$

Such states are similar to a bubble inside which the theory looks like that represented by the space $\mathcal{H}_-$. The state $|\psi\rangle$ however has a non zero energy which is at least proportional to the area of the boundary of the region $\mathcal{V}$ so that the region $\mathcal{V}$ cannot expand in time indefinitely. If the symmetry is broken so that $|0_-\rangle$ has a lower energy density then it is possible for the bubble to grow indefinitely, since the gain in energy proportional to the volume of $\mathcal{V}$ can compensate the energy involved in the boundary.

The scenario just described is valid in quantum field theory but it fails in ordinary quantum mechanics. To illustrate this we may consider the above example replacing the field ϕ by x. The Hamiltonian for this one dimensional model becomes

$$H = \tfrac{1}{2}\,p^2 + \tfrac{1}{24}\,g(x^2 - v^2)^2\,. \tag{2.20}$$

The Z_2 symmetry for $x \leftrightarrow -x$ is then the conventional parity symmetry. It is well known that in quantum mechanics that parity is always a good quantum number, if the potential is invariant under reflection, and that the energy eigenstates can be classified in terms of being even or odd parity. Near the minima of the potential the Hamiltonian in (2.20) may be approximated by a harmonic oscillator form $\frac{1}{2}p^2 + \frac{1}{2}\omega^2(x\mp v)^2$ with $\omega^2 = \frac{1}{3}gv^2$ so that there are two apparent degenerate ground state wave functions each with energy $\frac{1}{2}\omega$,

$$\psi_0(x\mp v)\,, \qquad \psi_0(x) = \Big(\frac{\omega}{\pi}\Big)^{\frac{1}{4}} e^{-\frac{1}{2}\omega x^2}\,. \tag{2.21}$$

In perturbation theory these states remain degenerate but there are non perturbative effects due to tunnelling through the potential barrier separating the two minima. The tunnelling amplitude is proportional to

$$e^{-\int_{-v}^{v}\mathrm{d}x\,\sqrt{2V(x)}} = e^{-\frac{2}{3\sqrt{3}}\sqrt{g}v^3}\,, \tag{2.22}$$

and the two low lying states are now non degenerate with approximate wave functions and energies of the form, if gv^6 is large,

$$\psi_\pm(x) \approx \frac{1}{\sqrt{2}}\Big(\psi_0(x-v)\pm\psi_0(x+v)\Big)\,, \quad E_\pm = \tfrac{1}{2}\omega \mp K e^{-\frac{2}{3\sqrt{3}}\sqrt{g}v^3}\,, \quad K>0\,. \tag{2.23}$$

Clearly these are now parity eigenstates with parity $\pm$. This tunnelling between the two ground states does not happen in quantum field theory. If the theory were quantised in a finite volume V then there would be a tunnelling amplitude so that

$$\langle 0_- | 0_+ \rangle \sim e^{-CV}\,. \tag{2.24}$$

This goes to zero, as also does the overlap between any state formed by applying products of field operators to the state $|0_+\rangle$ and any similar state formed from $|0_-\rangle$, as $V \to \infty$.

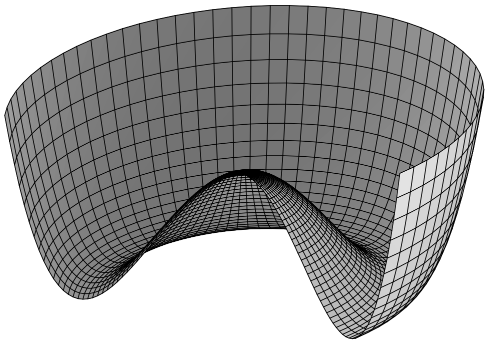
The above description of spontaneous symmetry breakdown for Z_2 generalises straightforwardly to any discrete symmetry group of order N. Any quantum field theory has a unique vacuum state chosen from N equivalent possibilities but there may be localised regions, bounded by domain walls of non zero energy density, where the state appears like one of the $N-1$ other vacua.

1.2 Spontaneous Symmetry Breakdown, Continuous Symmetries

We may also consider continuous symmetry group which may undergo spontaneous symmetry breakdown. As a simple illustration we first consider an n component scalar field theory with real fields $\phi = (\phi_1, \dots \phi_n)$. Defining $\phi^2 \equiv \phi{\cdot}\phi = \sum_r \phi_r\phi_r$ we postulate a Lagrangian density.

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi{\cdot}\partial_\mu\phi - V(\phi)\,, \quad V(\phi) = \tfrac{1}{8}g(\phi^2 - v^2)^2\,, \quad g>0\,. \tag{2.25}$$

For $n=2$ the potential has the form shown,



This Lagrangian is clearly invariant under the symmetry group $O(n)$ which rotates the n-vector ϕ. It is also evident that the classical ground state corresponding to the minimum of the potential in (2.25) is given by $\phi = \phi_0$ for any ϕ_0 such that

$$
{\phi_0}^2 = v^2 \,. \tag{2.26}
$$

This defines an $n-1$ dimensional sphere, S^{n-1}, which is not a single point left invariant by $O(n)$ (for $n = 2$ the classical ground state is a circle). At any point on the S^{n-1} defined by (2.26) there are directions where the potential energy remains unchanged. Although this is nearly obvious we can see this explicitly by expanding ϕ about a particular point on S^{n-1}, for example

$$
\phi_0 = (0, \dots, 0, v) \,, \tag{2.27}
$$

so that

$$
\phi = (\phi_\perp, v + f) \,, \quad \phi_\perp = (\phi_1, \dots, \phi_{n-1}) \,. \tag{2.28}
$$

The potential now becomes

$$
V(\phi) = \tfrac{1}{2} g v^2 f^2 + \tfrac{1}{2} g v \, ({\phi_\perp}^2 + f^2) f + \tfrac{1}{8} g \, ({\phi_\perp}^2 + f^2)^2 \,. \tag{2.29}
$$

There is only a quadratic piece for the field f, the $n-1$ fields $\phi_\perp$ have no quadratic contribution so that the frequencies of these modes for small fluctuations around ϕ_0 are zero. The quadratic terms in Lagrangian, after any linear terms have been removed by shifting the fields, determine the particle masses in the associated quantum field theory so that in this example there are $n-1$ massless fields after spontaneous symmetry breakdown and one massive field f. The fields which are massless are called Goldstone modes. The symmetry group $O(n)$ for this situation is then reduced to $O(n-1)$ which may be defined by those elements of $O(n)$ which leave ϕ_0 invariant, for ϕ_0 as in (2.27) the $O(n-1)$ group acts only on the first $n-1$ components $\phi_\perp$ of ϕ.

A more general discussion of spontaneous symmetry breakdown can be developed which is applicable to any field theory in which the Lagrangian is invariant under a continuous symmetry group G but the ground state is invariant under a subgroup H. We assume a Lagrangian density with a multi-component scalar field ϕ, belonging to a vector space V_ϕ, on which a representation of the group G (which for ϕ having n components must be a subgroup of $O(n)$) is defined, for $g \in G$ then $\phi \to g\phi$. The potential V is assumed to be invariant so that,

$$
V(g\phi) = V(\phi) \quad \text{for all} \quad g \in G \,. \tag{2.30}
$$

Classically spontaneous symmetry breakdown arises when the ground state is not a single point invariant under G but is a non trivial manifold,

$$
\Phi_0 = \{ \phi_0 \, : \, V(\phi_0) = V_{\rm min} \} \,. \tag{2.31}
$$

For any point $\phi_0 \in \Phi_0$ we may define its stability group $H \subset G$ by

$$h\phi_0 = \phi_0 \quad \text{for all} \quad h \in H\,. \tag{2.32}$$

It is convenient to assume that G acts transitively on Φ_0 (this need not be true, it is possible to tune the parameters in the potential V so that two or more local minima, on each of which G acts transitively, both have the same value which is the global minimum of V, but this is an unstable situation) which means

$$\phi_0{}', \phi_0 \in \Phi_0 \;\Rightarrow \phi_0{}' = g\phi_0 \quad \text{for some} \quad g \in G\,. \tag{2.33}$$

In this case the stability groups at each point are isomorphic, if H' is the stability group for ϕ_0' then $H' \simeq gHg^{-1}$, and we can identify Φ_0 with the coset G/H (for any subgroup $H \subset G$, G/H is defined as the set of equivalence classes under the equivalence relation $g_1 \sim g_2$ if $g_1 = g_2 h$ for some $h \in H$) since if $\phi_0{}' = g_1\phi_0 = g_2\phi_0$ then ${g_2}^{-1}g_1\phi_0 = \phi_0$ so that ${g_2}^{-1}g_1 = h$ for some $h \in H$. Thus,

$$\Phi_0 \simeq G/H\,. \tag{2.34}$$

In this context there is a crucial theorem, the Goldstone theorem, which states that in a quantum field theory when spontaneous symmetry breakdown of a continuous symmetry occurs there are zero mass particles, Goldstone bosons, whose numbers are determined by the dimensions of G and H. At the classical level this amounts to counting the number of zero frequency modes for small oscillations around the classical ground state. To demonstrate this result we first recast (2.30) in infinitesimal form,

$$V(\phi + \delta\phi) = V(\phi) \quad \text{for} \quad \delta\phi = \delta\lambda_a\theta_a\phi\,, \quad a = 1, \ldots \dim G\,, \tag{2.35}$$

where θ_a are the $\dim G$ generators of the Lie algebra of G in the representation defined by ϕ. (2.35) can obviously be rewritten as

$$\frac{\partial}{\partial\phi_r}V(\phi)\,(\theta_a\phi)_r = 0\,. \tag{2.36}$$

Since the kinetic term of the Lagrangian is also required to be invariant the generators should be antisymmetric or

$$\phi'{\cdot}(\theta_a\phi) = -(\theta_a\phi'){\cdot}\phi\,. \tag{2.37}$$

The frequencies of the oscillations of the fluctuations around the ground state are determined by the eigenvalues of the matrix formed by the second derivatives of V evaluated at the minimum. Choosing an arbitrary point $\phi_0 \in \Phi_0$ this matrix, which acts linearly on V_ϕ, is then defined by

$$\mathcal{M}_{sr} = \frac{\partial^2}{\partial\phi_s\partial\phi_r}V(\phi)\Big|_{\phi=\phi_0}\,. \tag{2.38}$$

Now from (2.36) we have

$$\frac{\partial^2}{\partial\phi_s\partial\phi_r}V(\phi)\,(\theta_a\phi)_r + \frac{\partial}{\partial\phi_r}V(\phi)\,(\theta_a)_{rs} = 0\,, \tag{2.39}$$

and since at a minimum the first derivatives of V must be zero we have

$$\mathcal{M}_{sr}(\theta_a\phi_0)_r = 0\,. \tag{2.40}$$

Thus $\theta_a\phi_0$ is a zero frequency eigenvector for the matrix $\mathcal{M}$.

To count the number of such zero eigenvectors we first note that from (2.32) if t_i is a basis in the appropriate representation for the generators of the Lie algebra of H which is the stability group at $\phi_0 \in \Phi_0$ then

$$t_i\phi_0 = 0\,, \quad i = 1, \ldots \dim H\,. \tag{2.41}$$

If G is compact and semi-simple (as is the case for most symmetry groups) we can define a positive definite group invariant scalar product on the Lie algebra of G. In this case we may then choose a basis for the Lie algebra such that

$$\theta_a = (t_i, \theta_{\hat{a}})\,, \tag{2.42}$$

with $\theta_{\hat{a}}$ orthogonal to t_i, which corresponds to $\mathrm{tr}(t_i\theta_{\hat{a}}) = 0$. With this result it is clear from (2.40) and (2.41) that there are $\dim G - \dim H$ linearly independent eigenvectors $\theta_{\hat{a}}\phi_0$ with zero eigenvalues for the matrix $\mathcal{M}$ (if $f_{\hat{a}}\theta_{\hat{a}}\phi_0 = 0$ for some linear combination $f_{\hat{a}}\theta_{\hat{a}}$ then this satisfies (2.41) and so should belong to the Lie algebra of H which is clearly impossible with the unique decomposition defined by (2.42)). If we apply this counting to the example given with $G = O(n)$, $H = O(n-1)$ then

$$\dim O(n) - \dim O(n-1) = \tfrac{1}{2}n(n-1) - \tfrac{1}{2}(n-1)(n-2) = n-1\,, \tag{2.43}$$

which is the correct number of Goldstone modes in this case. The group H is the manifest unbroken symmetry group of the theory after spontaneous symmetry breakdown. As an illustration we demonstrate that the eigenvectors of $\mathcal{M}$ with non zero eigenvalues may be classified in terms of the representations of H. To show this we apply a further derivative to (2.39) and then set $\phi = \phi_0$ to give,

$$\mathcal{M}_{sr}(\theta_a)_{rt} + \mathcal{M}_{tr}(\theta_a)_{rs} + \frac{\partial^3}{\partial\phi_t\partial\phi_s\partial\phi_r}V(\phi)\bigg|_{\phi=\phi_0}(\theta_a\phi_0)_r = 0\,, \tag{2.44}$$

and then take $\theta_a \to t_i$ using (2.41) and the antisymmetry of t_i to give

$$[t_i, \mathcal{M}] = 0\,. \tag{2.45}$$

The vector space V_ϕ may be decomposed into orthogonal irreducible subspaces under the action of the group H and in such a basis the matrix $\mathcal{M}$ takes a block diagonal form, each irreducible subspace giving eigenvectors of $\mathcal{M}$ with the same eigenvalue.

In quantum field theory there are similar features with the spontaneous symmetry breakdown of a continuous symmetry as for a discrete symmetry. If we introduce coordinates ξ for G/H (any group element $g \in G$ is specified by $g(u,\xi)$ where $g(u,\xi)h = g(u',\xi)$ for any $h \in H$) then the vacuum state can be labelled $|0_\xi\rangle$ and there is a explicit symmetry under which the vacuum is invariant $U(h)|0_\xi\rangle = |0_\xi\rangle$ for $h \in H_\xi$, the stability group at the point ξ. Application of field operators to the vacuum $|0_\xi\rangle$ defines a Hilbert space $\mathcal{H}_\xi$. All $\mathcal{H}_\xi$ give equivalent quantum field theories. For a continuous symmetry group the essential new feature when it undergoes spontaneous symmetry breakdown is the appearance of $\dim G - \dim H$ Goldstone bosons, exactly massless spinless particles if G was originally an exact symmetry.

1.3 Proof of the Goldstone Theorem in Quantum Field Theory

To demonstrate the existence of Goldstone bosons for spontaneous symmetry breakdown of a continuous symmetry it is possible to extend the previous discussion directly to the effective potential $V_{\text{eff}}(\phi)$ which is a quantum generalisation of the classical potential and whose minima determine the vacuum state. However an alternative proof makes the required assumptions more manifest. We assume the existence of conserved currents $j^\mu{}_a$, $a = 1, \ldots \dim G$, whose charges induce a representation of the Lie algebra of G on a set of scalar fields ϕ

$$\int \mathrm{d}^3x \, [j^0{}_a(x), \phi(0)] = -i\,\theta_a \phi(0) \,. \tag{2.46}$$

The requirement of spontaneous symmetry breakdown is made by assuming

$$\langle 0|\phi(0)|0\rangle = \phi_0 \,, \tag{2.47}$$

is non zero. To prove the theorem it is necessary to obtain a general expression for $\langle 0|[j^\mu{}_a(x), \phi(0)]|0\rangle$ so we first define

$$\begin{aligned} (2\pi)^3 \sum_n \delta^4(k-p_n)\langle 0|j^\mu{}_a(0)|n\rangle\langle n|\phi(0)|0\rangle &= ik^\mu \theta(k^0)\rho_a(k^2) \,, \\ (2\pi)^3 \sum_n \delta^4(k-p_n)\langle 0|\phi(0)|n\rangle\langle n|j^\mu{}_a(0)|0\rangle &= -ik^\mu \theta(k^0)\tilde{\rho}_a(k^2) \,, \end{aligned} \tag{2.48}$$

where the form of the right hand side is dictated by Lorentz invariance, $\theta(x)$ is the step function, $\theta(x) = 1, 0$ for $x \gtrless 0$, and $\rho_a(\sigma), \tilde{\rho}_a(\sigma)$ are non zero only for $\sigma \geq 0$. From these definitions we may obtain, using $j^\mu{}_a(x) = e^{iP\cdot x} j^\mu{}_a(0) e^{-iP\cdot x}$ with P^μ the 4-momentum operator,

$$\langle 0|[j^\mu{}_a(x), \phi(0)]|0\rangle = -\partial^\mu \frac{1}{(2\pi)^3} \int \mathrm{d}^4k \, e^{-ik\cdot x} \Big(\theta(k^0)\rho_a(k^2) - \theta(-k^0)\tilde{\rho}_a(k^2)\Big) \,. \tag{2.49}$$

The left hand side must vanish if $x^2 < 0$ and from the result

$$\Delta(x;\sigma) \equiv -\frac{i}{(2\pi)^3}\int \mathrm{d}^4k\, e^{-ik\cdot x}\epsilon(k^0)\delta(k^2-\sigma) = 0 \quad \text{if} \quad x^2 < 0\,, \tag{2.50}$$

for $\epsilon(x) = \pm 1$ if $x \gtrless 0$, we must require $\rho_a = \tilde{\rho}_a$ which then gives

$$\langle 0|[j^\mu{}_a(x), \phi(0)]|0\rangle = -i\partial^\mu \int \mathrm{d}\sigma\, \Delta(x;\sigma)\, \rho_a(\sigma)\,. \tag{2.51}$$

Using now

$$(\partial^2 + \sigma)\Delta(x;\sigma) = 0\,, \tag{2.52}$$

it is easy to see that

$$\partial_\mu j^\mu{}_a(x) = 0 \quad \Rightarrow \quad \sigma\rho_a(\sigma) = 0\,, \tag{2.53}$$

or

$$\rho_a(\sigma) = N_a\delta(\sigma)\,. \tag{2.54}$$

For a non zero vacuum expectation value in (2.47) N_a must be non zero since given the definition of $\Delta(x;\sigma)$ in (2.50),

$$\int \mathrm{d}^3x\, \Delta(x;0) = -x^0\,, \tag{2.55}$$

and therefore (2.46) implies

$$N_a = -\theta_a\phi_0\,. \tag{2.56}$$

For N_a non zero then (2.54) implies that there must be a contribution from zero mass states in the sum over intermediate states in (2.48). These are identified with the Goldstone bosons. With spin 0 massless particle states $|B,p\rangle$ then we define

$$\langle 0|j^\mu{}_a(0)|B,p\rangle = iF_a{}^B p^\mu\,, \qquad \langle B,p|\phi(0)|0\rangle = Z^B\,, \tag{2.57}$$

with Z^B a vector belonging to the space defined by the fields ϕ. Since

$$\int \frac{\mathrm{d}^3p}{2|\mathbf{p}|}\,\delta^4(k-p)\Big|_{p^0=|\mathbf{p}|} = \theta(k^0)\delta(k^2)\,, \tag{2.58}$$

we therefore find from the summation over these massless particle states in (2.48) and (2.54)

$$N_a = \sum_B F_a{}^B Z^B\,. \tag{2.59}$$

If the group G is spontaneously broken to a group H, defined by generators such that $t_i\phi_0 = 0$, then there must be $\dim G - \dim H$ linearly independent N_a from (2.56) which implies in turn that in (2.59) the matrix $F_a{}^B$ must have rank at least $\dim G - \dim H$ and hence that there must be this number of massless Goldstone bosons. From (2.56) clearly $N_i = 0$ which is in accord with the result that the charges formed from the unbroken symmetry currents annihilate the vacuum, $\int d^3x \, j^0{}_i(x)|0\rangle = 0$.

We should note that this theorem depends crucially on assuming manifest Lorentz invariance and also that the space of states has positive definite norm and for this reason fails in gauge theories when quantisation in general violates one or other of these assumptions. The theorem about the necessary existence of massless Goldstone bosons also breaks down for more technical reasons in two space-time dimensions.

2 Higgs Mechanism

In a gauge field theory spontaneous symmetry breakdown can lead to very different effects, essentially it provides a method of giving gauge fields a mass while maintaining gauge invariance. Although this effect was essentially first discovered in the theory of superconductivity and then generalised to relativistic theories by several different authors it is usually called the Higgs effect or Higgs mechanism. The essential features can be understood at the classical level although the justification for the physical relevance of the Higgs mechanism is in the context of quantum field theory. The exact gauge invariance is crucial in obtaining quantum field theories describing massive vector particles that can be renormaliseable and also have a positive norm Hilbert space of physical states. We first consider a simple example based on a $U(1)$ gauge theory and then analyse the general case for a relativistic non abelian gauge theory.

2.1 Higgs Mechanism in an Abelian Gauge Theory

The most elementary relativistic gauge field theory is the Maxwell theory of electromagnetism expressed in terms of the 4-vector gauge field A_μ. Here we consider its coupling to a complex scalar field ϕ with the Lagrangian density,

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D^\mu\phi)^* D_\mu\phi - V(\phi^*\phi) \,, \tag{2.60}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi \,. \tag{2.61}$$

This theory is invariant under local $U(1)$ gauge transformations where

$$A_\mu \to A_\mu + \frac{1}{e}\,\partial_\mu\lambda \,, \qquad \phi \to e^{i\lambda}\phi \,, \tag{2.62}$$

for arbitrary $\lambda(x)$. The covariant derivative is so constructed so that under (2.62) $D_\mu\phi \to e^{i\lambda} D_\mu\phi$ so that gauge invariance of (2.60) is trivially evident. In any gauge theory the invariance under gauge transformations implies that there is a redundancy in the initial Lagrangian, the physical dynamical variables must be defined modulo gauge transformations whose time evolution is unconstrained. In some cases it is possible to reformulate the theory explicitly in terms of gauge invariant variables but more generally it is necessary to impose some additional gauge fixing conditions which remove the gauge freedom but it is then necessary to carefully identify the physical results in a fashion which is independent of the gauge fixing procedure.

For the theory described by the Lagrangian (2.60) there are two phases with very different physics for which the natural physical variables are completely different.

1. The minimum of $V(\phi^*\phi)$ occurs at $\phi^*\phi = 0$, for instance

$$V(\phi^*\phi) = m^2\phi^*\phi + \tfrac{1}{2}g(\phi^*\phi)^2\,, \quad m^2, g > 0\,. \tag{2.63}$$

In this case in the classical theory in the ground state $\phi = 0$ and in the quantum theory we expect a unique vacuum state $|0\rangle$. The gauge field couples to a conserved current j^μ whose corresponding charge $Q = \int \mathrm{d}^3x\, j^0$ is conserved and generates a $U(1)$ symmetry with $Q|0\rangle = 0$. In a perturbative expansion the theory describes spinless charged particles, with to lowest order a mass $m\hbar$ and charges $\pm e\hbar$, interacting with massless photons. The physical degrees of freedom are then 2 for the field A_μ, corresponding to the two photon polarisation states after removal of gauge degrees of freedom, and 2 for the field ϕ, corresponding to the two charge states.

2. The minimum of $V(\phi^*\phi)$ occurs away from the origin at $\phi^*\phi = \frac{1}{2}v^2$, for instance we might take

$$V(\phi^*\phi) = \tfrac{1}{2}g(\phi^*\phi - \tfrac{1}{2}v^2)^2\,. \tag{2.64}$$

In this case the $U(1)$ gauge symmetry is broken by the ground state. To derive the physical consequences in this situation it is convenient to rewrite the fields if $\phi \neq 0$ in the form

$$A_\mu = A'_\mu + \frac{1}{e}\,\partial_\mu\theta\,, \qquad \phi = \frac{1}{\sqrt{2}}\,(v+f)e^{i\theta}\,, \tag{2.65}$$

with f, θ real. Under the action of gauge transformations in (2.62) it is easy to see that

$$\theta \to \theta + \lambda\,, \tag{2.66}$$

while A'_μ, f are gauge invariant. Using

$$D_\mu\phi = \frac{1}{\sqrt{2}}\,e^{i\theta}\Big(\partial_\mu f - ie\,A'_\mu(v+f)\Big)\,, \tag{2.67}$$

we may rewrite the Lagrangian in (2.60) in the form

$$\mathcal{L} = -\tfrac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}e^2(v+f)^2A'^\mu A'_\mu + \tfrac{1}{2}\partial^\mu f\,\partial_\mu f - \tfrac{1}{8}\,g(2vf+f^2)^2\,. \tag{2.68}$$

For small fluctuations around the ground state given by $f, A'_\mu = 0$ we may restrict this to just the quadratic terms giving

$$\mathcal{L}_{\text{quadratic}} = -\tfrac{1}{4}\, F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}e^2v^2\, A'^\mu A'_\mu + \tfrac{1}{2}\partial^\mu f\, \partial_\mu f - \tfrac{1}{2}gv^2 f^2 \,, \tag{2.69}$$

which results in the linearised equations of motion,

$$\partial^\mu F_{\mu\nu} + e^2v^2 A'_\nu = 0 \;\Rightarrow\; \begin{cases} \partial^\nu A'_\nu = 0 \\ (\partial^2 + e^2v^2)A'_\nu = 0 \end{cases}, \qquad (\partial^2 + gv^2)f = 0\,. \tag{2.70}$$

Thus A'_μ represents a massive vector field describing spin 1 particles with mass m_v, ${m_v}^2 = e^2v^2$ at lowest order, which has therefore 3 degrees of freedom, while the field f describes spinless particles of mass m_f, ${m_f}^2 = gv^2$. Unlike the case of spontaneous symmetry breakdown of a non gauged continuous symmetry there are no massless modes, in a sense the photon absorbs the Goldstone boson so as to ensure it has the right degrees of freedom to give a massive spin 1 particle. The Lagrangian in (2.68) has two basic couplings e, g which determine its interaction terms, their particular form plays a crucial role in ensuring renormaliseability of the perturbative expansion starting from the free field theory described by (2.69). There is of course no longer any conserved charge whose eigenvalues label the states.

In the example described above it was possible to rewrite the theory just in terms of gauge invariant variables, so that θ disappeared from the Lagrangian in (2.68). By a suitable gauge transformation as in (2.66) we could transform θ to zero. Alternatively we could impose a gauge condition on the fields. Equivalent results to the above may be obtained by applying the gauge condition,

$$\phi = \phi^* \,, \tag{2.71}$$

which makes ϕ real and hence $\theta = 0$.

2.2 Higgs Mechanism in Non Abelian Gauge Theories

A gauge theory Lagrangian may be defined for any Lie group G. The Lie algebra L_G for G forms a vector space on which can be defined a Lie bracket $[\,,\,]$ which maps $L_G \times L_G \to L_G$. Assuming a basis T_a, $a = 1, \dots \dim G$, this has the properties

$$[T_a, T_b] = -[T_b, T_a] = c_{abc}\, T_c \,, \tag{2.72}$$

and also we require the Jacobi identity

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] = 0\,. \tag{2.73}$$

This can also be written in terms of the structure constants c_{abc} as

$$c_{abd}\, c_{dce} + c_{bcd}\, c_{dae} + c_{cad}\, c_{dbe} = 0\,, \tag{2.74}$$

or equivalently in terms of the matrices,

$$(T^{\rm ad}{}_a)_{bc} = c_{acb}\,, \tag{2.75}$$

then (2.74) becomes

$$[T^{\rm ad}{}_a, T^{\rm ad}{}_b] = c_{abc}\, T^{\rm ad}{}_c\,, \tag{2.76}$$

where $[\,,\,]$ is here the standard matrix commutator so that the identity (2.73) is automatic. The matrices $T^{\rm ad}$ define the adjoint representation of the Lie algebra for which the representation space is the Lie algebra itself and so has dimension $\dim G$ (for a general element of the Lie algebra $X = X_a T_a$ then $[T_a, X] = (T^{\rm ad}{}_a X)_b T_b$).

The gauge field $A_{\mu a}$ belongs to the adjoint representation space since the corresponding infinitesimal gauge transformations are given by

$$\delta A_{\mu a} = \frac{1}{g}\,(D_\mu\lambda)_a\,, \qquad (D_\mu\lambda)_a = \partial_\mu\lambda_a + g\, c_{bca} A_{\mu b}\,\lambda_c\,, \tag{2.77}$$

for arbitrary $\lambda_a(x)$ and for a coupling g. In (2.77) $D_\mu = \partial_\mu + g A_{\mu a} T^{\rm ad}{}_a$ is the covariant derivative for the adjoint representation. The associated field strength is defined by

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g\, c_{bca} A_{\mu b} A_{\nu c}\,. \tag{2.78}$$

Under a gauge transformation as in (2.77) we have

$$\delta F_{\mu\nu a} = c_{bca}\, F_{\mu\nu b}\,\lambda_c = -\lambda_b (T^{\rm ad}{}_b F_{\mu\nu})_a\,, \tag{2.79}$$

which depends crucially on the Jacobi identity (2.74). Using (2.76) we may easily verify that acting on the gauge fields

$$[\delta_1, \delta_2] = \delta_3\,, \qquad \lambda_{3a} = c_{abc}\lambda_{1b}\lambda_{2c}\,. \tag{2.80}$$

If we now assume that the structure constants c_{abc} are completely antisymmetric, which can always be achieved if the group G is compact and semi-simple, then we can simply define a group invariant scalar product by $X.Y \equiv X_a Y_a$ since $X.(T^{\rm ad}{}_a Y) = -(T^{\rm ad}{}_a X).Y$. Then the usual gauge field Lagrangian extends to the non abelian case by taking,

$$\mathcal{L}_{\rm gauge} = -\tfrac{1}{4}\, F^{\mu\nu}.F_{\mu\nu}\,. \tag{2.81}$$

It is also straightforward to extend this gauge invariant Lagrangian to include scalar fields belonging to a vector space V_ϕ on which a representation of G is defined, with generators θ_a, $[\theta_a, \theta_b] = c_{abc}\theta_c$. Assuming a group invariant scalar product on V_ϕ, satisfying (2.37), the Lagrangian for the scalar field in (2.25) may be extended to be invariant under local gauge transformations if

$$\mathcal{L}_\phi = \tfrac{1}{2}\,(D^\mu\phi){\cdot}D_\mu\phi - V(\phi)\,, \tag{2.82}$$

since the covariant derivative,

$$D_\mu \phi = \partial_\mu \phi + g\, A_{\mu a} \theta_a \phi \,, \tag{2.83}$$

satisfies

$$\delta D_\mu \phi = -\lambda_a \theta_a D_\mu \phi \quad \text{if} \quad \delta\phi = -\lambda_a \theta_a \phi \,, \tag{2.84}$$

assuming the gauge field transforms as in (2.77). For gauge invariance of course the potential V must also satisfy (2.30) or infinitesimally (2.36) which can be equivalently be written as

$$V'(\phi) {\cdot} (\theta_a \phi) = 0 \,. \tag{2.85}$$

In order to discuss the Higgs effect in this theory we now assume that the potential determines a ground state corresponding to spontaneous symmetry breakdown of the group G, i.e. the minimum of the potential occurs for non zero $\phi_0 \in \Phi_0$ as in the discussion of spontaneous symmetry breakdown for continuous symmetries. Unlike the previous treatment however local gauge transformations do not represent physical degrees of freedom and to obtain a well defined dynamics we impose gauge conditions. With the gauge group G reduced at any point on Φ_0 to invariance under a subgroup H it is often convenient, although not necessary, first to consider $\dim G - \dim H$ gauge conditions which maintain local gauge invariance for H and then treat the resulting gauge field theory in the same fashion as any other conventional gauge theory. There is no unique necessary gauge fixing condition but a convenient choice which makes the physical degrees of freedom explicit is the so called unitarity gauge which just restricts the scalar fields ϕ,

$$(\theta_a \phi_0) {\cdot} \phi = 0 \,. \tag{2.86}$$

Since, for t_i the generators of H,

$$t_i \phi_0 = 0 \,, \tag{2.87}$$

we may restrict (2.86) to just $(\theta_{\hat a} \phi_0) {\cdot} \phi = 0$. Any ϕ can be arranged to satisfy (2.86) by applying a suitable gauge transformation $\phi \to g\phi$. This does not impose any restriction on H gauge transformations, infinitesimally when $\delta\phi = -\lambda_i t_i \phi$ this is evident from

$$(\theta_a \phi_0) {\cdot} (t_i \phi) = -(t_i \theta_a \phi_0) {\cdot} \phi = -([t_i, \theta_a] \phi_0) {\cdot} \phi = -c_{iab} (\theta_b \phi_0) {\cdot} \phi = 0 \,, \tag{2.88}$$

using (2.87).

With this gauge condition it is convenient to expand

$$\phi = \phi_0 + f \,, \qquad (\theta_{\hat a} \phi_0) {\cdot} f = 0 \,, \tag{2.89}$$

and to use as before in (2.42) the natural decomposition of the generators, and hence also for the gauge fields, into those belonging to the Lie algebra of H and those which are orthogonal,

$$\theta_a = (t_i, \theta_{\hat a}) \,, \qquad A_{\mu a} = (\mathcal{A}_{\mu i}, \hat A_{\mu \hat a}) \,. \tag{2.90}$$

With this decomposition and (2.89) the covariant derivative defined in (2.83) reduces to

$$D_\mu \phi = \mathcal{D}_\mu f + g\, \hat{A}_{\mu\hat{a}} \theta_{\hat{a}}(\phi_0 + f)\,, \quad \mathcal{D}_\mu f = \partial_\mu f + g\, \mathcal{A}_{\mu i} t_i f\,, \tag{2.91}$$

where $\mathcal{D}_\mu f$ is the H covariant derivative and $\mathcal{A}_\mu$ the corresponding gauge field. From (2.86) the gauge condition becomes

$$(\theta_{\hat{a}}\phi_0)\cdot f = 0\,, \tag{2.92}$$

which implies by virtue of (2.88)

$$(\theta_{\hat{a}}\phi_0)\cdot D^h{}_\mu f = 0\,, \tag{2.93}$$

and so the Lagrangian in (2.82) is now

$$\begin{aligned} \mathcal{L}_\phi &= \tfrac{1}{2}\,\mathcal{D}^\mu f\cdot\mathcal{D}_\mu f + \tfrac{1}{2}g^2\, \hat{A}^\mu{}_{\hat{a}} \hat{A}_{\mu\hat{b}}\, (\theta_{\hat{a}}(\phi_0+f))\cdot(\theta_{\hat{b}}(\phi_0+f)) \\ &\quad + g\, \hat{A}^\mu{}_{\hat{a}}\, (\theta_{\hat{a}} f)\cdot\mathcal{D}_\mu f - V(\phi_0 + f)\,. \end{aligned} \tag{2.94}$$

The structure constants for the Lie algebra of G defined by (2.72) can also be decomposed in this basis as

$$c_{abc} = \begin{cases} c_{ijk} \\ c_{i\hat{b}\hat{c}} \\ c_{\hat{a}\hat{b}\hat{c}} \end{cases}, \tag{2.95}$$

with c_{ijk} the structure constants for the Lie algebra of the subgroup H, so that $[t_i, t_j] = c_{ijk} t_k$, and they satisfy the appropriate version of the Jacobi identity in (2.74). We may now also correspondingly decompose the field strength given by (2.78) as

$$F_{\mu\nu a} = (\mathcal{F}_{\mu\nu i} + g\, c_{i\hat{b}\hat{c}}\, \hat{A}_{\mu\hat{b}} \hat{A}_{\nu\hat{c}}, \hat{F}_{\mu\nu\hat{a}})\,, \tag{2.96}$$

where

$$\begin{aligned} \mathcal{F}_{\mu\nu i} &= \partial_\mu \mathcal{A}_{\nu i} - \partial_\nu \mathcal{A}_{\mu i} + g\, c_{ijk} \mathcal{A}_{\mu j} \mathcal{A}_{\nu k}\,, \\ \hat{F}_{\mu\nu\hat{a}} &= \mathcal{D}_\mu \hat{A}_{\nu\hat{a}} - \mathcal{D}_\nu \hat{A}_{\mu\hat{a}} + g\, c_{\hat{a}\hat{b}\hat{c}} \hat{A}_{\mu\hat{b}} \hat{A}_{\nu\hat{c}}\,, \end{aligned} \tag{2.97}$$

with

$$\mathcal{D}_\mu \hat{A}_{\nu\hat{a}} = \partial_\mu \hat{A}_{\nu\hat{a}} + g\, \mathcal{A}_{\mu i} (\hat{T}_i)_{\hat{a}\hat{b}} \hat{A}_{\nu\hat{b}}\,, \quad (\hat{T}_i)_{\hat{a}\hat{b}} = -c_{i\hat{a}\hat{b}}\,, \tag{2.98}$$

the covariant derivative acting on $\hat{A}_\nu$ since $\hat{T}_i$ are the appropriate generators of the Lie algebra of H, $[\hat{T}_i, \hat{T}_j] = c_{ijk}\hat{T}_k$ by virtue of (2.74) and (2.95). The gauge field Lagrangian in (2.81) then becomes

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= -\tfrac{1}{4}\, \mathcal{F}^{\mu\nu}.\mathcal{F}_{\mu\nu} - \tfrac{1}{4}\, \hat{F}^{\mu\nu}.\hat{F}_{\mu\nu} \\ &\quad - \tfrac{1}{2} g\, c_{i\hat{a}\hat{b}}\, \hat{A}^\mu{}_{\hat{a}} \hat{A}^\nu{}_{\hat{b}} \mathcal{F}_{\mu\nu i} - \tfrac{1}{4} g^2\, c_{i\hat{a}\hat{b}} c_{i\hat{c}\hat{d}}\, \hat{A}^\mu{}_{\hat{a}} \hat{A}^\nu{}_{\hat{b}} \hat{A}_{\mu\hat{c}} \hat{A}_{\nu\hat{d}}\,. \end{aligned} \tag{2.99}$$

Although the complete theory is described by $\mathcal{L} = \mathcal{L}_{\rm gauge} + \mathcal{L}_\phi$ the physical particle states are given just by the quadratic terms in the expansion,

$$\begin{aligned}\mathcal{L}_{\rm quadratic} &= -\tfrac{1}{4}\,\mathcal{F}^{\mu\nu}.\mathcal{F}_{\mu\nu} - \tfrac{1}{4}\,(\mathcal{D}^\mu \hat{A}^\nu - \mathcal{D}^\nu \hat{A}^\mu).(\mathcal{D}_\mu \hat{A}_\nu - \mathcal{D}_\nu \hat{A}_\mu) \\ &\quad + \tfrac{1}{2}\,\mathcal{D}^\mu f{\cdot}\mathcal{D}_\mu f - \tfrac{1}{2} f{\cdot}(\mathcal{M} f) + \tfrac{1}{2}\hat{\mathcal{M}}_{\hat{a}\hat{b}} \hat{A}^\mu{}_{\hat{a}} \hat{A}_{\mu\hat{b}}\,, \end{aligned} \tag{2.100}$$

with the matrix $\mathcal{M}$ defined as in (2.38) and

$$\hat{\mathcal{M}}_{\hat{a}\hat{b}} = g^2(\theta_{\hat{a}}\phi_0){\cdot}(\theta_{\hat{b}}\phi_0)\,. \tag{2.101}$$

The matrix $\hat{\mathcal{M}}$ is positive definite since, by (2.101), it is clearly positive and any eigenvector with zero eigenvalue would have to satisfy $f_{\hat{a}}\theta_{\hat{a}}\phi_0 = 0$ which is impossible since it would then imply $f_{\hat{a}}\theta_{\hat{a}}$ was a generator of H. The result (2.99) or (2.100) is expressed in a manifestly gauge invariant form for the gauge group H, with infinitesimal gauge transformations given by

$$\delta\mathcal{A}_{\mu i} = \frac{1}{g}\,(\mathcal{D}_\mu \lambda)_i\,, \qquad \delta\hat{A}_{\mu\hat{a}} = -\lambda_i(\hat{T}_i \hat{A}_\mu)_{\hat{a}}\,, \qquad \delta f = -\lambda_i t_i f\,. \tag{2.102}$$

The matrices $\mathcal{M}$ and $\hat{\mathcal{M}}$ in (2.100) determine the masses of the physical scalar fields f and vector fields $\hat{A}_\mu$, apart from the H gauge fields A^h_μ which are massless. By virtue of (2.92) f is orthogonal to $\theta_{\hat{a}}\phi_0$ which are the eigenvectors of $\mathcal{M}$ with zero eigenvalue so there are no necessary massless Goldstone bosons in this case. For each eigenvalue the eigenvectors of $\mathcal{M}$ and $\hat{\mathcal{M}}$ form representation spaces for H. For $\mathcal{M}$ this follows by virtue of (2.45) and for $\hat{\mathcal{M}}$

$$\begin{aligned}[\hat{T}_i, \hat{\mathcal{M}}]_{\hat{a}\hat{b}} &= -g^2([t_i,\theta_{\hat{a}}]\phi_0){\cdot}(\theta_{\hat{b}}\phi_0) - g^2(\theta_{\hat{a}}\phi_0){\cdot}([t_i,\theta_{\hat{b}}]\phi_0) \\ &= -g^2(t_i\theta_{\hat{a}}\phi_0){\cdot}(\theta_{\hat{b}}\phi_0) - g^2(\theta_{\hat{a}}\phi_0){\cdot}(t_i\theta_{\hat{b}}\phi_0) = 0\,, \end{aligned} \tag{2.103}$$

using the definition of $\hat{T}_i$ in (2.98), (2.87) and the invariance of the scalar product as in (2.37). Since no linear combination of $\theta_{\hat{a}}$ annihilates ϕ_0 the matrix $\hat{\mathcal{M}}$ is positive definite. To obtain fields of definite mass we may introduce and orthogonal transformation $\hat{A}_{\mu\hat{a}} = \hat{\mathcal{R}}_{\hat{a}\hat{b}}\hat{A}'_{\mu\hat{b}}$ so that $\hat{\mathcal{R}}^T\hat{\mathcal{M}}\hat{\mathcal{R}}$ is diagonal, and similarly for the scalar fields f.

Part III

Weak Decay Processes

1 Weak Decays

The earliest manifestation of weak interactions to be identified was the β-decay of radioactive atoms which were shown to emit electrons with a continuous spectrum of energies. The basic process is the decay inside a nucleus of a neutron to a proton, electron and a neutrino (actually anti-neutrino)

$$n \to p + e^- + \overline{\nu}_e \ .$$

Since the neutron (939.565 MeV) by itself is heavier by 1.293 MeV than the proton (938.272 MeV) and the mass of the electron is only 0.511 MeV this decay can also occur for free neutrons. In the context of a nucleus the binding energies of the initial and final multi-particle systems can permit the reverse process

$$p \to n + e^+ + \nu_e$$

to occur. It was in order to ensure conservation of energy and angular momentum in β-decay that led Pauli to propose the effectively massless spin-$\frac{1}{2}$ neutrino as an essential part of the weak β-decay process.

In fact the neutrino that occurs in the above β-decay processes is specifically associated with the electron which is why we have given it the suffix e. Subsequently (in cosmic ray studies) the muon (105.66 MeV) was identified. This is a particle like the electron that participates only in weak and electromagnetic interactions. It is entirely similar to the electron except for its much greater mass. It was later confirmed in neutrino scattering experiments that there is a distinct neutrino ν_μ associated with the muon. The muon and the electron together with their associated neutrinos were the first leptons to be identified. They were joined by a much heavier third lepton, the τ (1777 MeV) and its associated neutrino ν_τ. At the present moment the three neutrinos are thought either to be massless or have very low masses. The upper bound on the mass of the ν_e is about 10 eV. A full discussion of neutrino masses is complicated by the possibility of neutrino mixing. It appears that there are only these three generations of leptons, at least associated with relatively light neutrinos.

1.1 Massless Dirac Field

The experimental existence of virtually massless spin-$\frac{1}{2}$ neutrinos, which interact solely through weak interactions, is a crucial fact in determining the detailed structure of the weak interaction. In the absence of mass neutrinos should be described by a massless Dirac field. Such a field satisfies the massless Dirac equation:

$$\gamma.\partial\psi(x) = 0 \ . \tag{3.1}$$

This has the important property that, because $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$, $\gamma_5\psi(x)$ also satisfies the same equation,

$$\gamma.\partial\gamma_5\psi(x) = 0 \ . \tag{3.2}$$

Since ${\gamma_5}^2 = 1$ it follows that the fields

$$\psi_L(x) = \tfrac{1}{2}(1-\gamma_5)\psi(x) \quad \text{and} \quad \psi_R(x) = \tfrac{1}{2}(1+\gamma_5)\psi(x) \tag{3.3}$$

do so as well. These fields are eigenvectors of γ_5 with eigenvalues ∓ 1 and describe particles of definite helicity. To show this we use the result that the angular momentum operator acting on Dirac wave functions $\psi(x)$ is $\mathbf{J} = -i\mathbf{x}\times\boldsymbol{\nabla} + \mathbf{S}$ where the spin $S_i = i\frac{1}{2}\epsilon_{ijk}\gamma^j\gamma^k$. Using $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ it is straightforward to see that $\gamma_5\mathbf{S} = \frac{1}{2}\gamma^0\boldsymbol{\gamma}$. We have therefore for a Dirac spinor of momentum p

$$\gamma.pu(p,\lambda) = 0 \quad \Rightarrow \quad (1 - 2\mathbf{S}.\hat{\mathbf{p}}\gamma_5)u(p,\lambda) = 0 \ , \tag{3.4}$$

where we have used $\hat{\mathbf{p}} = \mathbf{p}/E$ ($|\mathbf{p}| = E$ for massless particles). The component of the spin of a particle along the direction of motion, measured by $\mathbf{S}.\hat{\mathbf{p}}$, is referred to as the helicity. Hence for the projections in eq.(3.3)

$$(1 + 2\mathbf{S}.\hat{\mathbf{p}})u_L(p,\lambda) = 0 \ , \qquad (1 - 2\mathbf{S}.\hat{\mathbf{p}})u_R(p,\lambda) = 0 \ , \tag{3.5}$$

so that these describe left handed and right handed particles, with negative and positive helicity $\mp\frac{1}{2}$, respectively. The Dirac quantum field $\psi_L(x)$ annihilates massless particles of negative helicity.

Charge conjugation applied to the fields defined in eq.(3.3), which represent definite helicity, yields ${\psi_{L,R}}^C(x) = C\overline{\psi}_{L,R}(x)^t$. However since ${\gamma_5}^\dagger = \gamma_5$

$$\begin{aligned} {\psi_L}^C(x) &= C\tfrac{1}{2}(1+{\gamma_5}^t)\overline{\psi}(x)^t \\ &= \tfrac{1}{2}(1+\gamma_5)\psi^C(x) \ , \end{aligned} \tag{3.6}$$

using $C{\gamma_5}^tC^{-1} = \gamma_5$. It follows that the anti-neutrino corresponding to a left handed neutrino of negative helicity has positive helicity and *vice versa*.

1.2 Leptonic Processes

The most striking experimental discovery concerning weak interaction processes is that they are not invariant under parity. The sign that parity is not conserved is the appearance in experimental results of non-vanishing expectation values for *pseudo-scalar* quantities such as the projection of the electron momentum along the direction of the nuclear spin in β-decay. It turns out that the parity breakdown is in a certain sense "maximal". This can be demonstrated most directly by the fact that the neutrinos that participate in weak interactions have only negative helicity, that is they are purely *left-handed*. Correspondingly the anti-neutrinos are purely right-handed.

After much detailed analysis of experimental results it was concluded that the effective interaction Lagrangian controlling (low energy) leptonic weak interactions has the following form:

$$\mathcal{L}_W(x) = -\frac{G_F}{\sqrt{2}} J^\alpha(x)^\dagger J_\alpha(x) \ , \tag{3.7}$$

where G_F is the weak coupling constant and

$$J_\alpha(x) = \overline{\nu}_e(x)\gamma_\alpha(1-\gamma_5)e(x) + \overline{\nu}_\mu(x)\gamma_\alpha(1-\gamma_5)\mu(x) + \overline{\nu}_\tau(x)\gamma_\alpha(1-\gamma_5)\tau(x) \ . \tag{3.8}$$

The $\sqrt{2}$ in eq.(3.7) is conventional. The operator $J_\alpha(x)$ is referred to as the weak current and changes the electric charge by $\Delta Q = 1$ while $J^\alpha(x)^\dagger$ gives $\Delta Q = -1$. The Lagrangian $\mathcal{L}_W(x)$ in eq.(3.7) is described as a current-current interaction. The current $J_\alpha(x)$ can be decomposed into a vector part under parity transformations that is denoted by $V_\alpha(x)$

$$V_\alpha(x) = \overline{\nu}_e(x)\gamma_\alpha e(x) + \dots \ , \tag{3.9}$$

and an axial vector part $A_\alpha(x)$

$$A_\alpha(x) = \overline{\nu}_e(x)\gamma_\alpha\gamma_5 e(x) + \dots \ , \tag{3.10}$$

so that

$$J_\alpha(x) = V_\alpha(x) - A_\alpha(x) \ . \tag{3.11}$$

Since only the combination $V - A$ enters in $\mathcal{L}_W(x)$, which has the consequence that as indicated above the neutrino field enters only in its left-handed form, the theory is referred to as $V-A$ theory. Under parity transformation $V-A \to V+A$ which again is the justification for saying that weak interactions violate parity. The V, A currents however transform oppositely under charge conjugation so that $\mathcal{L}_W$ in (3.7) preserves CP invariance and hence also, by the PCT theorem, time reversal invariance.

Such an interaction with the leptonic current in eq.(3.8) gives rise to the decay processes

$$\mu^\pm \to e^\pm + \nu_e(\overline{\nu}_e) + \overline{\nu}_\mu(\nu_\mu) \tag{3.12}$$

and similar decays for the τ-lepton as well as neutrino electron scattering which has also been observed.

1.3 μ-Decay Rate and the Value of G_F

If $\mathcal{L}_I(x)$ is the Lagrangian density which gives rise to a coupling between a single particle state $|p\rangle$, of mass m, $p^2 = m^2$, which has the relativistically invariant normalisation $\langle p'|p\rangle = (2\pi)^3 2p^0 \delta^3(\mathbf{p}' - \mathbf{p})$, and states $|f\rangle$ which have a continuous mass spectrum then we may calculate the decay rate Γ to first order in $\mathcal{L}_I$ in terms of the differential decay rate given by

$$\mathrm{d}\Gamma = \frac{1}{2m}\, \mathrm{d}\rho_f\, |\mathcal{M}|^2 \,, \tag{3.13}$$

where

$$\mathcal{M} = \langle f|\mathcal{L}_I(0)|p\rangle \tag{3.14}$$

and $\mathrm{d}\rho_f$, called the phase space element, is defined, if $\sum_f |f\rangle\langle f| = 1$, by

$$\sum_f (2\pi)^4 \delta^4(P_f - p) \to \sum_{\text{spins}} \int_{\text{momenta}} \mathrm{d}\rho_f \,. \tag{3.15}$$

For $|f\rangle$ composed of particles with momenta p_r then, with standard normalisations,

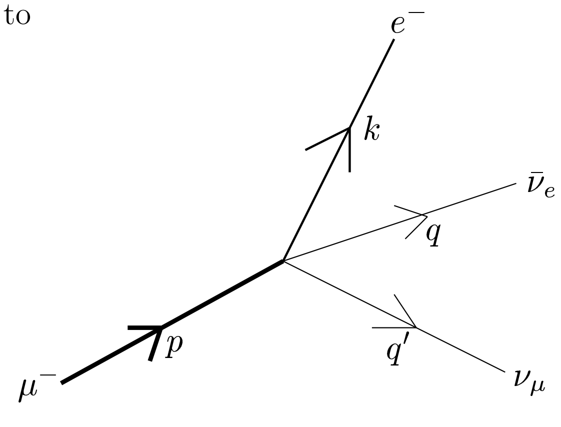
$$\mathrm{d}\rho_f = \prod_r \frac{\mathrm{d}^3 p_r}{(2\pi)^3 2p_r^0}\, (2\pi)^4 \delta^4(P_f - p) \,, \quad P_f = \sum_r p_r \,. \tag{3.16}$$

The differential decay rate for a particular decay process is then defined by summing or integrating eq.(3.13) over all unobserved final states, for the total decay rate Γ all states are summed over. If the decaying particle has spin but experimentally only decays of unpolarised particles are measured then the rate should be averaged over the initial spin.

In order to discuss the decay of the μ as a consequence of the weak interaction described by $\mathcal{L}_W$ in eq.(3.7) we choose the momenta of the particles so that

$$\mu^-(p) \to e^-(k) + \overline{\nu}_e(q) + \nu_\mu(q') \tag{3.17}$$

corresponding to



Mu decay to electron and neutrinos

The matrix element for the process, suppressing spin labels, is

$$\mathcal{M} = \langle e^-(k)\,\overline{\nu}_e(q)\,\nu_\mu(q')|\mathcal{L}_W(0)|\mu^-(p)\rangle \ . \tag{3.18}$$

In this process the Dirac fields can be regarded as free so that

$$\begin{aligned}\mathcal{M} &= -\frac{G_F}{\sqrt{2}}\langle e^-(k)\,\overline{\nu}_e(q)|\overline{e}\gamma^\alpha(1-\gamma_5)\nu_e|0\rangle\langle\nu_\mu(q')|\overline{\nu}_\mu\gamma_\alpha(1-\gamma_5)\mu|\mu^-(p)\rangle \\ &= -\frac{G_F}{\sqrt{2}}\,\overline{u}_e(k)\gamma^\alpha(1-\gamma_5)v_{\nu_e}(q)\,\overline{u}_{\nu_\mu}(q')\gamma_\alpha(1-\gamma_5)u_\mu(p) \ . \end{aligned}\tag{3.19}$$

To calculate the transition rate we need to compute the sum over spins of $|\mathcal{M}|^2$. These can be calculated using $\sum_\lambda u(p,\lambda)\overline{u}(p,\lambda) = \gamma.p + m$ if $p^2 = m^2$ and $\sum_\lambda \overline{u}(p,\lambda)Xu(p,\lambda) = \mathrm{tr}(X(\gamma.p+m))$ (for anti-particle spinors v similar formulae hold but with $m \to -m$) we find

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{G_F^{\,2}}{2} S_1^{\,\alpha\beta} S_{2\,\alpha\beta} \ , \tag{3.20}$$

where assuming the neutrinos have zero mass

$$\begin{aligned} S_1^{\,\alpha\beta} &= \mathrm{tr}\left\{(\gamma.k+m_e)\gamma^\alpha(1-\gamma_5)\gamma.q\gamma^\beta(1-\gamma_5)\right\} \ , \\ S_{2\,\alpha\beta} &= \mathrm{tr}\left\{(\gamma.p+m_\mu)\gamma_\alpha(1-\gamma_5)\gamma.q'\gamma_\beta(1-\gamma_5)\right\} \ . \end{aligned}\tag{3.21}$$

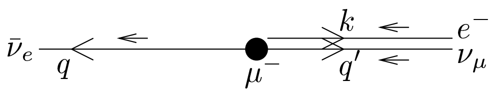
Using the standard rules for traces of γ-matrices ($\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) = 4i\epsilon^{\mu\nu\sigma\rho}$ with $\epsilon_{0123} = 1$) we find

$$\begin{aligned} S_1^{\,\alpha\beta} &= 8\left\{k^\alpha q^\beta + k^\beta q^\alpha - k.q\, g^{\alpha\beta} + i\epsilon^{\alpha\beta\sigma\rho}k_\sigma q_\rho\right\} \ , \\ S_{2\,\alpha\beta} &= 8\left\{p_\alpha q'_\beta + p_\beta q'_\alpha - p.q' g_{\alpha\beta} - i\epsilon_{\alpha\beta\lambda\tau}p^\lambda q'^\tau\right\} \ . \end{aligned}\tag{3.22}$$

Using the fact that the four momenta are linearly dependent ($p - k - q - q' = 0$) we have

$$S_1^{\,\alpha\beta} S_{2\,\alpha\beta} = 256\, p.q\, k.q' \ . \tag{3.23}$$

A consistency check for the result provided by eqs.(3.20,3.23) for $|\mathcal{M}|^2$ can be found by considering the case when all the 3-momenta are along the same direction $\hat{\mathbf{z}}$ in the limit $m_e \to 0$. If the initial μ is at rest and assuming the final electron and μ-neutrino are moving parallel to $\hat{\mathbf{z}}$ then $k \propto q'$, for $m_e = 0$, so that $|\mathcal{M}|^2 = 0$. This is essential for angular momentum conservation since as the e, ν_μ are left handed and the $\overline{\nu}_e$, which moves in direction $-\hat{\mathbf{z}}$, is right handed the total spin along $\hat{\mathbf{z}}$ is $-\frac{3}{2}$ which is incompatible with the initial μ having spin $\frac{1}{2}$.



Collinear decay of a mu forbidden by conservation of angular momentum

According to eq.(3.13) the decay rate Γ of the muon is given by

$$\Gamma = \frac{1}{2m_\mu}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\frac{\mathrm{d}^3q}{(2\pi)^3 2q^0}\frac{\mathrm{d}^3q'}{(2\pi)^3 2q'^0}\,(2\pi)^4\delta^4(p-k-q-q')\,\tfrac{1}{2}\sum_{\mathrm{spins}}|\mathcal{M}|^2\ , \tag{3.24}$$

where we average over the initial muon spin states. We have then

$$\Gamma = \frac{G_F^{\,2}}{8m_\mu\pi^5}\int \frac{\mathrm{d}^3k}{k^0}\frac{\mathrm{d}^3q}{q^0}\frac{\mathrm{d}^3q'}{q'^0}\,\delta^4(p-k-q-q')\,p.q\,k.q'\ . \tag{3.25}$$

We evaluate this phase space integral by first integrating over the neutrino momenta q, q' since these are unobserved. If we introduce a momentum $Q = q+q' = p-k$, $Q^2 = 2q.q' = \sigma^2 > 0$ with also $Q^0 > 0$ then, since for massless neutrinos $q^0 = |\mathbf{q}|$, $q'^0 = |\mathbf{q}'|$ the essential integral becomes

$$I_{\mu\nu}(Q) = \int \frac{\mathrm{d}^3q}{|\mathbf{q}|}\frac{\mathrm{d}^3q'}{|\mathbf{q}'|}\,\delta^4(Q-q-q')\,q_\mu q'_\nu\ . \tag{3.26}$$

Lorentz invariance requires that this has the form

$$I_{\mu\nu}(Q) = a\,Q_\mu Q_\nu + b\,g_{\mu\nu}Q^2\ . \tag{3.27}$$

To calculate a, b we may contract $I_{\mu\nu}(Q)$ with $g^{\mu\nu}$ and also $Q^\mu Q^\nu$ which then gives the equations

$$a+4b = \tfrac{1}{2}I\ , \quad a+b = \tfrac{1}{4}I\ , \qquad I = \int \frac{\mathrm{d}^3q}{|\mathbf{q}|}\frac{\mathrm{d}^3q'}{|\mathbf{q}'|}\,\delta^4(Q-q-q')\ . \tag{3.28}$$

The integral I can be easily evaluated in the centre of mass frame $Q^\mu = (\sigma, \mathbf{0})$ since it is Lorentz scalar,

$$I = \int \frac{\mathrm{d}^3q}{|\mathbf{q}|^2}\,\delta(\sigma - 2|\mathbf{q}|) = 4\pi\int_0^\infty \mathrm{d}q\,\delta(\sigma-2q) = 2\pi\ , \tag{3.29}$$

and hence

$$a = \tfrac{1}{3}\pi\ , \qquad b = \tfrac{1}{6}\pi\ . \tag{3.30}$$

Using eqs.(3.27,3.30) then eq.(3.25) becomes

$$\Gamma = \frac{G_F^{\,2}}{3m_\mu(2\pi)^4}\int \frac{\mathrm{d}^3k}{k^0}\,\Big(2p.(p-k)\,k.(p-k) + p.k\,(p-k)^2\Big)\ . \tag{3.31}$$

In the muon rest frame the integral can be reduced to one over the electron energy E using the result that $p.k = m_\mu E$ and $\mathrm{d}^3k/k^0 \to 4\pi|\mathbf{k}|\mathrm{d}E$. At this point it is also convenient to take advantage of the fact that $m_e/m_\mu = 0.0048 \ll 1$ to neglect the electron mass so that eq.(3.31) becomes

$$\Gamma = \frac{2G_F^{\,2}m_\mu}{3(2\pi)^3}\int_0^{\frac{1}{2}m_\mu}\mathrm{d}E\,E^2(3m_\mu - 4E)\ , \tag{3.32}$$

which is easily evaluated to give the final result for the muon decay rate

$$\Gamma_{\mu^- \to e^- + \overline{\nu}_e + \nu_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} \ . \tag{3.33}$$

The muon lifetime is measured to be $\tau_\mu = 2.1970 \times 10^{-6}$sec and, as this is the inverse of the decay rate, therefore $\Gamma_{\mu^- \to e^- + \overline{\nu}_e + \nu_\mu} = 0.2996 \times 10^{-18}$GeV since the muon has virtually only one decay channel (we have used for the conversion 1 $\text{GeV}^{-1} = 6.582 \times 10^{-25}$sec). Inserting the experimental numbers in eq.(3.33), with $m_\mu = 105.658$ MeV, we would find $G_F = 1.1638 \times 10^{-5}\text{GeV}^{-2}$. Including small radiative corrections the current experimental result is

$$G_F = 1.1664 \times 10^{-5}\text{GeV}^{-2} \ . \tag{3.34}$$

G_F is known as the *Fermi coupling constant.*

On replacing m_μ by m_τ in eq.(3.33) we obtain the estimate for the purely leptonic decays of the τ

$$\Gamma_{\tau^- \to e^- + \overline{\nu}_e + \nu_\tau} \sim \Gamma_{\tau^- \to \mu^- + \overline{\nu}_\mu + \nu_\tau} = 0.405 \times 10^{-12}\text{GeV} \ , \tag{3.35}$$

since there are no new parameters to be determined. Experimentally the τ decays 18% of the time into each of these channels. The lifetime of the τ is 0.295×10^{-12}sec so the total decay rate $\Gamma_\tau = 2.23 \times 10^{-12}$GeV and 18% of this total decay rate is $\sim 0.402 \times 10^{-12}$GeV which is very close to the estimate in eq.(3.35). This is therefore strong evidence that the same weak coupling constant controls all leptonic weak interactions.

1.4 Semi-Leptonic Processes

The β-decay of the neutron,

$$n \to p + e^- + \overline{\nu}_e \ , \tag{3.36}$$

is referred to as a *semi-leptonic* process because it involves hadrons as well as leptons. The initial and final state hadrons in general β-decay processes satisfy the selection rules:

$$\Delta B = 0 \ , \quad \Delta S = 0 \ , \quad \Delta Q = \Delta I_3 = \pm 1 \ , \quad |\Delta I| = 0, 1 \ , \tag{3.37}$$

where B = baryon number, S = strangeness and I_3 = 3-component of isospin. These characteristics are shared by π decays such as

$$\pi^\pm \to e^\pm + \nu_e(\overline{\nu}_e) \ , \quad \pi^\pm \to \mu^\pm + \nu_\mu(\overline{\nu}_\mu) \ . \tag{3.38}$$

Such processes can be accommodated in the current-current model for weak interactions by the inclusion of an additional hadronic part in the weak current.

This term also has a parity breaking $V-A$ structure. The weak interaction effective Lagrangian density is now extended to the form

$$\mathcal{L}_W(x) = -\frac{G_F}{\sqrt{2}} J^\alpha(x)^\dagger J_\alpha(x) \ , \quad J_\alpha(x) = J_\alpha(x)^{\text{lept.}} + J_\alpha(x)^{\text{had.}} \ , \tag{3.39}$$

where ${J_\alpha}^{\text{lept.}}$ is as in eq.(3.8) and

$$J_\alpha(x)^{\text{had.}} = V_\alpha(x)^{\text{had.}} - A_\alpha(x)^{\text{had.}} \ . \tag{3.40}$$

Only the cross terms $J^{\alpha\text{lept.}\dagger} {J_\alpha}^{\text{had.}} + h.c.$ in eq.(3.39) are of course relevant for semi-leptonic processes. In eq.(3.39) we have made the fundamental assumption that the same weak coupling governs the semi-leptonic decays as the purely leptonic ones. However to make this a meaningful restriction we must identify more precisely the structure of both the V and A parts of the weak hadronic current. At this point it is useful to employ hind-sight and exploit the modern understanding of hadronic structure that views the proton p and neutron n as made up of more fundamental quarks:

$$p \sim uud \ , \qquad n \sim udd \ . \tag{3.41}$$

The β-decay of the neutron is regarded as being induced by the β-decay of one of the d-quarks in the neutron:

$$d \to u + e^- + \overline{\nu}_e \ . \tag{3.42}$$

If we assumed that the quark contribution to the weak current is analogous to that of the leptons we would expect

$$J_\alpha(x)^{\text{had.}} \sim \overline{u}(x)\gamma_\alpha(1-\gamma_5)d(x) + \dots \ . \tag{3.43}$$

However there are also analogous semi-leptonic decay processes in which the strangeness of the hadrons changes by ± 1. A few examples of strangeness changing semi-leptonic weak interactions are

$$\Lambda \to p + e^- + \overline{\nu}_e \ , \quad \Sigma^- \to n + e^- + \overline{\nu}_e \ , \quad \Omega^- \to \Xi^0 + e^- + \overline{\nu}_e \ , \tag{3.44}$$

and $K_{\ell 3}$ decays such as

$$K^\pm \to \pi^0 + e^\pm + \nu_e(\overline{\nu}_e) \ . \tag{3.45}$$

In the quark model these particles have the structure $\Lambda \sim uds$, $\Sigma^- \sim dds$, $\Omega^- \sim sss$ and $\Xi^- \sim dss$ while $K^- \sim \overline{u}s$. Such processes can therefore be thought of as being due to the β-decay of a *strange* quark,

$$s \to u + e^- + \overline{\nu}_e \ . \tag{3.46}$$

The general selection rules for strangeness changing semi-leptonic decays resulting from $s \to u$ and also its charge conjugate $\overline{s} \to \overline{u}$ are

$$\Delta B = 0 \ , \quad \Delta S = \Delta Q = \pm 1 \ , \quad \Delta I_3 = \pm\tfrac{1}{2} \ , \quad |\Delta I| = \tfrac{1}{2} \ , \tag{3.47}$$

Such processes also exhibit a $V - A$ structure. However a key point is that, allowing in the appropriate way for the kinematic differences due to the different masses of the particles involved, the decay rates for strangeness changing processes are much less than for the corresponding $\Delta S = 0$ processes. It turns out however that the difference in strength is accounted for by modifying effective weak coupling for $\Delta S = \pm 1$ decays to $G_F \sin\theta_C$ while for $\Delta S = 0$ semi-leptonic processes it is $G_F \cos\theta_C$. θ_C is the Cabbibo angle and experimentally $\theta_C = 13°$ or $\sin\theta_C = 0.22$. The factor $\cos\theta_C = 0.975$ is also necessary to explain small differences between G_F measured in μ decay and the corresponding coupling in β-decays of radioactive nuclei. Actually we see later that this simple picture must be further elaborated when applied to c, b, t quarks as well. However for weak decays of low mass hadrons it is sufficient to take instead of eq.(3.43) a current which leads to $u \leftrightarrow d$ and $u \leftrightarrow s$ transitions

$$J_\alpha(x)^{\mathrm{had.}} = \overline{u}(x)\gamma_\alpha(1-\gamma_5)\left(\cos\theta_C\, d(x) + \sin\theta_C\, s(x)\right) + \ldots \ . \tag{3.48}$$

We have then for the associated vector and axial currents

$$\begin{aligned} V_\alpha(x)^{\mathrm{had.}} &= \overline{u}(x)\gamma_\alpha\left(\cos\theta_C\, d(x) + \sin\theta_C\, s(x)\right) + \ldots \\ A_\alpha(x)^{\mathrm{had.}} &= \overline{u}(x)\gamma_\alpha\gamma_5\left(\cos\theta_C\, d(x) + \sin\theta_C\, s(x)\right) + \ldots \ . \end{aligned}$$

Historically the structure of the hadronic weak current was postulated before quarks were generally accepted in terms of its algebraic properties. If we introduce the column vector

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \ , \tag{3.49}$$

then this forms a triplet of quark fields under the group $SU(3)$. Using the 3×3 Gell-Mann λ-matrices, which form a basis for the Lie algebra of $SU(3)$, we may define octets of vector and axial currents

$$V_{\alpha a}(x) = \overline{q}(x)\tfrac{1}{2}\lambda_a\gamma_\alpha q(x) \quad \text{and} \quad A_{\alpha a}(x) = \overline{q}(x)\tfrac{1}{2}\lambda_a\gamma_\alpha\gamma_5 q(x) \ . \tag{3.50}$$

Using the canonical equal time anti-commutation relations for the quark fields and $[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c$ it is easy to see that they satisfy commutation relations:

$$\begin{aligned} \left[V_a^0(x), V_b^0(x')\right]_{t=t'} &= if_{abc}V_c^0(x)\delta^3(\mathbf{x}-\mathbf{x}') \ , \\ \left[V_a^0(x), A_b^0(x')\right]_{t=t'} &= if_{abc}A_c^0(x)\delta^3(\mathbf{x}-\mathbf{x}') \ , \\ \left[A_a^0(x), A_b^0(x')\right]_{t=t'} &= if_{abc}V_c^0(x)\delta^3(\mathbf{x}-\mathbf{x}') \ . \end{aligned} \tag{3.51}$$

Since these commutation relations are inhomogeneous they fix the normalisation of the vector and axial currents independently of their detailed form in terms of quark fields. From the vector and axial currents we may construct charges $Q_{Va}(t) = \int \mathrm{d}^3x\, V_a^0(x)$ and $Q_{Aa}(t) = \int \mathrm{d}^3x\, A_a^0(x)$ so that from eq.(3.51) $Q_{\pm a}(t) = \frac{1}{2}(Q_{Va}(t) \pm Q_{Aa}(t))$ obey the algebra of $SU(3) \times SU(3)$ which is regarded as an approximate symmetry of the strong interaction Hamiltonian ($SU(3) \times SU(3)$ is supposed to be spontaneously broken to $SU(3)_V$ generated by $Q_{Va}(t)$). The generators of isospin $I_{1,2,3} = Q_{V1,2,3}$ and $Q_{V8} = \frac{\sqrt{3}}{2}Y$, where Y is the hypercharge, so that the electric charge for u, d, s quarks is given by $Q = I_3 + \frac{1}{2}Y$. Without assuming their construction in terms of quark fields the hadronic weak current given by eq.(3.48) can also be written as

$$J_\alpha{}^{\text{had.}} = \cos\theta_C \,(V_{\alpha 1+i2} - A_{\alpha 1+i2}) + \sin\theta_C \,(V_{\alpha 4+i5} - A_{\alpha 4+i5}) + \dots \,, \tag{3.52}$$

since

$$\tfrac{1}{2}(\lambda_1 + i\lambda_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \qquad \tfrac{1}{2}(\lambda_4 + i\lambda_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{3.53}$$

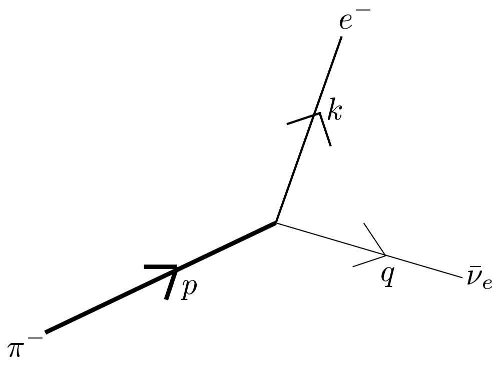
The structure of the hadronic weak current in (3.48) or (3.52) provides strong constraints on allowed semi-leptonic decays. In particular it forbids $\Delta S = -\Delta Q$ processes, such as $\Sigma^+ \to n e^+ \nu_e$, where Σ^+ is a suu baryon and n is formed from udd quarks, and $K^+ \to \pi^+\pi^+ e^- \overline{\nu}_e$, where K^+ and π^+ are $u\bar{s}$ and $u\bar{d}$ states. Such decays have never reliably been seen although closely related $\Delta S = \Delta Q$ decays, such as $\Sigma^- \to n e^- \overline{\nu}_e$ and $K^+ \to \pi^+\pi^- e^+ \nu_e$, are well known.

1.5 Decay of Pseudoscalars to Two Leptons

Much of the evidence for the picture we are describing, especially in relation to the vector part of the weak current, lies in the detailed analysis of baryonic weak interactions, in particular the decays of strange baryons. Additional useful information on the axial vector part of the weak Hamiltonian comes from the study of the two particle semi-leptonic weak decay of pseudo-scalar mesons which involves the axial current alone. This decay process is also technically much simpler to describe.

We consider then $\pi_{\ell 2}$ decays

$$\pi^\pm(p) \to e^\pm(k) + \nu_e(\overline{\nu}_e)(q) \;\; \text{and} \;\; \pi^\pm(p) \to \mu^\pm(k) + \nu_\mu(\overline{\nu}_\mu)(q) \,. \tag{3.54}$$



Pi decay to electron and neutrino

To lowest order the leptons can be taken as non interacting so the basic matrix element for the first of these processes is

$$\begin{aligned}\mathcal{M} &= \langle e^-(k)\,\overline{\nu}_e(q)|\mathcal{L}_W(0)|\pi^-(p)\rangle \\ &= -\frac{G_F}{\sqrt{2}}\langle e^-(k)\,\overline{\nu}_e(q)|\overline{e}\gamma^\alpha(1-\gamma_5)\nu_e|0\rangle\,\langle 0|J_\alpha(0)^{\text{had.}}|\pi^-(p)\rangle \\ &= \frac{G_F}{\sqrt{2}}\cos\theta_C\,\overline{u}_e(k)\gamma^\alpha(1-\gamma_5)v_{\nu_e}(q)\,\langle 0|A_{\alpha 1+i2}(0)|\pi^-(p)\rangle\ , \qquad (3.55)\end{aligned}$$

where we have used the form eq.(3.52) for the weak hadronic current and also, because of parity and isospin, only $A_{\alpha 1+i2}(0)$ has a non zero matrix element between the negative intrinsic parity or pseudoscalar π^- ($\hat{P}|\pi^-(p)\rangle = -|\pi^-(p_P)\rangle$) and the vacuum. Since the pion is spinless this matrix element can only have the form

$$\langle 0|A_{\alpha 1+i2}(0)|\pi^-(p)\rangle = i\sqrt{2}F_\pi p_\alpha\ , \qquad (3.56)$$

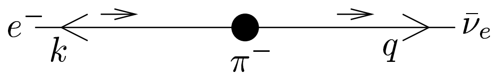
and under parity p_α transforms as a vector so the negative parity of the pion must be counterbalanced by the additional minus sign in the parity transformation of an axial current. This equation defines the pion weak decay constant F_π which has the dimensions of mass (the $\sqrt{2}$ is again conventional but sometimes $f_\pi = \sqrt{2}F_\pi$ is used instead). From eqs.(3.55) and (3.56) we find

$$\mathcal{M} = iG_F F_\pi \cos\theta_C\,\overline{u}_e(k)\gamma.p(1-\gamma_5)v_{\nu_e}(q)\ . \qquad (3.57)$$

If we take into account the fact that $p = k + q$ and make use of the results $\overline{u}_e(k)\gamma.k = \overline{u}_e(k)m_e$ and $\gamma.qv_{\nu_e}(q) = 0$ for massless neutrinos we find

$$\mathcal{M} = G_F F_\pi m_e \cos\theta_C\,\overline{u}_e(k)(1-\gamma_5)v_{\nu_e}(q)\ , \qquad (3.58)$$

so that the matrix element vanishes if $m_e = 0$. This is a consequence of angular momentum conservation since in this limit the electron has negative helicity while the anti-neutrino has positive helicity which, in their centre of mass frame, add up to a component of spin or angular momentum -1 along the electron direction of motion which is incompatible with an initial spinless pion.



Decay of pion forbidden by angular momentum for zero electron mass

If we sum the modulus squared of the matrix element over the final electron, neutrino spins we get

$$\sum_{\text{spins}} |\mathcal{M}|^2 = (G_F F_\pi m_e \cos\theta_C)^2 \operatorname{tr}\{(\gamma.k + m_e)(1-\gamma_5)\gamma.q\} \ , \tag{3.59}$$

where

$$\operatorname{tr}\{(\gamma.k + m_e)(1-\gamma_5)\gamma.q\} = 4k.q = 2(m_\pi^2 - m_e^2) \ . \tag{3.60}$$

From the general formula eq.(3.13) the decay rate

$$\Gamma = \frac{1}{2m_\pi}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\frac{\mathrm{d}^3q}{(2\pi)^3 2q^0}(2\pi)^4\delta^4(p-k-q)\sum_{\text{spins}}|\mathcal{M}|^2 \ , \tag{3.61}$$

Therefore from eqs.(3.59,3.60)

$$\Gamma = (G_F F_\pi m_e \cos\theta_C)^2 \frac{1}{m_\pi}(m_\pi^2 - m_e^2)\frac{1}{(4\pi)^2}\int \frac{\mathrm{d}^3k}{k^0|\mathbf{k}|}\delta(m_\pi - k^0 - |\mathbf{k}|) \ , \tag{3.62}$$

where $k^0 = (m_e^2 + \mathbf{k}^2)^{\frac{1}{2}}$ and we have restricted the integral to the π^- rest frame. The remaining integral may be easily evaluated giving

$$\Gamma_{\pi^-\to e^- + \overline{\nu}_e} = \frac{m_\pi}{4\pi}m_e^2\left(1 - \frac{m_e^2}{m_\pi^2}\right)^2 G_F^2 F_\pi^2 \cos^2\theta_C \ . \tag{3.63}$$

If we were to do the calculation for muons we would find

$$\Gamma_{\pi^-\to \mu^- + \overline{\nu}_\mu} = \frac{m_\pi}{4\pi}m_\mu^2\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 G_F^2 F_\pi^2 \cos^2\theta_C \ . \tag{3.64}$$

Although we are not in a position to predict F_π and hence the absolute values of these decay rates we do have a prediction from our $V-A$ theory for the ratio of the rates

$$R_0 = \frac{\Gamma(\pi^- \to e^- + \overline{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \overline{\nu}_\mu)} = \frac{m_e^2}{m_\mu^2}\frac{(m_\pi^2 - m_e^2)^2}{(m_\pi^2 - m_\mu^2)^2} \ . \tag{3.65}$$

Inserting the appropriate masses we find $R_0 = 1.28 \times 10^{-4}$ which should be compared with the experimental result $R_{\text{expt.}} = 1.23 \times 10^{-4}$. This very small number is a direct consequence of the $V-A$ theory. While there is a reasonably good comparison between theory and experiment it can be improved considerably by including appropriate radiative corrections (i.e. loop corrections due to virtual photons). If we accept this evidence then it goes to support the idea that the same hadronic axial current matrix element controls the two decay processes and hence supports the $V-A$ theory. There are many other decay processes that can be estimated with results that support the theory.

1.6 Non-Leptonic Processes

Many of the decays of strange particles involve no leptons at all. Such processes may in principle arise from the effective Lagrangian density in eq.(3.39), from the term $J^{\alpha \text{had.}\dagger} J_\alpha{}^{\text{had.}}$. Also the strangeness conserving part may induce small parity violating effects in nuclear physics which are potentially observable. However there is no completely well defined procedure for calculating theoretically such processes directly from $\mathcal{L}_W$ since the hadronic currents are not simply expressed in terms of free fields. They couple to strongly interacting particles which cannot be treated in any perturbative fashion such as was used for leptons. In the product of hadronic currents in $\mathcal{L}_W$ it would be necessary to introduce a complete set of intermediate states rather than just write it as a factorised product of single current matrix elements which was all that was necessary in the calculation of matrix elements for purely leptonic and semi-leptonic decays. In consequence the theory of non-leptonic weak decays is primarily phenomenological and does not determine much about the structure of weak interactions themselves.

2 CP Violation

A crucial non leptonic decay is that of neutral K's since in these it is possible to observe CP-violation. The K^0 and its anti-particle $\bar{K}^0$ are pseudoscalar mesons with quark structure $\bar{s}d$ and $\bar{d}s$, having strangeness 1 and -1 respectively. Under combined charge conjugation and parity the $K^0, \bar{K}^0$ states (at rest with zero 3-momentum) may be chosen so as to transform as

$$\hat{C}\hat{P}|K^0\rangle = |\bar{K}^0\rangle\,, \qquad \hat{C}\hat{P}|\bar{K}^0\rangle = |K^0\rangle\,, \tag{3.66}$$

so that we may define $CP = +1$ and -1 eigenstates by

$$|K_1{}^0\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle + |\bar{K}^0\rangle\Big)\,, \qquad |K_2{}^0\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle - |\bar{K}^0\rangle\Big)\,. \tag{3.67}$$

Assuming that weak interactions conserve CP the possible decays of $K_1{}^0$ and $K_2{}^0$ are very different. $K_1{}^0$ is allowed to decay to $\pi\pi$ whereas $K_2{}^0$ is not. Under C $\pi^+ \leftrightarrow \pi^-$ while $\pi^0 \to \pi^0$ and, in the centre of mass frame for two pions, P interchanges the two particles and hence $\hat{C}\hat{P}|\pi^+\pi^-\rangle = (-1)^\ell|\pi^+\pi^-\rangle$, $\hat{C}\hat{P}|\pi^0\pi^0\rangle = (-1)^\ell|\pi^0\pi^0\rangle$ where ℓ is the orbital angular momentum. In K decay we must take $\ell = 0$ as the K is spinless. In consequence, with CP conservation, $K_2{}^0$ is not allowed to decay to $\pi\pi$ states and only $CP = -1$ $\pi\pi\pi$ non leptonic final states are possible, as well as semi-leptonic decays. This is in apparent accord with experiment where neutral kaons have two characteristic lifetimes, the short lifetime $K_S{}^0$s are observed to decay almost entirely according to $K_S{}^0 \to \pi^+\pi^-, \pi^0\pi^0$, with a lifetime 0.89×10^{-10}sec, while $K_L{}^0$s have a much longer lifetime 5.18×10^{-8}sec, with a variety of decay modes including non leptonic 3π states. If neutral kaons are produced then beyond a few ${K_S}^0$ lifetimes only ${K_L}^0$s remain. The observation of ${K_L}^0 \to \pi\pi$ decays shows, with the definitions,

$$\eta_{+-} = \frac{\langle \pi^+\pi^- | H_W | {K_L}^0 \rangle}{\langle \pi^+\pi^- | H_W | {K_S}^0 \rangle}, \qquad \eta_{00} = \frac{\langle \pi^0\pi^0 | H_W | {K_L}^0 \rangle}{\langle \pi^0\pi^0 | H_W | {K_S}^0 \rangle}, \tag{3.68}$$

that $|\eta_{+-}|$, $|\eta_{00}|$ are non zero. Experimentally $|\eta_{+-}| \approx |\eta_{00}| \approx 2.28 \times 10^{-3}$.

Since this demonstrates that CP is not conserved we cannot identify ${K_S}^0, {K_L}^0$ with the ${K_1}^0, {K_2}^0$ CP eigenstates defined in eq.(3.67). Instead

$$|{K_S}^0\rangle = \frac{1}{(1+|\epsilon_1|^2)^{\frac{1}{2}}}\Big(|{K_1}^0\rangle + \epsilon_1 |{K_2}^0\rangle\Big), \quad |{K_L}^0\rangle = \frac{1}{(1+|\epsilon_2|^2)^{\frac{1}{2}}}\Big(|{K_2}^0\rangle + \epsilon_2 |{K_1}^0\rangle\Big), \tag{3.69}$$

with ϵ_1, ϵ_2 complex. These states are determined by diagonalisation of the 2×2 matrix

$$R = M - \tfrac{1}{2} i \Gamma = \begin{pmatrix} \langle K^0 | H' | K^0 \rangle & \langle K^0 | H' | \bar{K}^0 \rangle \\ \langle \bar{K}^0 | H' | K^0 \rangle & \langle \bar{K}^0 | H' | \bar{K}^0 \rangle \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{3.70}$$

where H' is the effective Hamiltonian describing processes to second order in the weak interaction H_W,

$$H' = H_W - \sum_f \frac{H_W|f\rangle\langle f|H_W}{E_f - m_W - i\epsilon}, \tag{3.71}$$

and we restrict to the subspace formed by $K^0, \bar{K}^0$ states at rest. In eq.(3.71) the states $|f\rangle$ satsfying $E_f = m_W$ are possible states to which $K^0, \bar{K}^0$ may decay, these give $i(H' - H'^\dagger) = 2\pi \sum_f \delta(E_f - m_W)\, H_W|f\rangle\langle f|H_W$. As a consequence H' and hence R in (3.70) are not hermitian, which is a reflection of the decay of $K^0, \bar{K}^0$. The eigenvectors of the non hermitian matrix $R = M - \frac{1}{2} i\Gamma$ determine the appropriate linear combinations appearing in (3.69) with (3.67). The corresponding complex eigenvalues may be written as $m_S - i\frac{1}{2}\gamma_S$, $m_L - i\frac{1}{2}\gamma_L$ respectively. Under time evolution we then have $|{K_S}^0(t)\rangle = e^{-i(m_S - i\frac{1}{2}\gamma_S)t}|{K_S}^0(0)\rangle$, $|{K_L}^0(t)\rangle = e^{-i(m_L - i\frac{1}{2}\gamma_L)t}|{K_L}^0(0)\rangle$ and hence, since $|e^{-i(m-i\frac{1}{2}\gamma)t}|^2 = e^{-\gamma t}$, γ_S, γ_L are the decay rates for the observed ${K_S}^0, {K_L}^0$ neutral kaons while m_S, m_L are their masses. Even without any CP violation second order $\Delta S = \pm 2$ weak interactions lead to non diagonal contributions to the matrix and give rise to a mass difference between the ${K_S}^0$ and ${K_L}^0$ states.

Assuming CPT invariance requires $\Theta H' \Theta^{-1} = H'^\dagger$ with $\Theta = \hat{C}\hat{P}\hat{T}$. Since we may take $\Theta|K^0\rangle = |\bar{K}^0\rangle$, $\Theta|\bar{K}^0\rangle = |K^0\rangle$, then from the antilinear property of Θ we have $\langle K^0|H'|K^0\rangle = \langle \bar{K}^0|H'^\dagger|\bar{K}^0\rangle^* = \langle \bar{K}^0|H'|\bar{K}^0\rangle$, or in 3.70 $R_{11} = R_{22}$, while $\langle K^0|H'|\bar{K}^0\rangle = \langle \bar{K}^0|H'^\dagger|K^0\rangle^*$ is unconstrained. Conversely, assuming T invariance, $\hat{T}H'\hat{T}^{-1} = H'^\dagger$ and $\hat{T}|K^0\rangle = |K^0\rangle$, $\hat{T}|\bar{K}^0\rangle = |\bar{K}^0\rangle$, leads to $\langle K^0|H'|\bar{K}^0\rangle = \langle \bar{K}^0|H'|K^0\rangle$, or $R_{12} = R_{21}$. The unnormalised eigenvector of the matrix R given by eq.(3.70) for ${K_S}^0$ is of the form $\begin{pmatrix} 1+\epsilon_1 \\ 1-\epsilon_1 \end{pmatrix}$ and for ${K_L}^0$, $\begin{pmatrix} 1+\epsilon_2 \\ -1+\epsilon_2 \end{pmatrix}$. Assuming CPT so that $R_{11} = R_{22}$ leads to

$$\epsilon_1 = \epsilon_2 = \frac{\sqrt{R_{12}} - \sqrt{R_{21}}}{\sqrt{R_{12}} + \sqrt{R_{21}}} \, . \tag{3.72}$$

Conversely T invariance alone and hence $R_{12} = R_{21}$ requires $\epsilon_1 = -\epsilon_2$. Experimentally these quantities may be measured and, within errors, CPT is conserved. If the only source of CP violation is through the mixing in eq.(3.69) then $\eta_{+-} = \eta_{00} = \epsilon_2$. Experimentally $|\eta_{+-}/\eta_{00}|^2 - 1 \approx 2 \times 10^{-2}$ which is small but non zero.

3 Intermediate Vector Bosons

The effective Lagrangian in (3.39) summarises a very large amount of experimental information concerning low energy weak interaction processes. Nevertheless it is theoretically unsatisfactory. Treated to first order weak processes rise rapidly with energy and violate general bounds. Regarded as an interaction in a quantum field theory then the perturbative expansion is unrenormaliseable and presents severe difficulties, essentially because the product of two currents has dimension 6. An amelioration of these problems is obtained if we replace the current-current interaction with an interaction between the weak current and an elementary complex vector field W_α,

$$\mathcal{L}_I = g_W \left(J^\alpha W_\alpha + J^{\alpha\dagger} {W_\alpha}^\dagger \right) \, . \tag{3.73}$$

The free field Lagrangian density for W_α is taken to be

$$\mathcal{L}_0 = -\tfrac{1}{2} F^{\alpha\beta\dagger} F_{\alpha\beta} + m_W^2 W^{\alpha\dagger} W_\alpha \, , \quad F_{\alpha\beta} = \partial_\alpha W_\beta - \partial_\beta W_\alpha \, , \tag{3.74}$$

which is an extension of the Lagrangian density for the electromagnetic field to include a mass term. From this form it is easy to derive the classical equation of motion

$$\partial^\alpha F_{\alpha\beta} + m_W^2 W_\beta = 0 \, , \tag{3.75}$$

which in turn requires

$$(\partial^2 + m_W^2) W_\beta = 0 \, , \qquad \partial^\alpha W_\alpha = 0 \, , \tag{3.76}$$

so that W_α is a free field of mass m_W with zero divergence. The field may be quantised by decomposing it into plane wave modes

$$W_\alpha(x) = \sum_{p,\lambda} \left(a(p,\lambda) \epsilon_\alpha(p,\lambda) e^{-ip.x} + b(p,\lambda)^\dagger \epsilon_\alpha(p,\lambda)^* e^{ip.x} \right) \, , \tag{3.77}$$

where the summation is over all 4-momenta that satisfy the mass-shell condition $p^2 = m_W^2$ and the λ summation is over labels identifying the three allowed polarization vectors that satisfy

$$p.\epsilon(p,\lambda) = 0 \quad \text{and} \quad \epsilon(p,\lambda)^*.\epsilon(p,\lambda') = -\delta_{\lambda\lambda'} \ . \tag{3.78}$$

In the rest frame when $p = (m_W, 0, 0, 0)$ then $\epsilon^\alpha(p,\lambda)$ has the form $(0, \boldsymbol{\epsilon}(\lambda))$ where $\{\boldsymbol{\epsilon}(\lambda)\}$ are three orthonormal 3-vectors. By contracting both sides with the linearly independent set $\{p, \epsilon(p,\lambda)\}$ we can verify the completeness identity

$$\sum_\lambda \epsilon_\alpha(p,\lambda)\epsilon_\beta(p,\lambda)^* = -g_{\alpha\beta} + \frac{p_\alpha p_\beta}{m_W^2} \ . \tag{3.79}$$

On quantisation $a, b, a^\dagger, b^\dagger$ become annihilation and creation operators which satisfy the commutation relations

$$[a(p,\lambda), a(p',\lambda')^\dagger] = \delta_{pp'}\delta_{\lambda\lambda'} \ , \quad [b(p,\lambda), b(p',\lambda')^\dagger] = \delta_{pp'}\delta_{\lambda\lambda'} \ , \tag{3.80}$$

where $a(p,\lambda)^\dagger, b(p,\lambda)^\dagger$ create massive spin 1 particle states. In the usual way from the quantised free fields we may construct the Feynman propagator

$$\begin{aligned}\langle 0|\mathcal{T}\{W_\alpha(x)^\dagger W_\beta(0)\}|0\rangle &= iD_{\alpha\beta}(x) \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, e^{-ip.x}\frac{i}{p^2 - m_W^2 + i\epsilon}\Big(-g_{\alpha\beta} + \frac{p_\alpha p_\beta}{m_W^2}\Big) . \end{aligned} \tag{3.81}$$

There is some subtlety in determining the propagator for the W field since using the mode expansion eq.(3.77) in the time ordered product in (3.81) leads to an extra non covariant piece when $\alpha = \beta = 0$. However a more careful treatment of the quantisation for the field theory arising from (3.74) shows that it is consistent to just take the form given by $iD_{\alpha\beta}(x)$ in eq.(3.81) for the W propagator. For application to weak interactions the crucial result is that as $m_W \to \infty$, or more physically if in any process the components of the momenta for virtual W's satisfy $|p^\alpha| \ll m_W$, then

$$D_{\alpha\beta}(x) \sim \frac{1}{m_W^2}\, g_{\alpha\beta}\, \delta^4(x) \ . \tag{3.82}$$

In the quantum field theory with the interaction in eq.(3.73) the corresponding S operator is as usual given by

$$S = \mathcal{T}\left\{e^{i\int \mathrm{d}^4x\, \mathcal{L}_I(x)}\right\} \ , \tag{3.83}$$

where the W field has the propagator in eq.(3.81). For processes in which no massive particles described by the W field are created or destroyed then, effectively by normal ordering S and dropping all terms involving the annihilation operators a, b and also the corresponding creation operators, we find

$$S \longrightarrow \mathcal{T}\left\{e^{-ig_W^2\int\int \mathrm{d}^4x\, \mathrm{d}^4x'\, J^\alpha(x)^\dagger D_{\alpha\beta}(x-x')J^\beta(x')}\right\} \ . \tag{3.84}$$

For low energy processes in which it is appropriate to take the limit given by eq.(3.82) then it is easy to see that the S operator takes the form corresponding to the weak interaction given by eq.(3.39) if

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{m_W^2} \, . \tag{3.85}$$

Clearly the coupling constant g_W, which is defined by the interaction eq.(3.73), is dimensionless which is necessary for this interaction to be at least potentially renormaliseable. However this theory of interacting massive vector fields is still not satisfactory since the large momentum behaviour of the vector propagator, which is exhibited in (3.81), contains the $p_\alpha p_\beta / m_W^2$ terms which lead to unrenormaliseable divergences. Such terms do not appear in quantum electrodynamics essentially as a consequence of gauge invariance. A renormaliseable quantum field theory describing weak interactions requires the construction of a suitable spontaneously broken non abelian gauge field theory. The first, and also the finally experimentally consistent such theory, is the Weinberg-Salam model.

Part IV

Weinberg-Salam Gauge Field Theory

1 Electro-Weak Theory

The electro-weak gauge theory is an illustration of the theoretical unification of physical phenomena which initially appeared very different. It brings together the theories of QED and weak interactions. In virtually its present form the theory was described S Weinberg in 1967 although it was also put forward by A Salam in the following year and earlier S Glashow had discussed many of the essential features of the final theory. In conjunction with J Iliopoulos and L Maiani, Glashow also showed how with the introduction of the charm quark, in addition to the already established u, d, s quarks, the theory could be successfully extended to hadrons.

1.1 Electro-Weak Theory for Leptons

Some of the essential experimental and theoretical ingredients used in constructing a unified theory of weak and electromagnetic interactions for leptons are:

a) Leptons have only electromagnetic and weak interactions.

b) QED is a *gauge* theory - a fact that is important for its renormalizability which allows for calculations of higher order corrections.

c) Weak interactions - like QED - involve vector-like currents which have a $V - A$ structure, giving rise to parity violation, while the electromagnetic current is pure V.

d) The current-current interaction, although very successful to first order as a phenomenological description of low energy weak interactions, does not allow higher order corrections to be calculated. It is not as it stands a renormalizable field theory and at energies of order $1/\sqrt{G_F}$ weak processes are no longer weak and the first order predictions of the current-current interaction must be drastically modified.

In order both to unify the forces governing weak interactions and to render the theory potentially renormalizable it is therefore natural to postulate that there should exist gauge bosons related to the weak currents in a manner similar to the way that the electromagnetic field is related to the electric current. However it is obvious that there are no *massless* vector particles that are partners to the photon.

It was shown by Higgs and others that this problem can be overcome by exploiting the ideas of spontaneous symmetry breaking which allows for massive vector particles in gauge theories. Using gauge theories with spontaneous symmetry breakdown, when some scalar field gains a non zero vacuum expectation value, also provides a mechanism for generating lepton masses when mass terms are forbidden by the gauge symmetry.

1.2 Chiral Structure of the Dirac Spinor Field

As a preliminary to describing the gauge theory we examine the chiral structure of the Dirac field. Such a field, for the electron say, satisfies the Dirac equation,

$$(i\gamma.\partial - m_e)\, e(x) = 0 \ . \tag{4.1}$$

This equation is obtained from the Lagrangian

$$\mathcal{L}(x) = \overline{e}(x)\, (i\gamma.\partial - m_e)\, e(x) \ . \tag{4.2}$$

Now split $e(x)$ into left and right chiral components

$$e(x) = e_R(x) + e_L(x) \ , \tag{4.3}$$

where

$$e_R(x) = \tfrac{1}{2}\,(1+\gamma_5)\, e(x) \quad \text{and} \quad e_L(x) = \tfrac{1}{2}\,(1-\gamma_5)\, e(x) \ . \tag{4.4}$$

The conjugate fields are

$$\overline{e_R}(x) = e_R(x)^\dagger \gamma^0 = e(x)^\dagger \tfrac{1}{2}\,(1+\gamma_5)\,\gamma^0 = \overline{e}(x)\tfrac{1}{2}\,(1-\gamma_5) \ , \tag{4.5}$$

and similarly

$$\overline{e_L}(x) = \overline{e}(x)\tfrac{1}{2}\,(1+\gamma_5) \ . \tag{4.6}$$

It follows that

$$\overline{e_R}(x) e_R(x) = \overline{e_L}(x) e_L(x) = 0 \ , \tag{4.7}$$

and

$$\overline{e_R}(x)\gamma^\alpha e_L(x) = \overline{e_L}(x)\gamma^\alpha e_R(x) = 0 \ . \tag{4.8}$$

The electron Lagrangian given by eq.(4.2) can then be rewritten as

$$\mathcal{L}(x) = \overline{e_R}(x) i\gamma.\partial e_R(x) + \overline{e_L}(x) i\gamma.\partial e_L(x) - m_e\left[\overline{e_R}(x) e_L(x) + \overline{e_L}(x) e_R(x)\right] \ . \tag{4.9}$$

The 'kinetic' part of the Lagrangian is therefore a sum of two terms involving the right and left chiral components separately, while the mass term couples right to left and left to right. From this point of view that the electron mass arises from an interaction that transforms left handed, or negative helicity, electrons into right handed, or positive helicity, electrons and vice versa. A massless neutrino has no interaction inducing such a $L-R$ flip so that it can therefore remain purely lefthanded

1.3 Weak Iso-Spin and Hypercharge

The starting point for the construction of a unified Electro-Weak gauge theory is to identify the appropriate gauge group G and also the corresponding representations under which the fields transform. For simplicity we initially restrict the theory solely to the electron and its associated neutrino which may be assumed to be massless and purely left-handed. Since the weak interactions violate parity the left handed and right handed fields are treated separately and may therefore belong to different representations of the gauge group. For the electromagnetic and weak interactions to be treated on a unified basis there has to be a close connection between the neutrino and the electron. The standard method for achieving such a relationship in quantum field theory is to combine the fields for the related particles into a multiplet that forms a representation of the appropriate symmetry group. In the present case the neutrino and the left handed electron form a doublet

$$L(x) = \begin{pmatrix} \nu_e(x) \\ e_L(x) \end{pmatrix} , \tag{4.10}$$

which is supposed to form a two dimensional representation of an $SU(2)$ group called *weak iso-spin*. The members of a symmetry multiplet must have the same numbers of degrees of freedom so only the *left* chiral component of the electron field is linked to the naturally left-handed neutrino field. The *right* chiral component of the electron field is taken as a *weak iso-singlet* which can be written as

$$R(x) = e_R(x) \ . \tag{4.11}$$

It is easily checked that the kinetic part of the electron-neutrino Lagrangian can be expressed as

$$\mathcal{L}_{\text{kin.}}(x) = \bar{L}(x) i\gamma.\partial L(x) + \bar{R}(x) i\gamma.\partial R(x) \ . \tag{4.12}$$

Clearly $\mathcal{L}_{\text{kin.}}(x)$ is invariant under weak iso-spin transformations when

$$L(x) \to e^{\frac{1}{2} i\boldsymbol{\alpha}.\boldsymbol{\tau}} L(x) \ , \quad \bar{L}(x) \to \bar{L}(x) e^{-\frac{1}{2} i\boldsymbol{\alpha}.\boldsymbol{\tau}} \ , \tag{4.13}$$

where $\boldsymbol{\tau}$ are the usual 2×2 Pauli matrices, and $R(x)$ is invariant

$$R(x) \to R(x) \ , \qquad \bar{R}(x) \to \bar{R}(x) \ . \tag{4.14}$$

The Lagrangian is also invariant under two independent $U(1)$ groups, or phase transformations, when L and R transform separately by multiplication by different complex numbers of modulus one. In constructing a gauge theory then classically any global symmetry of $\mathcal{L}_{\text{kin.}}$ may be made into a local symmetry by introducing suitable gauge fields and replacing the derivatives in eq.(4.12) by the appropriate covariant derivative. In order to accommodate electromagnetism it is essential that the gauge group G should contain the local $U(1)_Q$ electromagnetic gauge group generated by the electric charge Q. Acting on the left-handed doublet and right-handed singlet fields defined in eq.(4.10) and eq.(4.11) it is easy to see since the neutrino has no charge and the electron charge -1 that

$$QL(x) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} L(x) \ , \qquad QR(x) = -R(x) \ . \tag{4.15}$$

Acting on L clearly

$$Q = \tfrac{1}{2}\tau_3 - \tfrac{1}{2}1 \ , \tag{4.16}$$

which implies that a $U(1)_Q$ gauge transformation generated by Q is a product of an element belonging to a $U(1)$ subgroup of the weak $SU(2)$ and an element belonging to an independent commuting $U(1)$ group. If we denote the generators of the weak $SU(2)$ group in general by T_i, $i = 1, 2, 3$, $[T_i, T_j] = i\epsilon_{ijk}T_k$ and the commuting $U(1)$ group by Y then eq.(4.16) is generalised to

$$Q = T_3 + Y \ . \tag{4.17}$$

The minimal gauge group that is consistent with the requirements for an electroweak theory involving just e, ν_e may be therefore taken to be $G = SU(2)_T \times U(1)_Y$ where the factor $SU(2)_T$ refers to weak iso-spin, with generators $\mathbf{T}$, and the factor $U(1)_Y$ has as its generator the weak hypercharge Y. The generators of $SU(2)_T \times U(1)_Y$ then have the form $\mathbf{T} \times 1, 1 \times Y$. A typical element g of this group can be written

$$g(\boldsymbol{\alpha}, \beta) = \exp(i\boldsymbol{\alpha}.\mathbf{T}) \exp(i\beta Y) \ . \tag{4.18}$$

Acting on the $T = \frac{1}{2}$ representation defined by L we take $\mathbf{T} \to \frac{1}{2}\boldsymbol{\tau}$ while for the $T = 0$ singlet representation provided by R then $\mathbf{T} \to \mathbf{0}$. The irreducible representations of $U(1)_Y$ are one-dimensional and are determined by assigning a particular value to Y, acting on L as in eq.(4.16) $Y \to -\frac{1}{2}$ while for R from eq.(4.15) $Y \to -1$. In general the value of the weak hypercharge Y for an arbitrary multiplet is dictated by eq.(4.17) where Q takes equal values for the left and right handed chiral components of any charged field since the electromagnetic current is purely V.

1.4 $SU(2) \times U(1)$ Covariant Derivatives

If $\psi(x)$ is a field multiplet with weak hyper-charge Y then it transforms under a local or space dependent $SU(2)_T \times U(1)_Y$ gauge transformation according to

$$\psi(x) \to e^{i\boldsymbol{\alpha}(x).\mathbf{T} + i\beta(x)Y}\psi(x) \ , \tag{4.19}$$

where $\mathbf{T}$ belongs to the appropriate representation. For compatibility with the required form for weak interaction all left handed lepton and also quark fields belong to $T = \frac{1}{2}$ doublet representations while the right handed fermion fields are all singlets. The covariant derivative for $\psi(x)$ depends on gauge fields $\mathcal{A}_\mu$, B_μ and has the form

$$D_\mu \psi(x) = \left(\partial_\mu - ig\mathcal{A}_\mu(x).\mathbf{T} - ig' B_\mu(x) Y\right) \psi(x) \ , \tag{4.20}$$

where the non abelian $SU(2)_T$ vector gauge fields transform as

$$\mathcal{A}_\mu(x).\mathbf{T} \to e^{i\boldsymbol{\alpha}(x).\mathbf{T}} \mathcal{A}_\mu(x).\mathbf{T} e^{-i\boldsymbol{\alpha}(x).\mathbf{T}} + \frac{1}{g} e^{i\boldsymbol{\alpha}(x).\mathbf{T}} i\partial_\mu e^{-i\boldsymbol{\alpha}(x).\mathbf{T}} \ , \tag{4.21}$$

while the abelian $U(1)_Y$ gauge field transforms as

$$B_\mu(x) \to B_\mu(x) + \frac{1}{g'} \partial_\mu \beta(x) \ . \tag{4.22}$$

With these transformation properties for the vector fields the covariant derivatives transform in the same way as the multiplet itself in eq.(4.19). Thus

$$D_\mu \psi(x) \to e^{i\boldsymbol{\alpha}(x).\mathbf{T} + i\beta(x) Y} D_\mu \psi(x) \ , \tag{4.23}$$

Note that because of the direct product structure of the gauge group it is necessary to introduce two coupling constants g and g', one for each factor in the gauge group. The existence of two coupling parameters is crucial to the structure of electro-weak theory although it is an indication that the theory is not really fully unified.

Using the hyper-charge assignments for the various multiplets we see that the covariant derivatives for the lepton fields are then

$$\begin{aligned} D_\mu L(x) &= \left(\partial_\mu - ig\tfrac{1}{2}\mathcal{A}_\mu(x).\boldsymbol{\tau} + ig'\tfrac{1}{2}B_\mu(x)\right) L(x) \ , \\ D_\mu R(x) &= \left(\partial_\mu + ig' B_\mu(x)\right) R(x) \ . \end{aligned} \tag{4.24}$$

We can also define the field strengths for the gauge fields themselves by

$$\mathbf{F}_{\mu\nu}(x) = \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) - g\mathcal{A}_\mu(x) \times \mathcal{A}_\nu(x) \ , \tag{4.25}$$

where under a gauge transformation there is no inhomogeneous term as in eq.(4.21) $\mathbf{F}_{\mu\nu}(x).\mathbf{T} \to e^{i\boldsymbol{\alpha}(x).\mathbf{T}} \mathbf{F}_{\mu\nu}(x).\mathbf{T} e^{-i\boldsymbol{\alpha}(x).\mathbf{T}}$, and

$$G_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) \ , \tag{4.26}$$

which is invariant, just as the usual Maxwellian electromagnetic field strength is under gauge transformations.

The kinetic term for the lepton fields given by eq.(4.12) can now be extended to the local gauge invariant form as

$$\mathcal{L}_{\text{lept.}}(x) = \bar{L}(x) i\gamma^\mu D_\mu L(x) + \bar{R}(x) i\gamma^\mu D_\mu R(x) \ , \tag{4.27}$$

with covariant derivatives defines as in eq.(4.24), while the gauge fields are described by the usual generalisation of the Lagrangian for the electromagnetic field

$$\mathcal{L}_{\text{gauge}}(x) = -\tfrac{1}{4}\mathbf{F}^{\mu\nu}(x).\mathbf{F}_{\mu\nu}(x) - \tfrac{1}{4}B^{\mu\nu}(x)B_{\mu\nu}(x)\ . \tag{4.28}$$

Since the left handed and right handed lepton fields transform differently under both $SU(2)_T$ and $U(1)_Y$ there is no possibility of adding any mass terms to eq.(4.27) which is compatible with invariance under the gauge group.

1.5 Spontaneous Symmetry Breakdown

The gauge fields for the field theory described by the Lagrangian in eq.(4.28) correspond to massless vector, or spin 1, particles after quantisation, at least when treated in perturbation theory. For a theory of electro-weak interactions the only allowed massless vector particle is the photon corresponding to the usual Maxwell gauge field. The remaining vector fields must be given a mass. In order to ensure that the theory is renormaliseable this must be done in a way which preserves gauge invariance under the gauge group $G = SU(2)_T \times U(1)_Y$. This can be achieved by using the mechanism of spontaneous symmetry breaking when the Lagrangian remains invariant under the symmetry group but the vacuum state of the theory does not. The simplest way of achieving the required spontaneous symmetry breakdown is to introduce an elementary scalar Higgs field ϕ whose potential $V(\phi)$ is invariant under gauge transformations on ϕ but is such that its minimum is obtained for non zero values of the field. In the ground state of the field theory the Higgs field is restricted to a subset $\mathcal{V}_{\min}$ on which G acts in a non trivial fashion. In the quantum field theory the vacuum is defined, to lowest order in perturbation theory, by choosing a particular point ϕ_0 belonging $\mathcal{V}_{\min}$ and then expanding about it. In general ϕ_0 is not invariant under the action of group transformations belonging to G but those elements of G which leave ϕ_0 invariant, $h\phi_0 = \phi_0$, define a subgroup $H \subset G$ which is then the unbroken gauge group. The gauge fields linked to the generators of the Lie algebra of H remain massless while those corresponding to the coset G/H gain a mass. In the present case it is necessary to preserve a residual $U(1)$ invariance to ensure that there remains a massless photon. This is ensured by choosing the Higgs field to be a $T = \frac{1}{2}$ weak iso-spin doublet which also carries weak hypercharge $Y = \frac{1}{2}$. Hence

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}\ , \tag{4.29}$$

when the covariant derivative is

$$D_\mu\phi(x) = \left(\partial_\mu - ig\tfrac{1}{2}\mathcal{A}_\mu(x).\boldsymbol{\tau} - ig'\tfrac{1}{2}B_\mu(x)\right)\phi(x)\ . \tag{4.30}$$

A Lagrangian for ϕ that is invariant under the local gauge group $SU(2)_T \times U(1)_Y$ is then

$$\mathcal{L}_{\text{Higgs}}(x) = (D^\mu\phi(x))^\dagger D_\mu\phi(x) - V(\phi(x))\ , \tag{4.31}$$

if the potential $V(\phi)$ has the form

$$V(\phi) = F(\phi^\dagger\phi) \ . \tag{4.32}$$

The Lagrangian defined by eqs.(4.31,4.32) is invariant under $SU(2)_T \times U(1)_Y$ for any value of the weak hyper-charge of ϕ but choosing $Y = \frac{1}{2}$ is crucial later to allow for coupling of ϕ to the lepton fields. For spontaneous symmetry breakdown the potential V, or F, is assumed to have a minimum at a point where $\phi^\dagger\phi = \frac{1}{2}v^2$. For renormaliseability $V(\phi)$ should be at most quartic in the field ϕ so, if we choose $V_{\min} = 0$, we may take

$$V(\phi) = \tfrac{1}{2}\lambda\left(\phi^\dagger\phi - \tfrac{1}{2}v^2\right)^2 \ . \tag{4.33}$$

As a particular ground state which realises the minimum of $V(\phi)$ we choose

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \ , \tag{4.34}$$

where v is real and $v > 0$. All other solutions of the minima condition $\phi^\dagger\phi = \frac{1}{2}v^2$ can be obtained from ϕ_0 by an application of suitable transformations belonging to the symmetry group of V, $SU(2)_T \times U(1)_Y$. In the quantum field theory of course ϕ_0 is the vacuum expectation value of the Higgs doublet. With the choice in eq.(4.34) it is easy to see that

$$(\tfrac{1}{2}\tau_3 + \tfrac{1}{2}1)\phi_0 = Q\phi_0 = 0 \ , \tag{4.35}$$

where the charge Q is defined in general by eq.(4.17). Thus the unbroken subgroup under which the ground state or vacuum is invariant is $U(1)_Q$ generated by Q. The coupling to the Higgs field, as in eq.(4.31) then gives masses to all gauge fields other than that corresponding to the photon.

1.6 The Electro-Weak Lagrangian and the Physical Degrees of Freedom

The physical fields after spontaneous symmetry breakdown may be identified most easily by using a gauge transformation to ensure that the Higgs field is orthogonal to the massless Goldstone boson fields. These Goldstone modes can be regarded as belonging to the coset space $SU(2)_T \times U(1)_Y/U(1)_Q$ and are effectively absorbed into the gauge fields by the gauge transformation. The resulting form for the Higgs field is equivalent to writing in this case

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \rho(x))\begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{4.36}$$

where ρ is a real scalar field which represents the fluctuations of the Higgs field around the ground state value. The choice in eq.(4.36) is equivalent to imposing three gauge conditions on the Higgs field of the form

$$\phi(x)^\dagger \boldsymbol{\tau}\phi_0 - {\phi_0}^\dagger \boldsymbol{\tau}\phi(x) = 0\,, \qquad \phi(x)^\dagger \phi_0 - {\phi_0}^\dagger \phi(x) = 0\,, \tag{4.37}$$

where ϕ_0 is given by eq.(4.34). Although (4.37) contains apparently four linear conditions on ϕ because of eq.(4.35) one is redundant so there remains one real degree of freedom represented by ρ in eq.(4.36). With the definition of the covariant derivative in eq.(4.30) we then find

$$\begin{aligned} D_\mu\phi &= \frac{1}{\sqrt{2}}\begin{pmatrix} -i\,\frac{1}{2}g(v+\rho)(A_{\mu 1} - iA_{\mu 2}) \\ \partial_\mu\rho + i\,\frac{1}{2}(v+\rho)(gA_{\mu 3} - g'B_\mu) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\Big(\partial_\mu\rho + i\,\frac{g}{2\cos\theta_W}(v+\rho)Z_\mu\Big)\begin{pmatrix}0\\1\end{pmatrix} - i\,\tfrac{1}{2}g(v+\rho)W_\mu\begin{pmatrix}1\\0\end{pmatrix}\ , \end{aligned} \tag{4.38}$$

where the Weinberg angle θ_W is defined by

$$\tan\theta_W = \frac{g'}{g}\ , \quad \cos\theta_W = \frac{g}{(g^2+g'^2)^{\frac{1}{2}}}\ , \tag{4.39}$$

and we introduce the linear combinations

$$\begin{aligned} W_\mu &= \tfrac{1}{\sqrt{2}}\,(A_{1\mu} - iA_{2\mu})\ , \\ Z_\mu &= \cos\theta_W A_{3\mu} - \sin\theta_W B_\mu\ . \end{aligned} \tag{4.40}$$

The Higgs Lagrangian which is given by eqs.(4.31,4.33) then becomes

$$\mathcal{L}_{\text{Higgs}} = \tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho + \tfrac{1}{4}g^2(v+\rho)^2\Big(\frac{1}{\cos^2\theta_W}\,\tfrac{1}{2}Z^\mu Z_\mu + W^{\mu\dagger}W_\mu\Big) - \tfrac{1}{8}\lambda(\rho^2+2v\rho)^2\,. \tag{4.41}$$

The field $\rho(x)$ represents the degrees of freedom associated with the Higgs boson whose mass satisfies ${m_\rho}^2 = \lambda v^2$. Actually the corresponding Higgs particle has not been observed experimentally. Its mass is considered to be very large lying between 100 and 200 GeV. The most important aspect of eq.(4.41) for the construction of a viable electro-weak theory is that it generates a mass term for the vector fields W_μ, Z_μ when $\rho \to 0$

$$m_W^2 W^{\mu\dagger}W_\mu + \tfrac{1}{2}\,m_Z^2 Z^\mu Z_\mu\ , \tag{4.42}$$

where

$$m_W^2 = \tfrac{1}{4}\,g^2v^2\ , \qquad m_Z^2 = \tfrac{1}{4}\,(g^2+g'^2)v^2 = \frac{m_W^2}{\cos^2\theta_W}\ . \tag{4.43}$$

The orthogonal combination to Z_μ in eq.(4.40) given by

$$A_\mu = \sin\theta_W A_{3\mu} + \cos\theta_W B_\mu\ , \tag{4.44}$$

has no mass term, i.e. there is no term of the form $\frac{1}{2}A^\mu A_\mu$, and is the gauge field for the unbroken $U(1)_Q$ gauge symmetry. The result in eq.(4.41) is in fact independent of the particular weak hypercharge assignment to the Higgs field ϕ although the definition of the Weinberg angle θ_W would have to be modified from eq.(4.39).

Using the definitions in eqs.(4.40,4.44) we may now decompose the gauge field Lagrangian in eq.(4.28) in terms of the physical gauge fields W_μ, Z_μ, A_μ selected by the mass terms generated by the Higgs field. It is convenient to define

$$F^A{}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \quad F^Z{}_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \ , \tag{4.45}$$

and we may write for $\mathbf{F}_{\mu\nu}$

$$\begin{aligned} F_{\mu\nu 3} &= \sin\theta_W \, F^A{}_{\mu\nu} + \cos\theta_W \, F^Z{}_{\mu\nu} - ig(W_\mu {W_\nu}^\dagger - W_\nu {W_\mu}^\dagger) \ , \\ F^W{}_{\mu\nu} &= \tfrac{1}{\sqrt{2}}(F_{\mu\nu 1} - iF_{\mu\nu 2}) = d_\mu W_\nu - d_\nu W_\mu \ , \\ & \quad d_\mu = \partial_\mu - igA_{\mu 3} = \partial_\mu - ieA_\mu - ig\cos\theta_W \, Z_\mu \ , \end{aligned} \tag{4.46}$$

where we define

$$e = g\sin\theta_W \ . \tag{4.47}$$

We may now rewrite eq.(4.28) in the form

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= -\tfrac{1}{2}F^{W\mu\nu\dagger}F^W{}_{\mu\nu} - \tfrac{1}{4}F^{A\mu\nu}F^A{}_{\mu\nu} - \tfrac{1}{4}F^{Z\mu\nu}F^Z{}_{\mu\nu} \\ & \quad + iW^\mu W^{\nu\dagger}\left(e\,F^A{}_{\mu\nu} + g\cos\theta_W F^Z{}_{\mu\nu}\right) \\ & \quad + \tfrac{1}{2}g^2\left(W^2 W^{\dagger 2} - (W.W^\dagger)^2\right) \ . \end{aligned} \tag{4.48}$$

Since the relations given in eqs.(4.40,4.44) between the gauge fields $A_{\mu 3}, B_\mu$ and the physical fields A_μ, Z_μ, which are the natural basis for the mass terms so that they take the form in eq.(4.42), is just an orthogonal rotation

$$\begin{pmatrix} A_{3\mu} \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \ , \tag{4.49}$$

the quadratic terms in $\mathcal{L}_{\text{gauge}}$ remain diagonal. Clearly the piece $-\frac{1}{4}F^{A\mu\nu}F^A{}_{\mu\nu}$ in $\mathcal{L}_{\text{gauge}}$ represents the usual Lagrangian for the electromagnetic field. There is no coupling between A_μ and Z_μ reflecting that the massive Z particle is neutral, with electric charge zero, like the photon. The complex vector field W_μ is coupled to the electromagnetic gauge field with a coupling e, defined in eq.(4.47), so that the corresponding spin-1 particles in the quantised theory have charge $\pm e$.

To complete the construction of the Lagrangian for the electro-weak theory of the electron and its neutrino it remains only to consider the coupling of the leptons to the Higgs field. If the $T = \frac{1}{2}$ doublet ϕ has weak hypercharge $Y = \frac{1}{2}$ then there is an invariant Yukawa like coupling

$$\mathcal{L}_{\text{lept},\phi}(x) = -\sqrt{2}G_e\left[\bar{L}(x)\phi(x)R(x) + \bar{R}(x)\phi(x)^\dagger L(x)\right] \ . \tag{4.50}$$

With the choice of gauge when the Higgs field takes the form in eq.(4.36) this becomes

$$\mathcal{L}_{\text{lept},\phi} = -G_e(v+\rho)\,[\overline{e}_L e_R + \overline{e}_R e_L] = -G_e(v+\rho)\,\overline{e}e\ . \tag{4.51}$$

In the ground state when $\rho \to 0$ the lepton Lagrangian acquires an effective mass term so that

$$m_e = G_e v\ . \tag{4.52}$$

The mass of the electron is thus determined by the coupling of the Higgs field to the lepton fields and by the vacuum expectation value v of the Higgs field which sets the basic mass scale of the theory. It is important to recognise that the full Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{lept}} + \mathcal{L}_{\text{lept},\phi} + \mathcal{L}_{\text{Higgs}}\ , \tag{4.53}$$

as given by eqs.(4.28,4.27,4.50,4.31,4.33), contains all terms allowed by renormaliseability and $SU(2)_T \times U(1)_Y$ gauge invariance so it is the most general renormaliseable gauge invariant Lagrangian for the fields e, ν_e, ϕ with the assumed representations of $SU(2)_T$ and assignments of weak hypercharge Y.

1.7 Massive Vector Bosons

Neglecting its interactions with the other fields the field Z_μ has a Lagrangian

$$\mathcal{L}_Z = -\tfrac{1}{4}F^{Z\mu\nu}F^{Z}{}_{\mu\nu} + \tfrac{1}{2}m_Z^2 Z^\mu Z_\mu\ . \tag{4.54}$$

The Lagrangian in eq.(4.54) gives rise to the equation of motion

$$\partial^\mu F^{Z}{}_{\mu\nu} + m_Z^2 Z_\nu = \partial^2 Z_\nu - \partial_\nu \partial.Z + m_Z^2 Z_\nu = 0\ . \tag{4.55}$$

Taking the divergence we find at once that

$$m_Z^2\,\partial.Z = 0\ . \tag{4.56}$$

In turn this implies that

$$(\partial^2 + m_Z^2)Z_\nu = 0\ . \tag{4.57}$$

When expressed in terms of annihilation and creation operators for states of definite momentum the vector field becomes

$$Z_\mu(x) = \sum_{p,\lambda}\left(a(p,\lambda)\epsilon_\mu(p,\lambda)e^{-ip.x} + a(p,\lambda)^\dagger \epsilon_\mu(p,\lambda)^* e^{ip.x}\right)\ , \tag{4.58}$$

where the summation is over 4-momenta that satisfy the mass-shell condition

$$p^2 = m_Z^2\ , \tag{4.59}$$

and the λ summation is over labels identifying the three allowed polarization vectors that satisfy

$$p.\epsilon(p,\lambda) = 0 \quad \text{and} \quad \epsilon(p,\lambda)^*.\epsilon(p,\lambda') = -\delta_{\lambda\lambda'}\ . \tag{4.60}$$

If we look at the particle state in its rest frame then $p = (m_Z, 0, 0, 0)$ and $\epsilon(p, \lambda)$ has the form $(0, \boldsymbol{\epsilon}(\lambda))$ where $\{\boldsymbol{\epsilon}(\lambda)\}$ are three orthonormal 3-vectors. By contracting both sides with the linearly independent set $\{p, \epsilon(p, \lambda)\}$ we can verify the useful identity

$$\sum_\lambda \epsilon_\mu(p, \lambda)\epsilon_\nu(p, \lambda)^* = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2} \ . \tag{4.61}$$

After quantisation the annihilation and creation operators satisfy the commutation relations

$$[a(p, \lambda), a(p', \lambda')^\dagger] = \delta_{pp'}\delta_{\lambda\lambda'} \ . \tag{4.62}$$

The field $Z_\mu(x)$ therefore is associated with a *vector* particle.

To perform perturbative calculations in which the vector particle appears on internal lines of a Feynman graph it is necessary to determine its propagator $D_{\mu\nu}(x - y)$. This can be defined by introducing an extra coupling to an external current $j^\mu(x)Z_\mu(x)$ in the the Lagrangian $\mathcal{L}_Z(x)$. The classical equations of motion then become

$$\partial^2 Z^\nu(x) - \partial^\nu \partial.Z(x) + m_Z^2 Z^\nu(x) = -j^\nu(x) \ , \tag{4.63}$$

with the consequence this time that

$$m_Z^2 \partial.Z(x) = \partial.j(x) \ , \quad (\partial^2 + m_Z^2)Z_\mu(x) = -\Big(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{m_Z^2}\Big)j^\nu(x) \ . \tag{4.64}$$

The solution is then written as

$$Z_\mu(x) = \int \mathrm{d}^4y \, D_{\mu\nu}(x - y)j^\nu(y) \ . \tag{4.65}$$

If we use Fourier transforms with Feynman boundary conditions the differential operator is easily inverted giving find

$$D_{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, e^{-ip.(x-y)} \tilde{D}_{\mu\nu}(p) \ , \quad \tilde{D}_{\mu\nu}(p) = \frac{1}{p^2 - m_Z^2 + i\epsilon} \Big(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}\Big) \ . \tag{4.66}$$

In Feynman diagrams the propagator for each internal line corresponding to a virtual neutral vector boson Z_μ is $iD_{\mu\nu}(x - y)$ or in momentum space $i\tilde{D}_{\mu\nu}(p)$.

The charged vector boson field $W_\mu(x)$ can be treated in the same way. The propagator is exactly the same with the mass m_Z replaced by the mass m_W.

For low energy processes, as for weak decays, the mass of the vector boson m_Z or m_W is very large relative to the momentum components $\{p_\mu\}$ and it is appropriate to make the approximation in which the momentum is neglected. In the neutral Z boson case for example

$$\tilde{D}_{\mu\nu}(p) \sim \frac{g_{\mu\nu}}{m_Z^2} \ , \qquad D_{\mu\nu}(x - y) \sim \frac{1}{m_Z^2} g_{\mu\nu}\delta^4(x - y) \ . \tag{4.67}$$

1.8 Interactions between Fields

The most important interactions from the viewpoint of an experimentally successful electro-weak theory are:

i) the charged gauge fields $W_\mu(x), W_\mu(x)^\dagger$ with the leptonic weak currents,

ii) the electromagnetic field $A_\mu(x)$ with the electric current,

iii) the massive neutral vector boson $Z_\mu(x)$ current with the new neutral weak current.

All these couplings arise from the gauge invariant extension of the kinetic part of the leptonic Lagrangian as given in eq.(4.27),

$$\begin{aligned}
\mathcal{L}_{\text{lept.}} &= \mathcal{L}_{\text{kin.}} + g\bar{L}\gamma^\mu \tfrac{1}{2}\boldsymbol{\tau}L.\mathbf{A}_\mu - g'(\tfrac{1}{2}\bar{L}\gamma^\mu L + \bar{R}\gamma^\mu R)\,B_\mu \\
&= \mathcal{L}_{\text{kin.}} + \frac{g}{2\sqrt{2}}(J^\mu W_\mu + J^{\mu\dagger}W_\mu{}^\dagger) + ej^\mu_{\text{e.m.}}A_\mu + \frac{g}{2\cos\theta_W}J^\mu_n Z_\mu \,. \quad (4.68)
\end{aligned}$$

The couplings of the charged vector mesons arise from the terms involving $A_{1\mu}$ and $A_{2\mu}$. Using the definition of W_μ in eq.(4.40) it is easy to see that

$$J^\mu = 2\bar{L}\gamma^\mu\tau_+ L = \overline{\nu}_e\gamma^\mu(1-\gamma_5)e \ , \quad (4.69)$$

using the definition of the lepton doublet L in eq.(4.10) and

$$\tau_+ = \tfrac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ . \quad (4.70)$$

Hence W_μ couples to the $\Delta Q = 1$ weak current for the electron and its associated neutrino with an effective coupling is $g/2\sqrt{2}$. Later this result will be used to relate g to the weak coupling constant G_F.

Using the results in eq.(4.49) for $A_{\mu 3}, B_\mu$ in terms of the electromagnetic gauge field A_μ and the massive neutral vector field Z_μ, and also the definition of e in eq.(4.47), we may find expressions for the electromagnetic current $j^\mu_{\text{e.m.}}$ and also the neutral current J^μ_n. For the former it is easy to obtain

$$j^\mu_{\text{e.m.}} = \bar{L}\gamma^\mu\tfrac{1}{2}(\tau_3 - 1)L - \bar{R}\gamma^\mu R = -\bar{e}\gamma^\mu e \ , \quad (4.71)$$

which is of course the required form for the contribution to the electromagnetic current arising from the electron Dirac field. For the neutral current we can similarly read off the required contributions from the electron, neutrino fields, using eq.(4.39) to eliminate g',

$$\begin{aligned}
J^\mu_n &= \bar{L}\gamma^\mu(\cos^2\theta_W\tau_3 + \sin^2\theta_W 1)L - 2\sin^2\theta_W\bar{R}\gamma^\mu R \\
&= \tfrac{1}{2}\left[\overline{\nu}_e\gamma^\mu(1-\gamma_5)\nu_e - \overline{e}\gamma^\mu(1-\gamma_5 - 4\sin^2\theta_W)e\right] \ . \quad (4.72)
\end{aligned}$$

The neutral current allows the Z to decay into $\overline{\nu}_e \nu_e$ or e^+e^-. Of course the weak neutral current receives similar contributions from muons, τ's and their associated neutrinos which also provide decay channels for the Z.

The Weinberg Salam electro-weak theory predicts the existence of a neutral vector boson with a mass m_Z as well as a charged vector boson with mass m_W. Well after the theory was well established by relatively low energy experiments these particles were discovered experimentally and the latest values for their masses are $m_W = 80.4 \pm 0.2$ GeV and $m_Z = 91.187 \pm 0.007$ GeV. The ratio $m_W/m_Z = 0.8798 \pm 0.0028$ which yields from eq.(4.43) an estimate of the Weinberg angle $\sin^2\theta_W = 0.232$, although at this level of accuracy it is necessary to consider higher order corrections and specify precisely the exact definition of θ_W.

To analyse the theory at low energies we may expand the S operator for the interaction of the W, Z fields with the charged, neutral currents to second order when there is a contribution due to virtual W, Z's with propagators $iD^W_{\mu\nu}, iD^Z_{\mu\nu}$ respectively,

$$\begin{aligned} S &= \mathcal{T}\Big\{ \exp i \int \mathrm{d}^4x \Big(\frac{g}{2\sqrt{2}} (J^\mu W_\mu + J^{\mu\dagger} W_\mu{}^\dagger) + \frac{g}{2\cos\theta_W} J^\mu_n Z_\mu \Big) \Big\} \\ &= \mathcal{T}\Big\{ 1 - i\frac{1}{8} g^2 \int \mathrm{d}^4x \mathrm{d}^4x' \, \Big(J^\mu(x)^\dagger D^W_{\mu\nu}(x-x') J^\nu(x') \\ &\qquad + \frac{1}{\cos^2\theta_W} J^\mu_n(x) D^Z_{\mu\nu}(x-x') J^\nu_n(x') \Big) + \ldots \Big\} , \end{aligned} \tag{4.73}$$

neglecting terms which involve operators which create or destroy W, Z particles. For low energy processes the momenta of the virtual W, Z is small compared with their masses m_W, m_Z and so it is valid to use the approximate form given in eq.(4.67) and correspondingly for $D^W_{\mu\nu}$. Hence we find a low energy effective current-current interaction given by

$$\mathcal{L}_{W\mathrm{eff}} = -\frac{g^2}{8m_W^2} \Big(J^{\mu\dagger} J_\mu + \rho \, J^\mu_n J_{n\mu} \Big) \ , \quad \rho = \frac{m_W^2}{\cos^2\theta_W m_Z^2} \ . \tag{4.74}$$

From the expressions for the W, Z masses and the Weinberg angle in eq.(4.43)

$$\rho = 1 \ . \tag{4.75}$$

This result is a direct consequence of the choice of the $SU(2)_T$ and weak hypercharge quantum numbers of the Higgs field ϕ which gives rise to spontaneous symmetry breakdown. Comparing eq.(4.74) with the phemenonological form for the weak interaction deduced from an analysis of weak decays we find

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} \ . \tag{4.76}$$

From the formula eq.(4.43) for m_W the essential energy scale v set by the vacuum expectation value of the Higgs field in eq.(4.34) can be determined

$$v = \Big(\sqrt{2} G_F \Big)^{-\frac{1}{2}} \approx 250 \text{ GeV}. \tag{4.77}$$

The electro-weak theory is defined by three parameters, the energy scale v and the two couplings g, g' or at low energies the electric charge e, the Fermi constant G_F and the Weinberg angle θ_W. To determine the latter requires experimental investigation of neutral current processes. With the extension to quarks these were found to be in accord with the predictions of the Weinberg Salam model and to give a value for θ_W agreeing with the values of the masses of the W, Z which were later discovered directly.

1.9 Coupling to Quarks

The electro-weak coupling of gauge fields to hadrons is similar to that for leptons when it is assumed that it is sufficient to use a Lagrangian involving the fundamental quark fields although some of the details are more intricate due to the need to incorporate a non zero Cabibbo angle. For the moment we consider a multi-component fermion field ψ which forms a representation of $SU(2)_T \times U(1)_Y$. The gauge invariant coupling to the gauge fields $\mathbf{A}_\mu, B_\mu$ is given by

$$\mathcal{L}_\psi = \overline{\psi} i\gamma^\mu D_\mu \psi \ , \quad D_\mu \psi = (\partial_\mu - ig\mathcal{A}_\mu . \mathbf{T} - ig' B_\mu Y)\, \psi \ , \tag{4.78}$$

where the covariant derivative, as in eq.(4.20), is determined by the matrix generators $\mathbf{T}$ of $SU(2)_T$ and also the hypercharge Y. If we assume that only left-handed fermion fields have non trivial representations of $SU(2)_T$ then we can write

$$\mathbf{T} = \mathbf{T}_L \tfrac{1}{2}(1-\gamma_5) \ , \tag{4.79}$$

while the hypercharge is determined by eq.(4.17)

$$Y = Q - T_3 = Y_L \tfrac{1}{2}(1-\gamma_5) + Y_R \tfrac{1}{2}(1+\gamma_5) \ , \quad Y_L = Q - T_{L3} \, , \quad Y_R = Q \, , \tag{4.80}$$

since Q is purely vector, not involving γ_5. Writing as in eq.(4.68)

$$\mathcal{L}_\psi = \overline{\psi} i\gamma^\mu \partial_\mu \psi + \frac{g}{2\sqrt{2}} \left(J^\mu W_\mu + J^{\mu\dagger} W_\mu{}^\dagger \right) + e\, j^\mu_{\text{e.m.}} A_\mu + \frac{g}{2\cos\theta_W} J^\mu_n Z_\mu \ , \tag{4.81}$$

we may determine the contribution in general for the fermion field ψ to the weak currents. Essentially by construction we have

$$j^\mu_{\text{e.m.}} = \overline{\psi} \gamma^\mu Q \psi \ , \tag{4.82}$$

while for the charged current

$$J^\mu = \overline{\psi} \gamma^\mu (1-\gamma_5) T_{L+} \psi \ , \quad T_{L+} = T_{L1} + iT_{L2} \ . \tag{4.83}$$

Using eq.(4.49) the neutral current then has the general form

$$J^\mu_n = 2J^\mu_3 - 2\sin^2\theta_W j^\mu_{\text{e.m.}} = \overline{\psi} \gamma^\mu \Big((1-\gamma_5) T_{L3} - 2\sin^2\theta_W\, Q \Big) \psi \ . \tag{4.84}$$

In order to reproduce the observed low energy weak interactions of hadrons it would be necessary to assume the left-handed quarks with low mass, u, d, s, form a $T = \frac{1}{2}$ weak doublet of the form

$$\begin{pmatrix} u_L \\ \cos\theta_C\, d_L + \sin\theta_C\, s_L \end{pmatrix} , \tag{4.85}$$

where θ_C is the Cabibbo angle. The right handed quarks are singlets as usual and, in the absence of any other quarks, so must also be the orthogonal combination $-\sin\theta_C d_L + \cos\theta_C s_L$. While this gives the accepted form for the charged weak current

$$J^\mu = \overline{u}\gamma^\mu(1-\gamma_5)(\cos\theta_C d + \sin\theta_C s) , \tag{4.86}$$

it leads to an immediate problem with the neutral current since now

$$J^\mu{}_3 = \tfrac{1}{2}\overline{u_L}\gamma^\mu u_L - \tfrac{1}{2}\left(\cos\theta_C\overline{d_L} + \sin\theta_C\overline{s_L}\right)\gamma^\mu\left(\cos\theta_C d_L + \sin\theta_C s_L\right) . \tag{4.87}$$

This contains terms which, for $\theta_C \neq 0$, give rise to the 'flavour changing' transition $d \leftrightarrow s$ which are strictly forbidden by experiment for neutral current processes. They would lead to decays like $K^0 \to \mu\overline{\mu}$ which would easily have been observed. The resolution of this paradox is the so called GIM mechanism, after Glashow, Iliopoulos and Maiani, which involves the charge $\frac{2}{3}$ charm quark (undiscovered at the time the idea was put forward). The essential assumption is that there are two left handed quark $T = \frac{1}{2}$ doublets

$$\begin{pmatrix} u_L \\ \cos\theta_C\, d_L + \sin\theta_C\, s_L \end{pmatrix} , \quad \begin{pmatrix} c_L \\ -\sin\theta_C\, d_L + \cos\theta_C\, s_L \end{pmatrix} . \tag{4.88}$$

The charged weak current can then be written as

$$J^\mu = \begin{pmatrix} \overline{u} & \overline{c} \end{pmatrix}\gamma^\mu(1-\gamma_5)\begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}\begin{pmatrix} d \\ s \end{pmatrix} . \tag{4.89}$$

It is easy to see that for processes without any charm quarks that this is identical to the experimentally successful form in eq.(4.86). Since $\sin\theta_C$ is small it is also clear that hadrons containing the charm quark, with non zero charm quantum numbers, decay predominantly to strange hadrons. With two $T = \frac{1}{2}$ doublets as in eq.(4.88) we may also see that

$$\begin{aligned} J^\mu{}_3 &= \tfrac{1}{2}\begin{pmatrix} \overline{u_L} & \overline{c_L} \end{pmatrix}\gamma^\mu\begin{pmatrix} u_L \\ c_L \end{pmatrix} \\ &\quad -\tfrac{1}{2}\begin{pmatrix} \overline{d_L} & \overline{s_L} \end{pmatrix}\begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}^{-1}\gamma^\mu\begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}\begin{pmatrix} d_L \\ s_L \end{pmatrix} \end{aligned} \tag{4.90}$$

where it is now evident that the unwanted $d \leftrightarrow s$ terms cancel so that the neutral current is diagonal (the part given by the electromagnetic current is diagonal by construction).

The contribution of the u, d, s, c quarks to the weak neutral current $J_n^\mu = 2J_3^\mu - 2\sin^2\theta_W j^\mu_{\rm e.m.}$ is then clear from eq.(4.90) since the electric current is straightforwardly given by

$$j^\mu_{\rm e.m.} = \tfrac{2}{3}\left(\,\overline{u}\gamma^\mu u + \overline{c}\gamma^\mu c\right) - \mathrm{tr}\left(\,\overline{d}\gamma^\mu d + \overline{s}\gamma^\mu s\right) \ . \tag{4.91}$$

These results imply that the Z boson can decay into hadrons through its coupling to the weak neutral current. The decay rate can be estimated by treating the quarks as free particles even though free quarks do not appear in the final state. The introduction of further generations of quarks requires a more systematic treatment which also shows how the GIM mechanism becomes more natural.

The distinction between the different charge $\frac{2}{3}$ quarks u, c, t and also between the different $-\mathrm{tr}$ charge quarks d, s, b is due to the fact that they have different masses, all other interactions are essentially identical. If the d, s quarks had the same mass then the Cabibbo angle would be without significance since there would be no independent distinction between these quarks. In fact the physical significance of the Cabibbo angle in the weak current depends on assuming that the quark fields are defined so that the mass terms are of the conventional form $\mathcal{L}_m = -\sum_q m_q \overline{q} q$ with m_q all different. Hovever, just as for leptons, there are no possible $SU(2)_T$ or $U(1)_Y$ invariant mass terms so these can only arise from the coupling to the Higgs field through the mechanism of spontaneous symmetry breakdown when the Higgs field gains a vacuum expectation value.

Without further input we can only assume the most general form for the coupling of the $T = \frac{1}{2}, Y = \frac{1}{2}$ Higgs field ϕ to the quark fields. It is important to recognise that from ϕ it is possible to form a conjugate field ϕ^c which transforms under $SU(2)_T \times U(1)_Y$ as a weak iso-doublet with a weak hyper-charge $Y = -\frac{1}{2}$. This is defined by

$$\phi^c = i\tau_2\phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} {\phi_1}^\dagger \\ {\phi_2}^\dagger \end{pmatrix} = \begin{pmatrix} {\phi_2}^\dagger \\ -{\phi_1}^\dagger \end{pmatrix} \ . \tag{4.92}$$

The hyper-charge Y of ϕ^c is obvious and the $SU(2)$ transformation properties follow from the fact that complex conjugation leads to an equivalent $T = \frac{1}{2}$ representation which is given by ϕ^c. This can be seen by using

$$i\tau_2\,\boldsymbol{\tau}^*\,i\tau_2 = \boldsymbol{\tau} \ , \tag{4.93}$$

with the result that

$$i\tau_2\left[e^{\frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}\right]^* = e^{\frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}} i\tau_2 \ . \tag{4.94}$$

Note that the vacuum expectation value of ϕ^c, taking v to be real, from eq.(4.34) is,

$$ {\phi_0}^c = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} \ . \tag{4.95}$$

For many flavours of quarks we assume that all quark fields are assembled in a multi-component column vector ψ. We assume that ψ_L forms a reducible $T=\frac{1}{2}$ representation of $SU(2)_T$ while the components of ψ_R are all $T=0$ singlets. It is convenient to write

$$\psi = \begin{pmatrix} q_+ \\ q_- \end{pmatrix} , \quad q_\pm = \begin{pmatrix} q_{1\pm} \\ \vdots \\ q_{N\pm} \end{pmatrix} , \tag{4.96}$$

for N $T=\frac{1}{2}$ multiplets. In this basis, for $\boldsymbol{\tau}$ the 2×2 Pauli matrices and 1 the $N\times N$, unit matrix

$$\mathcal{T}_L = \tfrac{1}{2}\boldsymbol{\tau} \times 1 \ . \tag{4.97}$$

In this basis we also take

$$Y_L = \tfrac{1}{6}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times 1 \ , \qquad Y_R = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\mathrm{tr} \end{pmatrix} \times 1 \ , \tag{4.98}$$

so that, since $Q = T_3 + Y$ as in eq.(4.17), after spontaneous symmetry breakdown leaving just the massless photon q_- are the charge $-\mathrm{tr}$ quark fields while q_+ are the $\frac{2}{3}$ charge quark fields. It is easy to see that $\overline{\psi_L} q_{-R}$ and $\overline{v}\psi_L q_{+R}$ transform as $T=\frac{1}{2}$ representations with $Y=-\frac{1}{2}$ and $Y=\frac{1}{2}$ respectively. Hence the general gauge invariant expression for the coupling of the Higgs field to the quark fields is of the form

$$\mathcal{L}_{\psi,\phi} = -\sqrt{2}\left(\overline{\psi_L}\Gamma_- q_{-R}\,\phi + \overline{\psi_L}\Gamma_+ q_{+R}\,\phi^c + \text{hermitian conjugate}\right) \ , \tag{4.99}$$

where Γ_-, Γ_+ are complex matrices acting on q_-, q_+. When the Higgs field is replaced by its vacuum expectation values in eqs.(4.34, 4.95) (when the generator of the unbroken $U(1)_Q$ gauge group is given by eq.(4.17)) then this becomes a mass term

$$\mathcal{L}_{\psi,m} = -\left(\overline{\psi_L} m_- q_{-R} + \overline{\psi_L} m_+ q_{+R} + \text{hermitian conjugate}\right) \ , \tag{4.100}$$

where $m_- = \Gamma_- v$ and $m_+ = \Gamma_+ v$ are potentially arbitrary complex mass matrices. Of course there can be no term mixing q_- and q_+ since this would violate charge conservation and $U(1)_Q$ is an unbroken gauge symmetry. In the absence of the mass terms the basis chosen for q_- and q_+ is arbitrary. To select the basis of physical quark fields with definite mass it is necessary to diagonalise the mass matrices. For the physical case of $N=3$ the eigenvalues of ${m_-}^\dagger m_-$ are taken to be m_d^2, m_s^2, m_b^2 while the eigenvalues of ${m_+}^\dagger m_+$ are m_u^2, m_c^2, m_t^2, assuming three generations of quarks. It is a theorem on matrices that m_- can be brought to diagonal form using two unitary matrices L_-, R_-, and similarly for m_+, (to prove this use the result that a hermitian matrix can always be diagonalised by a unitary transformation R to write $Rm^\dagger m R^{-1} = D^2$, where D is diagonal with real positive entries, and then define $mR^{-1}D^{-1} = L^\dagger$ which ensures $LL^\dagger = 1$ so that L is unitary and $LmR^{-1} = D$)

$$L_-m_-R_-^{-1} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad L_+m_+R_+^{-1} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \tag{4.101}$$

where we may require the masses $m_d, m_s, m_b, m_u, m_c, m_t$ to be all positive. We now make a change of basis of the $-\frac{1}{3}$ and $\frac{2}{3}$ charged quarks through separate unitary transformations on the left and right handed quark fields,

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix}_L = L_-q_{-L}\,, \quad \begin{pmatrix} d \\ s \\ b \end{pmatrix}_R = R_-q_{-R}\,, \quad \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L = L_+q_{+L}\,, \quad \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R = R_+q_{+R}\,, \tag{4.102}$$

so that the mass terms in eq.(4.100) are diagonal,

$$\mathcal{L}_{\psi,m} = -\sum_{q=d,s,b,u,c,t} m_q\,\overline{q}q\ . \tag{4.103}$$

It is also easy to see that the kinetic term also remains diagonal

$$\mathcal{L}_{\psi,K} = \overline{\psi}i\gamma.\partial\psi = \sum_{q=d,s,b,u,c,t} \overline{q}i\gamma.\partial q\ . \tag{4.104}$$

Thus $m_d, m_s, m_b, m_u, m_c, m_t$ are the physical masses of the quark fields. Clearly the coupling to the Higgs field can generate arbitrary masses for each quark so there is no understanding of the bizarre mass ratios required for agreement with experiment. However the charged weak current now contains a matrix V in the basis of the physical quark fields which is a generalisation of the orthogonal matrix specified by the Cabibbo angle appearing in eq.(4.89)

$$J^\mu = \overline{q_+}\gamma^\mu(1-\gamma_5)q_- = \begin{pmatrix} \overline{u} & \overline{c} & \overline{t} \end{pmatrix}\gamma^\mu(1-\gamma_5)V\begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad V = L_+L_-^{-1}\,. \tag{4.105}$$

As required experimentally the neutral current also becomes a sum of terms diagonal in the quark fields where, from eq.(4.84),

$$\begin{aligned} J_n^\mu &= \overline{\psi_L}\gamma^\mu\tau_3\psi_L - 2\sin^2\theta_W\overline{\psi}\gamma^\mu Q\psi \\ &= \overline{q_+}\gamma^\mu\tfrac{1}{2}\left(1-\tfrac{8}{3}\sin^2\theta_W-\gamma_5\right)q_+ - \overline{q_-}\gamma^\mu\tfrac{1}{2}\left(1-\tfrac{4}{3}\sin^2\theta_W-\gamma_5\right)q_- \\ &= \sum_{q=u,c,t}\overline{q}\gamma^\mu\tfrac{1}{2}\left(1-\tfrac{8}{3}\sin^2\theta_W-\gamma_5\right)q - \sum_{q=d,s,b}\overline{q}\gamma^\mu\tfrac{1}{2}\left(1-\tfrac{4}{3}\sin^2\theta_W-\gamma_5\right)q\,. \end{aligned} \tag{4.106}$$

The above formalism can be applied to a theory with any number of generations. For N generations the unitary matrices $L_\pm, R_\pm$, and hence V, are $N\times N$

so V contains N^2 parameters. However the N charge $-$tr quarks and the N charge $\frac{2}{3}$ quark fields contain $2N$ unobservable complex phases but the current J^μ is invariant under a common phase transformation on all the $-$tr and $\frac{2}{3}$ charge quark fields, so that $q_\pm \to e^{i\theta} q_\pm$. Thus $2N-1$ complex phases in the matrix V are physically irrelevant leaving N^2-2N+1 parameters in general. If we consider the GIM two generation model then the unitary matrix V can be restricted to a real orthogonal 2×2 matrix depending solely on an angle θ_C when it takes the form

$$V = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} . \tag{4.107}$$

Of course θ_C is the Cabibbo angle which on phemenonological grounds is not zero.

In the realistic case of a three generation model then V has 4 relevant parameters. A real orthogonal 3×3 matrix is determined by 3 angles so in general V must contain a complex phase. There are many ways of choosing the 4 parameters. The first people to construct such a parameterisation were Kobayashi and Maskawa. A version of the CKM matrix is

$$V \equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 + s_2 s_3 e^{i\delta} & c_1 c_2 s_3 - s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 - c_2 s_3 e^{i\delta} & c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix} , \tag{4.108}$$

using the convention that $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ $i = 1,2,3$. The 4 parameters are then the three angles $\theta_1,\ \theta_2,\ \theta_3$ and one complex phase δ. The presence of the phase $\delta \neq 0$ in the matrix shows that in general it cannot be reduced to purely real form for three generations of quarks. This lack of reality corresponds to a breakdown of CP invariance or equivalently of T invariance. In this picture clearly three generations are necessary in order to have CP violation. Assuming that all the angles are small we can identify θ_1 with θ_C. To verify the unitarity of the CKM matrix it is easiest to note that it can be written as a product of three obviously unitary matrices, thus

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix} . \tag{4.109}$$

Independent of any specific parameterisation it is a major experimental challenge to determine the elements of the matrix V, up to the phase arbitrariness $V_{rs} \sim e^{i(\theta_r - \phi_s)} V_{rs}$. At present only $|V_{ud}|, |V_{us}|, |V_{cd}|$ are reasonably well known. Note that from the unitarity of V, $V^\dagger V = I$, there is a condition on the elements in the first and third columns of V of the form

$$V_{ub} V_{ud}^* + V_{cb} V_{cd}^* + V_{tb} V_{td}^* = 0 . \tag{4.110}$$

The three complex numbers in eq.(4.110) form a closed triangle whose area is $\frac{1}{2}|J|$ where

$$J = \mathrm{Im}(V_{cb}V_{cd}^{*}V_{ub}^{*}V_{ud}) \ , \tag{4.111}$$

which is a parameterisation independent measure of the CP violation arising from V, with the form in eq.(4.108) $J = c_1 c_2 c_3 {s_1}^2 s_2 s_3 \sin\delta$. Note that the triangles defined from other pairs or columns of V have the same area.

A similar discussion is possible for the leptons so that in general we may expect a mixing matrix V_{lept} as well as V_{quark}. However if no right handed neutrinos are contained within the lepton sector of the standard model, so that the coupling to the Higgs field has no piece corresponding to the ϕ^c term in eq.(4.99), then the neutrinos are massless. In this case the electron neutrino ν_e can be defined by its coupling to the electron in the charged weak current, and similarly for ν_μ, ν_τ. When right handed neutrino fields are incorporated into the theory there can be non zero neutrino masses and also mixing angles. The weak interaction eigenstates ν_e, ν_μ, ν_τ are no longer mass eigenstates and so after they have been formed in weak decays of hadrons we may expect neutrino oscillations in which there are transitions such as $\nu_e \leftrightarrow \nu_\mu$.

Part V
QCD, perturbative aspects

1 QCD as a non abelian gauge theory

Hadrons, which are particles that undergo strong interactions, can be regarded as composite bound states whose constituents are fractionally charged quarks, $u, d, s, \ldots$, just like nuclei are formed from protons and neutrons. Historically, when quarks were first introduced, they were regarded by many as a convenient fiction which motivated the appearance of particular representations of the approximate symmetry group $SU(3)_F$ with appropriate values of the quantum numbers for P, C and dynamical problems such as the non appearance of free quarks were neglected. The advent of QCD showed how quarks could be described as interacting particles by a fundamental quantum field theory. QCD is a non abelian gauge field theory based on the gauge group $SU(3)_{\text{colour}}$ and the particles corresponding to the gauge field are referred to as gluons, reflecting their role in binding hadrons together. As far as the basic quantum field theory is concerned quarks in QCD appear in a very similar fashion to electrons in QED, quantum electrodynamics, while gluons are analogous to the photon. Nevertheless there are of course very real differences since, unlike electrons and photons, quarks and gluons never appear as physical particles. This phenomenon is resolved by the dogma of confinement which asserts that the dynamics of QCD are such that only $SU(3)_{\text{colour}}$ singlet states are present in the space of finite energy physical states which provides a representation space for the associated quantum field theory. Furthermore there are no massless states except perhaps the pions and associated pseudoscalar particles if the quark masses vanish. These features cannot be described in conventional perturbation theory since the starting point is then a theory of free quarks and free massless gluons with no restrictions on allowed colour quantum numbers but depends on understanding how the non perturbative dynamics of QCD require a new confinement phase which is very different from previous quantum field theories. However in perturbation theory we may show that non abelian gauge theories uniquely, in four space-time dimensions, have the property of asymptotic freedom which justifies the application of perturbation theory calculations to predict quantitatively some measurable aspects of scattering cross sections in suitable contexts, usually some high energy limit. Thus many perturbative calculations in QCD have been carried out, at two or more loops, which allow detailed comparison with experiment so that now QCD has been tested to a high precision. The detailed hadron mass spectrum is outside perturbation theory, since it cannot incorporate confinement, although careful analysis of QCD as a renormaliseable field theory shows how a mass scale can be generated even if no mass parameter is present in the original lagrangian. Any discussion of QCD, after introduction of the basic lagrangian, should therefore start by a consideration of its properties after renormalisation and its dependence on mass scales.

1.1 Basic Lagrangian

A gauge field theory for the non abelian gauge group $SU(3)$ has eight gauge fields $A_{\mu a}$, $a = 1, \dots 8$, and a corresponding field strength

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g\, f_{abc} A_{\mu b} A_{\nu c} \,, \tag{5.1}$$

where g is the coupling and f_{abc} are the totally antisymmetric structure constants of $SU(3)$. For λ_a the Gell-Mann λ-matrices, 3×3 generalisations of the Pauli matrices, we have $[\frac{1}{2}\lambda_a, \frac{1}{2}\lambda_b] = i f_{abc} \frac{1}{2}\lambda_c$. The quark fields belong to the complex 3-dimensional representation of $SU(3)$ defined by the λ-matrices so that the basic QCD lagrangian is simply

$$\mathcal{L}_{\mathrm{QCD}} = -\tfrac{1}{4} F^{\mu\nu}{}_a F_{\mu\nu a} + \sum_f \overline{q}_f (i\gamma^\mu D_\mu - m_f) q_f \,, \tag{5.2}$$

where the covariant derivative is defined by

$$D_\mu q_f = \partial_\mu q_f - i g A_{\mu a} \tfrac{1}{2}\lambda_a q_f \,, \tag{5.3}$$

and colour indices, as well as spinor indices, for the quark fields are suppressed. The sum over f is over the different quark flavours, so that $q_f = u, d, s, \dots$, which are distinguished by their differing masses m_f. This lagrangian is easily seen to be invariant under local $SU(3)$ gauge transformations when infinitesimally

$$\delta A_{\mu a} = \frac{1}{g}(\partial_\mu \xi_a + g f_{abc} A_{\mu b} \xi_c) \,, \qquad \delta q_f = i \xi_a \tfrac{1}{2}\lambda_a q_f \,. \tag{5.4}$$

The quantisation of non abelian gauge theories, such as described by (5.2), is nowadays standard. It is necessary to add extra gauge fixing terms which break the gauge invariance to $\mathcal{L}_{\mathrm{QCD}}$ in order to set up a perturbative expansion starting from a zeroth order free field theory of quarks and gluons. In order to ensure this is done consistently it is necessary also to introduce ghost fields and then amplitudes for physical processes, or matrix elements involving gauge invariant operators, are independent of any gauge fixing parameters and the theory defines a space of physical states, with positive definite norm, invariant under time evolution. The Feynman rules involve quark and gluon propagators, three and four gluon vertices, which are proportional to g and g^2 respectively, and also an $\mathrm{O}(g)$ vertex when a gluon couples to a quark. Furthermore there are ghost propagators which, with standard choices of gauge fixing, couple to other lines in a Feynman graph through a single gluon vertex $\propto g$. The perturbative expansion defines a renormaliseable quantum field theory so that no new parameters, beyond those present in the initial classical lagrangian and gauge fixing terms need be introduced.

1.2 General Features of Renormalisation

QCD is a renormaliseable quantum field theory with a single coupling g. For simplicity we here neglect the quark masses m_f although the treatment can be extended to include them. In practice m_u, m_d are very small compared with typical hadronic scales and naively one would expect that mass terms would be irrelevant in high energy limits when all components of the momenta become large. In general setting mass terms to zero may generate additional infra red divergences in Feynman amplitudes but with appropriate prescriptions, and due caveats to be made clearer later, these can be avoided and the massless limit of perturbative QCD exists. Because of short distance ultra-violet divergences it is necessary to introduce some regularisation for the loop integrals which appear in the perturbative expansion of physical amplitudes. Ideally a convenient regularisation should preserve as many as possible of the general requirements of quantum field theory and also the symmetries of a particular theory (it cannot preserve them all since if that were possible the regularised quantum field theory would itself be a bona fide quantum field theory). Without specifying any details we suppose there is a cut off M which renders Feynman integrals finite and preserves Lorentz invariance, unitarity, etc for energy scales $\ll M$. Any regularisation introduces a mass scale like M even if the original theory has no mass parameters such as QCD in the massless limit (for QCD dimensional regularisation is virtually universally used since this preserves gauge invariance, in this case the regularisation mass scale is more subtle but is present since the coupling g is no longer dimensionless if $d \neq 4$).

Let us now consider some physical amplitude f, which we take to be characterised by a set of momenta p_i, and which has a perturbative expansion so that we may write $f(g, M; p_i)$ where we display explicitly the necessary dependence on the cut off M. The fundamental requirement of renormaliseability, which may be proven order by order in the perturbative expansion, asserts that if we let $g \to g_0(M)$ and, if the overall normalisation of $f(g, M; p_i)$ is not constrained by some identity we also introduce a suitable overall rescaling $Z(M)$ which is independent of the momenta, then we may take the limit $M \to \infty$ so that

$$Z f(g_0, M; p_i) \longrightarrow F(g, \mu; p_i) \quad \text{as} \quad M \to \infty \,, \tag{5.5}$$

where $F(g, \mu; p_i)$ is finite and obeys the general axioms of quantum field theory (more generally Z becomes a matrix corresponding to a set of physical amplitudes such that $Z_{ab} f_b(g_0, M; p_i) \to F_a(g, \mu; p_i)$). The statement (5.5) is valid order by order in a perturbative expansion in the finite coupling parameter g so that

$$g_0\Big(g, \frac{M}{\mu}\Big) = g + \mathrm{O}(g^3) \,, \qquad Z\Big(g, \frac{M}{\mu}\Big) = 1 + \mathrm{O}(g^2) \,, \tag{5.6}$$

are also given as an expansion in g. In (5.5) and (5.6) we have introduced a finite mass scale μ which is essential in order to consistently define g_0, Z and also the renormalised amplitude F. Its appearance is tied up the precise definition of g which is essentially arbitrary other than being required to satisfy (5.6). Any precise definition of g compatible with this is permissible, in dimensional regularisation the standard prescription is termed minimal subtraction when only the poles in $4-d$ are subtracted to define the finite physical amplitude in the limit $d \to 4$. In the present context we may alternatively choose some physical amplitude f_g which has a perturbative expansion

$$f_g(g, M; p_i) = g + \dots\,, \tag{5.7}$$

and where we require that f_g does not need any overall factor such as Z in (5.5) in order to obtain a sensible limit for $M \to \infty$. Then we may define

$$g = \lim_{M\to\infty} f_g(g_0, M; \mathring{p}_i)\,, \tag{5.8}$$

where $\mathring{p}_i$ are an arbitrarily chosen set of momenta specified in terms the arbitrary scale μ, which therefore becomes a variable on which g depends. We may similarly precisely determine Z in (5.5) if $F(g, \mu; p_i)$ is prescribed in some fashion compatible with the lowest order perturbative result for some convenient choice of the momenta p_i in terms of μ. In older discussions the scale μ was usually not introduced explicitly but was essentially replaced by some physical mass, in QCD this could be some quark mass, but then it is impossible to take the zero mass limit. In zero mass QCD the presence of μ is essential to avoid infra red divergences.

Although μ plays an essential role in the definition of finite physical amplitudes its particular value is unimportant. This is reflected in $F(g, \mu; p_i)$ obeying a so called renormalisation group equation reflecting its invariance under any rescaling $\mu \to e^t\mu$. Initially this appears an almost trivial identity but it is in fact a deep consequence of the fundamental property of renormaliseability and leads to significant physical consequences. To derive this we note that, for fixed g_0, M, $f(g_0, M; p_i)$ is independent of μ or

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} f(g_0, M; p_i) = 0\,. \tag{5.9}$$

Using (5.5) this becomes

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}F(g,\mu;p_i) = \Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\Big)F(g,\mu;p_i) = -\gamma(g)F(g,\mu;p_i)\,, \tag{5.10}$$

where, since the total μ derivative is defined for fixed g_0,

$$\beta(g) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g\Big|_{g_0}\,, \qquad \gamma(g) = -\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z\Big|_{g_0}Z^{-1}\,. \tag{5.11}$$

Since F is independent of the cut off M, $\beta(g), \gamma(g)$ must also be independent of M and hence μ, and depend only on g (the definition of $\gamma(g)$ extends to the case when Z and hence $\gamma(g)$ are matrices). The result (5.10) is a version of the renormalisation group equation and is essentially similar in form to the so called Callan-Symanzik equation which has a very similar content (a more systematic derivation entails that the momenta p_i should be restricted so that all linear combinations $p_{i_1} + \ldots + p_{i_n}$ are non zero and also that there are no singularities present in any Lorentz scalar combination).

In a perturbative treatment we expect that $g_0(g, M/\mu)$ has an expansion of the form, dropping any inverse powers of M,

$$g_0 = g + g^3\Big(b \ln \frac{M}{\mu} + a\Big) + g^5\Big(\tfrac{3}{2}b^2\Big(\ln \frac{M}{\mu}\Big)^2 + \ldots\Big) + \mathrm{O}(g^7)\,. \qquad (5.12)$$

Differentiating with respect to μ we find from the definition of the β-function in (5.11)

$$0 = \beta(g)\Big(1 + 3g^2\Big(b \ln \frac{M}{\mu} + a\Big) + \ldots\Big) - g^3 b - 3g^5 b^2 \ln \frac{M}{\mu} + \ldots \qquad (5.13)$$

giving

$$\beta(g) = g^3 b + \mathrm{O}(g^5)\,. \qquad (5.14)$$

It is important to note that consistency determines the $g^5(\ln(M/\mu))^2$ term in (5.12). This property extends to higher orders, all powers of $\ln(M/\mu)$ beyond first order, which determine $\beta(g)$, are fixed by the renormalisation group equation. Similarly we expect to lowest order in the perturbative expansion

$$Z = 1 + g^2\Big(c \ln \frac{M}{\mu} + d\Big) + \mathrm{O}(g^4)\,, \qquad (5.15)$$

which in (5.11) gives

$$\gamma(g) = g^2 c + \mathrm{O}(g^4)\,. \qquad (5.16)$$

The coefficients b, c in (5.12,5.15) are determined by the short distance divergences while a, d depend on the precise definition of g and the conditions which specify the finite part of Z. In general beyond lowest order $\beta(g), \gamma(g)$ are not unique but depend on the choice of renormalisation scheme, different schemes correspond to couplings which are related by a reparameterisation, $g \to g'(g) = g + \mathrm{O}(g^3)$. It is important to use the same scheme for calculations of different processes, such as consistently using dimensional regularisation with minimal subtraction, or to take account of the appropriate redefinition when comparing calculations according to differing regularisation schemes.

1.3 Solution of Renormalisation Group Equation and its Physical Consequences

The renormalisation group equation (5.10) reflects the fact that the overall scale for μ is immaterial. This becomes evident from its explicit solution. To obtain this we first recast (5.10) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}F(g_t, e^t\mu; p_i) = -\gamma(g_t)F(g_t, e^t\mu; p_i)\,, \tag{5.17}$$

where g_t is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}g_t = \beta(g_t)\,, \quad g_t = g \ \text{ for } \ t = 0\,. \tag{5.18}$$

Eq. (5.17) is an ordinary differential equation which is readily integrated to give

$$F(g_t, e^t\mu; p_i) = e^{-\rho(t)}F(g, \mu; p_i) \quad \text{for} \quad \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \gamma(g_t)\,, \ \rho(0) = 0\,. \tag{5.19}$$

The solution for g_t in (5.18) and $\rho(t)$ in (5.19) may alternatively be given by

$$\int_g^{g_t} \mathrm{d}x\, \frac{1}{\beta(x)} = t\,, \qquad \rho(t) = \int_0^t \mathrm{d}s\, \gamma(g_s) = \int_g^{g_t} \mathrm{d}x\, \frac{\gamma(x)}{\beta(x)}\,. \tag{5.20}$$

These equations have physical content if we assume the dimensional scaling rule, since apart from the momenta μ provides the only scale,

$$F(g, \mu; e^t p_i) = e^{pt}F(g, e^{-t}\mu; p_i)\,, \tag{5.21}$$

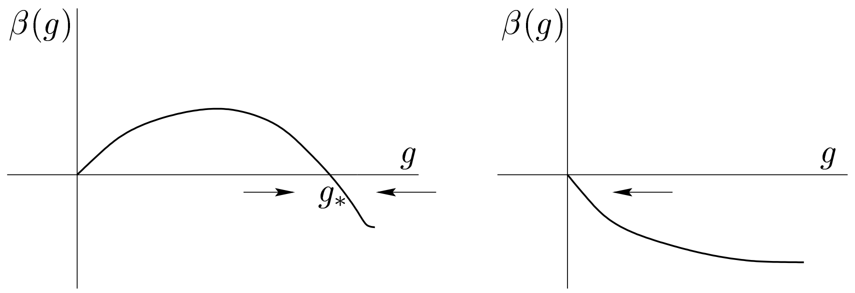
where p is the scaling dimension of F (in mass units). Combining (5.21) with (5.19) for $\mu \to e^{-t}\mu$ we find

$$F(g, \mu; e^t p_i) = e^{pt+\rho(t)}F(g_t, \mu; p_i)\,. \tag{5.22}$$

This result makes clear that the behaviour of $F(g, \mu; p_i)$ when all momenta become large simultaneously and the neglect of mass terms is justified, or in (5.22) $t \to \infty$, is controlled by the properties of g_t and also $\rho(t)$, which are defined by the solutions to (5.18) or (5.20), as $t \to \infty$. Conversely for a massless quantum field theory the behaviour in the infra red limit of small momenta is given in terms of the limit $t \to -\infty$.

The main features of these limits depends only on the qualitative form of $\beta(g)$. Although analysis of the infra red limit in terms of the renormalisation group equation is also of vital theoretical importance, especially in the context of statistical physics, we here concentrate on the high energy ultra violet limit. The justification of the renormalisation group equations depend on perturbation theory, at least for equations of the form (5.10), but they are assumed to transcend such limitations so that $\beta(g)$, and also $\gamma(g)$, are presumed to be general functions of g with only the first few terms in an expansion at $g = 0$ known (the perturbative expansion is at best asymptotic and there may well be non perturbative contributions).

The possibilities for the behaviour of g_t as t increases is shown below



Beta functions and renormalisation flow for non zero fixed point and also for asymptotically free theories

In the first case the β-function has a zero such that

$$\beta(g_*) = 0\,, \qquad \beta'(g_*) < 0\,. \tag{5.23}$$

Then if g is in some neighbourhood of g_*, so that for $g > g_*$, $\beta(g) < 0$ and for $g < g_*$, $\beta(g) > 0$, then it is easy to see that solution of (5.18) requires

$$g_t \longrightarrow g_* \quad \text{as} \quad t \to \infty\,. \tag{5.24}$$

In this situation g_* is referred to as an ultra-violet fixed point. From (5.20) we may expect

$$\rho(t) \sim \gamma(g_*)t\,, \tag{5.25}$$

assuming that $\gamma(g_*) \neq 0$. Hence from (5.22) we then have

$$F(g,\mu;\lambda p_i) \sim \lambda^{p+\gamma(g_*)} F(g_*,\mu;p_i) \quad \text{as} \quad \lambda \to \infty\,, \tag{5.26}$$

with $F(g_*,\mu;\lambda p_i) = \lambda^{p+\gamma(g_*)} F(g_*,\mu;p_i)$. This result represents an exact scaling relation with $\gamma(g_*)$ corresponding to an anomalous dimension.

Such ultra-violet fixed points, while theoretically feasible, are beyond the scope of any perturbative analysis. Another relevant possibility is if

$$\beta(0) = 0\,, \qquad \beta(g) < 0 \ \text{ for } g \text{ in some neighbourhood of } g = 0\,. \tag{5.27}$$

For an initial g in this region we then have

$$g_t \longrightarrow 0 \quad \text{as} \quad t \to \infty\,, \tag{5.28}$$

so that the origin is an ultra-violet fixed point. Such a circumstance is called asymptotic freedom and it provides a justification for the validity of using perturbation theory for high energy processes. Whether the behaviour in (5.27) holds may readily be found from perturbative calculations, in (5.14) it just corresponds to $b < 0$. If the $\mathrm{O}(g^5)$ terms in (5.14) are neglected, as is appropriate if g is small, then the differential equation in (5.18) may be written explicitly in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{g_t^2} = -2b\,, \tag{5.29}$$

which can easily be solved to give

$$\frac{1}{g_t^2} - \frac{1}{g^2} = -2bt\,. \tag{5.30}$$

With $b < 0$ then clearly $g_t^2 \sim 1/(2|b|t) \to 0$ as $t \to \infty$.

If we take the lowest order result for $\gamma(g)$ in (5.16) in conjunction with that for $\beta(g)$ then from (5.20) we may easily find

$$\rho(t) = -\frac{c}{b}\,\ln\frac{g}{g_t}\,. \tag{5.31}$$

To apply this we may also assume that to lowest order in perturbation theory the amplitude $F(g,\mu;p_i)$ has the form

$$F(g,\mu;p_i) \sim g^N F_0(p_i)\,, \tag{5.32}$$

where from (5.21) $F_0(\lambda p_i) = \lambda^p F_0(p_i)$. Using this, as well as (5.31), in (5.22) we find

$$F(g,\mu;e^t p_i) \sim g_t^N \left(\frac{g_t}{g}\right)^{\frac{c}{b}} F_0(e^t p_i)\,. \tag{5.33}$$

This result, which depends only on lowest order perturbative calculations, becomes asymptotically exact as $t \to \infty$ in the asymptotic freedom case, i.e. $b < 0$, so that the form of $F(g,\mu;p_i)$ when all components of all momenta become large simultaneously is thereby determined.

The general solution of (5.18) $g_t \equiv g(e^t\mu)$ defines a running coupling constant which no longer has a fixed value, $g(\mu)$ depends on the arbitrary scale μ. The lowest order solution (5.30) may be rewritten as

$$\frac{1}{g(\mu)^2} - \frac{1}{g(\mu')^2} = -b\ln\frac{\mu^2}{\mu'^2}\,. \tag{5.34}$$

Any measurable physical amplitude must be independent of μ but the detailed form of $g(\mu)$ may be exploited to justify and extend the scope of a perturbative analysis beyond its initially apparent region of validity.

1.4 β-function in Non Abelian Gauge Theories

In any renormaliseable quantum field theory it is straightforward to calculate the β-function to one or two, or sometimes more, loops. For a non abelian gauge theory, with a simple gauge group so that there is a single gauge coupling g, the corresponding β-function may be writtens as

$$\beta(g) = -\beta_0 \, \frac{g^3}{16\pi^2} + \mathrm{O}(g^5) \, . \tag{5.35}$$

We suppose that, as in QCD, the gauge field is only coupled to fermion fields ψ_i through covariant derivatives $D_\mu \psi_i = \partial_\mu \psi_i - igA_{\mu a} t_{ia} \psi_i$, where t_{ia} are matrix generators of the Lie algebra of the gauge group for the irreducible representation defined by ψ_i, $[t_{ia}, t_{ib}] = if_{abc} t_{ic}$. In this case the general formula for β_0 (this assumes that the gauge field coupling does not distinguish between left and right handed fermions, there is no γ_5 involved) is

$$\beta_0 = \tfrac{11}{3}\, C - \tfrac{4}{3} \sum_f T_f \, , \tag{5.36}$$

where C, T_f are group theory factors defined by

$$f_{acd} f_{bcd} = C \delta_{ab} \, , \qquad \mathrm{tr}(t_{fa} t_{fb}) = T_f \delta_{ab} \, . \tag{5.37}$$

For gauge group $SU(N)$ then $C = N$ while if the fermions are in the fundamental representation, as are the quark fields for $SU(3)_{\mathrm{colour}}$, then $T_f = \frac{1}{2}$. For QCD the formula therefore becomes

$$\beta_0 = 11 - \tfrac{2}{3} N_{\mathrm{fl}} \, , \tag{5.38}$$

where N_{fl} is the number of quark flavours which contribute to the β-function, $N_{\mathrm{fl}} = 3$ for the light u, d, s quarks while $N_{\mathrm{fl}} = 4$ if the c quark is added as well (in determining the running coupling $g(\mu)$ those quarks with masses $\gtrsim \mu$ should not contribute to the β-function). Clearly from (5.38) $\beta_0 > 0$, so that we have asymptotic freedom, since the total number of flavours is just $N_{\mathrm{fl}} = 6$.

It is convenient to define for the QCD coupling, like in QED,

$$\alpha_s = \frac{g^2}{4\pi} \, , \tag{5.39}$$

and then the lowest order solution (5.30) of (5.18), which may alternatively be written as

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g = \beta(g) \, , \tag{5.40}$$

for the running coupling $g(\mu)$ can be equivalently expressed as

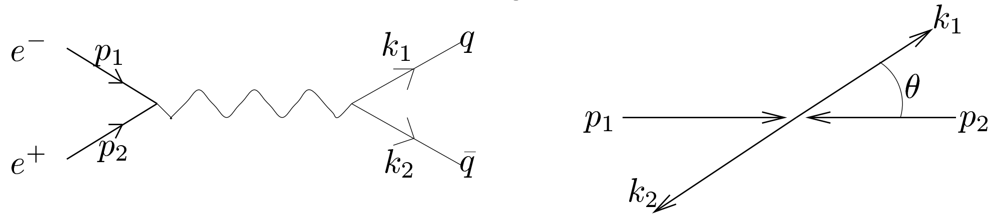
$$\frac{1}{\alpha_s(\mu^2)} = \frac{\beta_0}{4\pi} \ln \frac{\mu^2}{\Lambda^2} \, , \tag{5.41}$$

where Λ, which may be regarded as a constant of integration in the solution of (5.40), provides a basic QCD mass scale even in the absence of any quark masses. Λ is in essence the fundamental strong interaction mass scale which replaces the coupling g or α_s as a free parameter. The result (5.41) shows clearly how $\alpha_s(\mu^2) \to 0$ as $\mu \to \infty$, although Λ cannot really be determined precisely from (5.41) since any rescaling of Λ modifies the result by terms $\mathrm{O}((\ln \mu^2/\Lambda^2)^{-2})$ which are of the same order as contributions from higher orders in $\beta(g)$. Such rescalings of Λ result from redefinitions of the coupling corresponding to different regularisation schemes, so that the precise value of Λ has significance only in the context of a particular scheme. In a modified minimal subtraction scheme $\Lambda \approx 200 - 250$ MeV, although the uncertainty is quite large.

2 $e^-e^+ \to$ hadrons

In many ways the cleanest application of asymptotic freedom in QCD is to the total cross section for $e^-(p_1) + e^+(p_2) \to$ hadrons. To lowest order in the electromagnetic coupling e the e^-e^+ annihilate to produce a virtual photon, with momentum $q = p_1 + p_2$, which then forms physical hadron states.

As a precursor to discussion of this we consider $e^-(p_1)+e^+(p_2) \to q(k_1)+\overline{q}(k_2)$ for $q, \overline{q}$ free quarks, anti-quarks with charges $\pm Q$, in units of e, and $k_1 + k_2 = q$.



Electron positron annihilation to quark, antiquark

The amplitude is then simply

$$i\mathcal{M} = (-ie)^2 Q\, \overline{u}(k_1)\gamma^\mu v(k_2)\, i\frac{-g_{\mu\nu}}{q^2}\, \overline{v}(p_2)\gamma^\nu u(p_1)\,. \tag{5.42}$$

The sum over $e^\mp$ and $q, \overline{q}$ spins in $|\mathcal{M}|^2$ can be converted to Dirac traces in the conventional fashion, neglecting electron and quark masses, so that ${p_1}^2 = {p_2}^2 = {k_1}^2 = {k_2}^2 = 0$, we get

$$\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}|^2 &= e^4 \frac{Q^2}{(q^2)^2}\, \mathrm{tr}(\gamma{\cdot}k_1\, \gamma^\mu\, \gamma{\cdot}k_2\, \gamma^\nu)\, \mathrm{tr}(\gamma{\cdot}p_1\, \gamma_\mu\, \gamma{\cdot}p_2\, \gamma_\nu) \\
&= 16\, e^4 \frac{Q^2}{(q^2)^2}\, ({k_1}^\mu {k_2}^\nu + {k_1}^\nu {k_2}^\mu - g^{\mu\nu} k_1{\cdot}k_2)\, (p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - g_{\mu\nu}p_1{\cdot}p_2) \\
&= 32\, e^4 \frac{Q^2}{(q^2)^2}\, (p_1{\cdot}k_1\, p_2{\cdot}k_2 + p_2{\cdot}k_1\, p_1{\cdot}k_2)\,. \qquad (5.43)
\end{aligned}$$

In the centre of mass frame ${p_1}^\mu = (|\mathbf{p}|, \mathbf{p})$, ${p_2}^\mu = (|\mathbf{p}|, -\mathbf{p})$ so that $q^\mu = (\sqrt{q^2}, \mathbf{0})$ with $\sqrt{q^2} = 2|\mathbf{p}|$. The quark momenta k_1, k_2 may be represented similarly in terms of $\mathbf{k}$ with $|\mathbf{k}| = |\mathbf{p}|$. Assuming a scattering angle θ, so that $\mathbf{k}{\cdot}\mathbf{p} = \frac{1}{4}q^2\cos\theta$, we have

$$p_1{\cdot}k_1 = p_2{\cdot}k_2 = \tfrac{1}{4}q^2(1-\cos\theta)\,, \qquad p_2{\cdot}k_1 = p_1{\cdot}k_2 = \tfrac{1}{4}q^2(1+\cos\theta)\,, \tag{5.44}$$

and hence

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 4e^4 Q^2 (1+\cos^2\theta)\,. \tag{5.45}$$

The formula for the differential cross section in this case becomes

$$\mathrm{d}\sigma = \frac{1}{F}\,\frac{\mathrm{d}^3 k_1}{(2\pi)^3 2{k_1}^0}\,\frac{\mathrm{d}^3 k_2}{(2\pi)^3 2{k_2}^0}\,(2\pi)^4\delta^4(q-k_1-k_2)\,\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\,, \tag{5.46}$$

where F is the flux factor for the initial states and the factor of $\frac{1}{4}$ is for averaging over the initial spins. With standard normalisations consistent with $\mathcal{M}$ in (5.42) we have

$$F = 4{p_1}^0{p_2}^0 v = 2q^2\,, \qquad v = |\mathbf{v}_1 - \mathbf{v}_2|\,, \tag{5.47}$$

since the relative speed $v = 2$ in the C.M. frame neglecting electron masses. Substituting this and (5.45) into the cross section formula (5.46) we have

$$\mathrm{d}\sigma = \frac{e^4 Q^2}{2(2\pi)^2(q^2)^2}\,\mathrm{d}^3 k\,\delta\Big(\sqrt{q^2} - 2|\mathbf{k}|\Big)\,(1+\cos^2\theta)\,. \tag{5.48}$$

Since $\mathrm{d}^3 k = |\mathbf{k}|^2\mathrm{d}|\mathbf{k}|\,\mathrm{d}\Omega$, where $\mathrm{d}\Omega$ is the solid angle element for the direction $\hat{\mathbf{k}}$, we have finally for the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4q^2}\,Q^2\,(1+\cos^2\theta)\,, \tag{5.49}$$

where $\alpha = e^2/4\pi$. It is easy to integrate this to find the total cross section

$$\sigma_{\mathrm{tot},\,e^-e^+\to q\overline{q}} = \frac{4\pi\alpha^2}{3q^2}\,Q^2\,, \tag{5.50}$$

which would be valid for $\sqrt{q^2} \gg m_q$.

It is not immediately obvious how the above calculation, which assumes free quarks, applies to the experimental observed process $e^-e^+ \longrightarrow$ hadrons. To show the relevance of the result (5.50) to the total cross section for $e^-e^+ \longrightarrow$ hadrons we first derive a general formula for this, to lowest order in e. For a final hadronic state X the amplitude is

$$\mathcal{M}_X = e^2\,\frac{1}{q^2}\,\langle X|J_{\mathrm{h}}^{\mu}|0\rangle\,\overline{v}(p_2)\gamma_\mu u(p_1)\,, \tag{5.51}$$

where J_{h}^{μ} is the hadronic contribution to the electromagnetic current which may be expressed in term of quark fields by

$$J_{\mathrm{h}}^{\mu} = \overline{q}\gamma^\mu Q q\,, \tag{5.52}$$

for Q the diagonal matrix of quark charges. Extending (5.46) to this case gives, for $q = p_1 + p_2$,

$$\sigma_{\mathrm{tot},\, e^-e^+\to\mathrm{hadrons}} = \frac{1}{F}\sum_X \frac{1}{4}\sum_{\mathrm{spins}} (2\pi)^4 \delta^4(q - p_X)\, |\mathcal{M}_X|^2 \,. \qquad (5.53)$$

The sum over hadronic states X may be subsumed in a single function of q^2 by virtue of

$$\begin{aligned} \rho_{\mathrm{h}}^{\nu\mu}(q) &\equiv (2\pi)^3 \sum_X \delta^4(q-p_X) \langle 0| J_{\mathrm{h}}^\nu |X\rangle \, \langle X| J_{\mathrm{h}}^\mu |0\rangle \\ &= (-g^{\nu\mu} q^2 + q^\nu q^\mu)\, \theta(q^0)\, \rho_{\mathrm{h}}(q^2)\,, \end{aligned} \qquad (5.54)$$

using Lorentz invariance and current conservation, $\partial_\mu J_{\mathrm{h}}^\mu = 0$. In consequence, neglecting the electron mass,

$$\sum_X \sum_{\mathrm{spins}} (2\pi)^4 \delta^4(q-p_X)\, |\mathcal{M}_X|^2 = \frac{8\pi e^4}{(q^2)^2} (q^2\, p_1{\cdot}p_2 + 2 q{\cdot}p_1\, q{\cdot}p_2) \rho_{\mathrm{h}}(q^2) = 8\pi e^4\, \rho_{\mathrm{h}}(q^2)\,, \qquad (5.55)$$

and hence, with the result (5.47) for the flux factor F,

$$\sigma_{\mathrm{tot},\, e^-e^+\to\mathrm{hadrons}} = \pi e^4 \, \frac{1}{q^2}\, \rho_{\mathrm{h}}(q^2)\,. \qquad (5.56)$$

It remains to understand $\rho_{\mathrm{h}}(q^2)$ which for general $q^2 > 0$ is potentially very non trivial. If we once again consider the quark fields composing J_{h}^μ in (5.52) to be free, and restrict $\sum_X$ to a sum over $q, \overline{q}$ states, then

$$\begin{aligned} \rho_{\mathrm{h}}^{\nu\mu}(q) &= \sum_f Q_f^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0} \, \frac{\mathrm{d}^3 k'}{(2\pi)^3 2k'^0}\, (2\pi)^4 \delta^4(q-k-k') \\ &\qquad \times \mathrm{tr}\Big((\gamma{\cdot}k + m_f)\gamma^\mu (\gamma{\cdot}k' - m_f)\gamma^\nu \Big)\Big|_{k^2=k'^2=m_f^2} \,. \end{aligned} \qquad (5.57)$$

The phase space integral can be evaluated with the aid of

$$\begin{aligned} &\int \frac{\mathrm{d}^3 k}{k^0}\, \frac{\mathrm{d}^3 k'}{k'^0}\, \delta^4(q-k-k')\, k^\mu k'^\nu \Big|_{k^2=k'^2=m_f^2} \\ &= \theta(q^0)\theta(q^2 - 4m_f^2)\, \pi \Big(1 - \frac{4m_f^2}{q^2}\Big)^{\frac{1}{2}} \Big\{ \frac{q^2 + 2m_f^2}{3q^2} (-g^{\nu\mu} q^2 + q^\nu q^\mu) + \tfrac{1}{2}\, g^{\nu\mu} q^2 \Big\} \end{aligned} \qquad (5.58)$$

Hence (5.54) and (5.57) give

$$\rho_{\mathrm{h}}(q^2) = \frac{1}{12\pi^2} \sum_f Q_f^2\, \theta(q^2 - 4m_f^2)\, \frac{q^2 + 2m_f^2}{q^2} \Big(1 - \frac{4m_f^2}{q^2}\Big)^{\frac{1}{2}} . \qquad (5.59)$$

It is easy to see that for $q^2 \gg m_f^2$ (5.56) and (5.59) are compatible with (5.50).

The application of QCD is based implicitly on the postulate, whose justification is essentially beyond the scope of perturbation theory, that

$$\sum_{X=\text{hadrons}} |X\rangle\langle X| = \sum_{X=q,\overline{q},g\,\text{states}} |X\rangle\langle X| \,, \tag{5.60}$$

at least in application to high energy processes. If we restrict $\sum_f$ to those quarks such that $q^2 \gg m_f^2$ then the assumption (5.60) leads to

$$\rho_{\text{h}}(q^2) = \frac{1}{12\pi^2}\Big(N_{\text{col}} \sum_f Q_f^2 \, R\Big(\frac{q^2}{\mu^2}, \alpha_s\Big) + \Big(\sum_f Q_f\Big)^2 S\Big(\frac{q^2}{\mu^2}, \alpha_s\Big)\Big) \,, \tag{5.61}$$

or, with N_{col} the number of colours (3 for QCD),

$$\sigma_{\text{tot},\, e^-e^+\to\text{hadrons}} = \frac{4\pi\alpha^2}{3q^2}\Big(N_{\text{col}} \sum_f Q_f^2 \, R\Big(\frac{q^2}{\mu^2}, \alpha_s\Big) + \Big(\sum_f Q_f\Big)^2 S\Big(\frac{q^2}{\mu^2}, \alpha_s\Big)\Big) \,, \tag{5.62}$$

where we have set the quark masses to zero, $m_f = 0$, in which case each quark contributes identically and and the result depends only on two functions of q^2/μ^2 and α_s which may be calculated in terms of a Feynman diagram expansion. The zero mass limit is well defined in perturbation theory so long as we introduce the arbitrary scale μ and then in (5.61) and (5.62) $\alpha_s \to \alpha_s(\mu^2)$, the QCD running coupling. Since we have factored off explicitly the number of colours N_{col} it is evident from the above results for free quarks that

$$R(x, 0) = 1 \,. \tag{5.63}$$

Furthermore by drawing Feynman diagrams (or considering states such that $X =$ 3 gluons) we also have

$$S(x, \alpha_s) = \mathrm{O}({\alpha_s}^3) \,. \tag{5.64}$$

In calculating R and also S no overall factor like Z in (5.5) is necessary to remove divergences when the limit $M \to \infty$ for the cut off is taken, since the conserved current J_{h}^μ does not require independent renormalisation. Assuming R satisfies a renormalisation group equation of the form (5.22), reflecting the arbitrariness in the scale μ, we then have

$$R\Big(\frac{q^2}{\mu^2}, \alpha_s(\mu^2)\Big) = R\Big(1, \alpha_s(q^2)\Big) \sim 1 + \frac{\alpha_s(q^2)}{\pi} \quad \text{as} \quad q^2 \to \infty \,, \tag{5.65}$$

where the first QCD correction to (5.63) has been exhibited. From (5.64) it is similarly clear that S is asymptotically unimportant in the large q^2 limit.

In principle the result (5.65) allows the coupling α_s to be determined by comparing with experimental results for $\sigma_{\text{tot},\, e^-e^+\to\text{hadrons}}$ at high energies although in practice this is difficult to achieve accurately.

2.1 Space-like and Time-like Asymptotic Limits

Theoretically the application of the renormalisation group to $\rho_\mathrm{h}(q^2)$ and neglect of mass terms, which was assumed in obtaining (5.61) and (5.62), is not really justified. $\rho_\mathrm{h}(q^2)$ is not analytic and has potential discontinuities in q^2 whenever the threshold for producing new states X in (5.54) is achieved and also has peaks when resonances with the appropriate energy are present. To overcome such difficulties we consider first the Feynman amplitude

$$i\int \mathrm{d}^4x\, e^{iq\cdot x}\langle 0|J_\mathrm{h}^\nu(x)J_\mathrm{h}^\mu(0)|0\rangle = (-g^{\nu\mu}q^2+q^\nu q^\mu)\,\Pi_\mathrm{h}(q^2)\,, \tag{5.66}$$

which can be calculated in terms of the contributions of Feynman graphs. $\Pi_\mathrm{h}(q^2)$ is related to $\rho_\mathrm{h}(q^2)$ by the Lehmann representation

$$\Pi_\mathrm{h}(q^2) = \int_0^\infty \mathrm{d}s\,\frac{\rho_\mathrm{h}(s)}{s-q^2-i\epsilon}\,, \tag{5.67}$$

so $\Pi_\mathrm{h}(s)$ may be extended to an analytic function of s throughout the complex s-plane except for a cut along the positive real axis. For s real and negative the zero mass limit may be justified order by order in the perturbation expansion and we may apply renormalisation group methods to the limit $-s\to\infty$. From (5.67) $\rho_\mathrm{h}(s)$ for $s>0$, which is directly measureable in e^-e^+-scattering, can be related to $\Pi_\mathrm{h}(s)$ by

$$\rho_\mathrm{h}(s) = \frac{1}{2\pi i}\Big(\Pi_\mathrm{h}(s+i\epsilon)-\Pi_\mathrm{h}(s-i\epsilon)\Big) = -\frac{1}{2\pi i}\int_C \mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\Pi_\mathrm{h}(x)\,, \tag{5.68}$$

where C is a contour from $s+i\epsilon$ to $s-i\epsilon$ around the branch point at $x=0$. It is convenient to define

$$D(-s) = -s\frac{\mathrm{d}}{\mathrm{d}s}\Pi_\mathrm{h}(s)\,, \tag{5.69}$$

and then (5.68) becomes

$$\rho_\mathrm{h}(s) = \frac{1}{2\pi i}\int_C \frac{\mathrm{d}x}{x}\,D(-x) = \frac{1}{2\pi}\int_{-\pi+\epsilon}^{\pi-\epsilon}\mathrm{d}\theta\,D(se^{i\theta})\,, \tag{5.70}$$

choosing for C the circular contour $x=-se^{i\theta}$, $-\pi+\epsilon<\theta<\pi-\epsilon$.

The result (5.70) allows the asymptotic result for $D(s)$ in the space-like limit $s\to\infty$ where $D(s)$ is analytic, assuming it to be also valid for $|s|\to\infty$ with $-\pi+\epsilon<\arg(s)<\pi-\epsilon$ (which need not be true as $\epsilon\to 0$ reflecting the fact that $\rho_\mathrm{h}(s)$ need not be a smooth function), to be applied to derive the asymptotic behaviour of $\rho_\mathrm{h}(s)$. From the renormalisation group we get

$$D(s) \sim D_0\Big(1+\frac{\alpha_s(s)}{\pi}+d_2\,\alpha_s(s)^2+\dots\Big)\,. \tag{5.71}$$

From the leading behaviour (5.41) of the running coupling the relevant integral in (5.70) is of the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \, \frac{1}{\ln \frac{se^{i\theta}}{\Lambda^2}} = \frac{1}{\pi} \tan^{-1} \frac{\pi}{\ln \frac{s}{\Lambda^2}} \sim \frac{1}{\ln \frac{s}{\Lambda^2}} - \frac{\pi^2}{3\left(\ln \frac{s}{\Lambda^2}\right)^3} + \dots \,. \tag{5.72}$$

Hence the asymptotic expansion of $\rho_{\mathrm{h}}(s)$ differs from $D(s)$, as in (5.71), at order ${\alpha_s}^3$.

3 Deep Inelastic Scattering

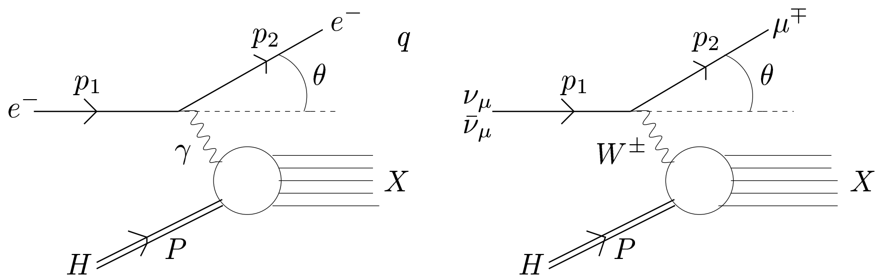
The most detailed results which are described by perturbative QCD are found in deep inelastic scattering for proton or neutron targets. Historically experimental results for deep inelastic scattering were interpretated in terms of point-like free constituents or partons. The parton model predicted Bjorken scaling, at high energies cross sections depended on functions of particular dimensionless variables, and led to many relations which agreed with experiment if the partons were identified with fractionally charged quarks. Subsequently QCD provided a justification for the assumptions made in the parton model, which was essentially a zeroth approximation to the full QCD result, and gave calculable corrections to Bjorken scaling.

3.1 Kinematics

The essential process for deep inelastic electron scattering on a hadron H, of mass M, is

$$e(p) + H(P) \longrightarrow e(p') + X \,, \tag{5.73}$$

where X is an unobserved final state. To lowest order in e the electron couples to the hadron through a virtual photon,



Deep inelastic electron, neutrino scattering on a hadron

The amplitude is

$$i\mathcal{M} = (ie)^2 \overline{u}(p')\gamma^\mu u(p) \, i\frac{-g_{\mu\nu}}{q^2} \langle X|J_{\mathrm{h}}^\nu|H,P\rangle \,, \quad q = p - p' \,. \tag{5.74}$$

In the hadron rest frame $P^\mu = (M, \mathbf{0})$, $p^\mu = (E, \mathbf{p})$ and $p'^\mu = (E', \mathbf{p}')$ and the basic dynamical variables are

$$\nu \equiv P{\cdot}q = M(E - E')\,, \qquad Q^2 \equiv -q^2 = 2p{\cdot}p' = 2EE'(1-\cos\theta)\,, \tag{5.75}$$

where we have neglected the electron mass, so that $E = |\mathbf{p}|$, $E' = |\mathbf{p}'|$, and θ is the electron scattering angle. Clearly $Q^2 \geq 0$ and also

$$M_X^2 = (P+q)^2 \geq M^2 \quad \Rightarrow \quad Q^2 \leq 2\nu\,. \tag{5.76}$$

The standard expression for the differential cross section gives

$$\mathrm{d}\sigma = \frac{1}{F}\,\frac{\mathrm{d}^3 p'}{(2\pi)^3 2p'^0}\,\sum_X (2\pi)^4 \delta^4(q+P-p_X)\,\tfrac{1}{2}\sum_{e\ \text{spins}} |\mathcal{M}|^2\,, \tag{5.77}$$

where F is the flux factor

$$F = 4p^0 P^0 |\mathbf{v}_e - \mathbf{v}_H| \to 4EM\,, \tag{5.78}$$

in the hadron rest frame. It is easy to see from (5.74) that

$$\sum_{e\ \text{spins}} |\mathcal{M}|^2 = \frac{e^4}{(q^2)^2}\, L_{\nu\mu}\, \langle H, P| J_{\mathrm{h}}^\nu |X\rangle\, \langle X| J_{\mathrm{h}}^\mu |H, P\rangle\,, \tag{5.79}$$

where, setting $m_e = 0$,

$$\begin{aligned} L_{\nu\mu} &= \sum_{e\ \text{spins}} \overline{u}(p)\gamma_\nu u(p')\,\overline{u}(p')\gamma_\mu u(p) = \mathrm{tr}(\gamma{\cdot}p\,\gamma_\nu\,\gamma{\cdot}p'\,\gamma_\mu) \\ &= 4(p_\nu p'_\mu + p_\mu p'_\nu - g_{\nu\mu}\,p{\cdot}p')\,. \end{aligned} \tag{5.80}$$

If we define

$$W_H^{\nu\mu}(q,P) = \frac{1}{4\pi}\sum_X (2\pi)^4\delta^4(q+P-p_X)\,\langle H, P| J_{\mathrm{h}}^\nu |X\rangle\, \langle X| J_{\mathrm{h}}^\mu |H, P\rangle\,, \tag{5.81}$$

then the cross section formula (5.77) becomes

$$E'\frac{\mathrm{d}\sigma}{\mathrm{d}^3 p'} = \frac{e^2}{8(2\pi)^2 EM}\,\frac{1}{(q^2)^2}\, L_{\nu\mu} W_H^{\nu\mu}(q,P)\,. \tag{5.82}$$

If the hadron H has spin, as in the realistic case of a proton or most nuclei, then in (5.81) the spin should be averaged over.

By virtue of current conservation $(p_X - P)_\mu \langle X| J_{\mathrm{h}}^\mu |H, P\rangle = 0$ which implies $q_\mu W_H^{\nu\mu}(q,P) = q_\nu W_H^{\nu\mu}(q,P) = 0$ and hence

$$W_H^{\nu\mu}(q,P) = \Big(-g^{\nu\mu} + \frac{q^\nu q^\mu}{q^2}\Big)W_1 + \Big(P^\nu - \frac{P{\cdot}q}{q^2}\,q^\nu\Big)\Big(P^\mu - \frac{P{\cdot}q}{q^2}\,q^\mu\Big)W_2\,, \tag{5.83}$$

with $W_{1,2}$ Lorentz scalars, called structure functions for the hadron H, and which depend on the two variables Q^2 and ν. In writing (5.83) we have neglected a possible term involving the ϵ-tensor but this can be excluded by using parity invariance. To calculate the contraction in (5.82) we may use $L_{\nu\mu}q^\nu = L_{\nu\mu}q^\mu = 0$ so that from (5.80) and (5.83) we have,

$$\begin{aligned} L_{\nu\mu}W_H^{\nu\mu}(q,P) &= 8p'{\cdot}p\,W_1 + 4(2p{\cdot}P\,p'{\cdot}P - M^2p'{\cdot}p)W_2 \\ &= 4Q^2\,W_1 + 2M^2(4EE' - Q^2)\,W_2 \\ &\sim 8EM\Big(xy\,W_1 + \frac{1-y}{y}\,\nu W_2\Big)\,, \end{aligned} \tag{5.84}$$

where in the second line we have used $p{\cdot}p' = -\frac{1}{2}q^2$, if $m_e = 0$, together with $p{\cdot}P = ME$, $p'{\cdot}P = ME'$ and in last line we have assumed the limit $Q^2 = \mathrm{O}(\nu) \to \infty$ with x, y dimensionless variables (in this limit $Q^2 \ll EE'$),

$$x = \frac{Q^2}{2\nu}\,, \qquad y = \frac{\nu}{ME} = 1 - \frac{E'}{E}\,, \tag{5.85}$$

which stay fixed. It is easy to see that

$$0 \le x \le 1\,, \qquad 0 \le y \le 1\,. \tag{5.86}$$

Since

$$\mathrm{d}^3p' \to 2\pi\, E'^2\mathrm{d}(\cos\theta)\,\mathrm{d}E' = \pi E'\,\mathrm{d}Q^2\,\mathrm{d}y = 2\pi E'\nu\,\mathrm{d}x\,\mathrm{d}y\,, \tag{5.87}$$

we have in the high energy limit from (5.82), (5.84) and (5.85)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y} = \frac{4\pi\alpha^2}{Q^4}\,2ME\Big((1-y)F_2(x,Q^2) + xy^2F_1(x,Q^2)\Big)\,, \tag{5.88}$$

where as usual $\alpha = e^2/4\pi$ and

$$F_2(x,Q^2) = \nu W_2\,, \qquad F_1(x,Q^2) = W_1\,, \tag{5.89}$$

are dimensionless quantities. Clearly comparison of cross section measurements with (5.88) allows $W_{1,2}$ or $F_{1,2}$ to be disentangled.

In the basic process (5.73) the electron e may be replaced by a muon without changing any of the subsequent results. A very similar analysis also holds for inelastic scattering of neutrinos, or anti-neutrinos, when

$$\nu_\mu(p) + H(P) \longrightarrow \mu^-(p') + X \quad \text{or} \quad \overline{\nu}_\mu(p) + H(P) \longrightarrow \mu^+(p') + X\,. \tag{5.90}$$

For such processes the incoming neutrinos or anti-neutrinos are produced by weak decay of pions, so they are almost entirely ν_μ or $\overline{\nu}_\mu$, and their energies have to be inferred from the total energy of the final state. The scattering is now mediated by a virtual W^+ or W^- instead of a virtual γ so that to first order in the weak interaction the amplitude is similar to (5.74) but

$$-\frac{e^2}{q^2} \longrightarrow \frac{1}{8}\,\frac{g^2}{m_W^2 - q^2} = \frac{G_F}{\sqrt{2}}\,\frac{m_W^2}{m_W^2 + Q^2}\,. \tag{5.91}$$

If we assume $Q^2 \ll m_W^2$ then instead of (5.82) we have

$$E'\frac{\mathrm{d}\sigma_{\nu H,\overline{\nu} H}}{\mathrm{d}^3 p'} = \frac{G_F^2}{2}\,\frac{1}{4(2\pi)^2 EM}\,L^{\mp}_{\nu\mu} W_H^{\pm,\nu\mu}(q,P)\,, \tag{5.92}$$

since for neutrino, anti-neutrino beams no spin averaging is necessary as they have definite helicity. Neglecting m_μ we have

$$\begin{aligned} L^-_{\nu\mu} &= \sum_{e\ \text{spins}} \overline{u}(p)\gamma_\nu(1-\gamma_5)u(p')\,\overline{u}(p')\gamma_\mu(1-\gamma_5)u(p) \\ &= \mathrm{tr}(\gamma{\cdot}p(1-\gamma_5)\,\gamma_\nu\,\gamma{\cdot}p'(1-\gamma_5)\,\gamma_\mu) \\ &= 8(p_\nu p'_\mu + p_\mu p'_\nu - g_{\nu\mu}\,p{\cdot}p' + i\epsilon_{\nu\mu\alpha\beta}p^\alpha p'^\beta)\,, \end{aligned} \tag{5.93}$$

where $\epsilon^{1230} = 1$. Similarly

$$L^+_{\nu\mu} = \sum_{e\ \text{spins}} \overline{v}(p')\gamma_\nu(1-\gamma_5)v(p)\,\overline{v}(p)\gamma_\mu(1-\gamma_5)v(p') = L^-_{\mu\nu}\,. \tag{5.94}$$

With $m_\mu = 0$, $q^\nu L^\pm_{\nu\mu} = 0$. Instead of (5.81) and (5.83) we now have, if $J_{\mathrm{h}}^{\pm\mu}$ are the hadronic $\Delta Q = \pm 1$ weak currents, $(J_{\mathrm{h}}^{+\mu})^\dagger = J_{\mathrm{h}}^{-\mu}$,

$$\begin{aligned} W_H^{\pm\nu\mu}(q,P) &= \frac{1}{4\pi}\sum_X (2\pi)^4\delta^4(q+P-p_X)\,\langle H,P|J_{\mathrm{h}}^{\mp\nu}|X\rangle\,\langle X|J_{\mathrm{h}}^{\pm\mu}|H,P\rangle \\ &= \Big(-g^{\nu\mu} + \frac{q^\nu q^\mu}{q^2}\Big)W_1^\pm + \Big(P^\nu - \frac{P{\cdot}q}{q^2}\,q^\nu\Big)\Big(P^\mu - \frac{P{\cdot}q}{q^2}\,q^\mu\Big)W_2^\pm - \tfrac{1}{2}i\,\epsilon^{\nu\mu\alpha\beta}q_\alpha P_\beta W_3^\pm \\ &\quad + \text{asymptotically unimportant terms}\,. \end{aligned} \tag{5.95}$$

The neglected terms in the last line of (5.95) are present since $\partial_\mu J_{\mathrm{h}}^{\pm\mu} \propto m_q$ is non zero but such contributions are expected to be small and to fall off faster as $Q^2,\nu \to \infty$. The additional term, beyond those which appear in (5.81), proportional to $W_3^\pm$ is present since the weak current contains both vector and axial pieces so that the ϵ-tensor is not ruled out by parity. By its definition in (5.95) $W_H^{\pm\nu\mu}(q,P)^\dagger = W_H^{\pm\mu\nu}(q,P)$ so that $W^\pm_{1,2,3}$ are real. Using $\frac{1}{2}\epsilon^{\nu\mu\alpha\beta}\epsilon_{\nu\mu\gamma\delta} = -\delta_\gamma{}^\alpha\delta_\delta{}^\beta + \delta_\gamma{}^\beta\delta_\delta{}^\alpha$ we find, instead of (5.84),

$$\begin{aligned} L^{\mp}_{\nu\mu}W_H^{\pm\nu\mu}(q,P) &= 16p{\cdot}p'\,W_1^\pm + 8(2P{\cdot}p'\,P{\cdot}p - M^2 p{\cdot}p')W_2^\pm \pm 8p{\cdot}p'\,P{\cdot}(p+p')\,W_3^\pm \\ &= 8Q^2\,W_1^\pm + 4M^2(4EE' - Q^2)\,W_2^\pm \pm 4Q^2 M(E+E')\,W_3^\pm \\ &\sim 16EM\Big(xy\,W_1^\pm + \frac{1-y}{y}\,\nu W_2^\pm \pm x(1-\tfrac{1}{2}y)\,\nu W_3^\pm\Big)\,. \end{aligned} \tag{5.96}$$

Combining (5.96) with (5.92) and following similar steps which led to (5.88) we find in the limit $Q^2, \nu \to \infty$,

$$\frac{\mathrm{d}\sigma_{\nu H,\overline{\nu}H}}{\mathrm{d}x\mathrm{d}y} = \frac{G_F^2}{2\pi} 2ME\Big((1-y)F_2^{\pm}(x,Q^2) + xy^2 F_1^{\pm}(x,Q^2) \pm xy(1-\tfrac{1}{2}y)F_3^{\pm}(x,Q^2)\Big)\,, \tag{5.97}$$

where now

$$F_2^{\pm}(x,Q^2) = \nu W_2^{\pm}\,, \quad F_1^{\pm}(x,Q^2) = W_1^{\pm}\,, \quad F_3^{\pm}(x,Q^2) = \nu W_3^{\pm}\,. \tag{5.98}$$

More generally (5.97) should contain a factor $(1+Q^2/m_W^2)^{-2}$.

3.2 Light Cone Variables

In order to analyse the behaviour of $W_H^{\nu\mu}(q,P)$ defined in (5.81), or $W_H^{\pm\nu\mu}(q,P)$ given in (5.95), in the deep inelastic limit, $Q^2, \nu \to \infty$ with $x = Q^2/2\nu = \mathrm{O}(1)$, it is very convenient to introduce an alternative basis for 4-vectors which give what are termed light cone variables. For an arbitrary 4-vector V^μ we define

$$V^{\pm} = V^0 \pm V^3\,, \qquad \mathbf{V}_\perp = (V^1, V^2)\,, \tag{5.99}$$

and then the Lorentz invariant scalar product for two 4-vectors V^μ and U^μ becomes

$$V{\cdot}U = \tfrac{1}{2}(V^+U^- + V^-U^+) - \mathbf{V}_\perp{\cdot}\mathbf{U}_\perp\,. \tag{5.100}$$

In this basis therefore $g_{+-} = g_{-+} = \frac{1}{2}$, $g_{++} = g_{--} = 0$ and $g_{ij} = -\delta_{ij}$ for $i,j = 1,2$. Under Lorentz boosts along the 3-direction $V^{\pm} \to e^{\pm\theta}V^{\pm}$ while $\mathbf{V}_\perp$ is unchanged.

To discuss $W_H^{\nu\mu}(q,P)$ and $W_H^{\pm\nu\mu}(q,P)$ we choose a frame such that

$$\mathbf{P}_\perp = \mathbf{q}_\perp = \mathbf{0}\,, \tag{5.101}$$

(note that $P^{\pm} > 0$) and then

$$Q^2 = -q^+q^-\,, \qquad \nu = \tfrac{1}{2}(q^+P^- + q^-P^+)\,. \tag{5.102}$$

The deep inelastic limit is realised by letting $q^- \to \infty$ with $q^+ = \mathrm{O}(P^+)$ so that $\nu \sim \frac{1}{2}q^-P^+$ and therefore

$$x \sim -\frac{q^+}{P^+}\,, \qquad \nu \sim \tfrac{1}{2}q^-P^+\,. \tag{5.103}$$

In this frame from (5.83)

$$\begin{aligned} W_H^{+-}(q,P) &= -W_1 + \Big(P - \frac{P{\cdot}q}{q^2}\,q\Big)^2 W_2 \\ &= -W_1 + \Big(M^2 + \frac{\nu^2}{Q^2}\Big)W_2 \equiv F_L(x,Q^2)\,. \end{aligned} \tag{5.104}$$

The other 'longitudinal' components are determined by current conservation,

$$W_H^{++}(q,P) = \frac{(q^+)^2}{Q^2}\, F_L(x,Q^2)\,, \qquad W_H^{--}(q,P) = \frac{(q^-)^2}{Q^2}\, F_L(x,Q^2)\,. \tag{5.105}$$

Since $\epsilon^{12+-} = 2$ and $V_\pm = \frac{1}{2}V^\mp$ the 'transverse' components of (5.95) are given by

$$W_H^{\pm ji}(q,P) = \delta^{ji} W_1^{\pm} - i\,\epsilon^{ji} F\, W_3^{\pm}\,, \quad F = \tfrac{1}{2}(q^- P^+ - q^+ P^-)\,, \tag{5.106}$$

where ϵ^{ji} is the two-dimensional antisymmetric symbol, $\epsilon^{12} = 1$, and it is easy to see with (5.102) that

$$F^2 = \nu^2 + M^2 Q^2\,. \tag{5.107}$$

In the deep inelastic limit (5.104) becomes

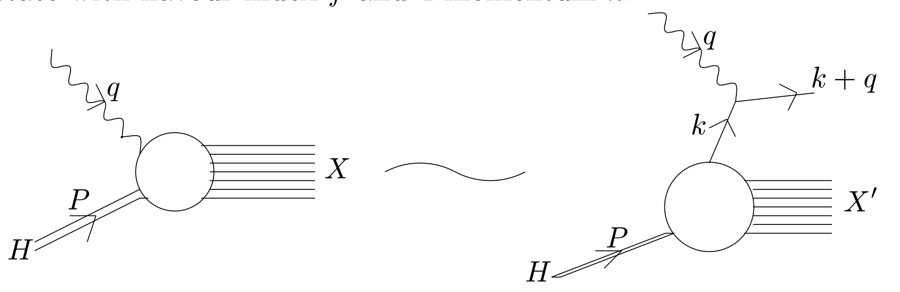
$$F_L(x,Q^2) \sim -F_1(x,Q^2) + \frac{1}{2x}\, F_2(x,Q^2)\,, \tag{5.108}$$

while (5.106) simplifies to

$$W_H^{\pm ji}(q,P) \sim \delta^{ji} F_1^{\pm}(x,Q^2) - i\,\epsilon^{ji} F_3^{\pm}(x,Q^2)\,. \tag{5.109}$$

3.3 Parton Model

As mentioned earlier the parton model was developed prior to the advent of QCD and depends only on taking seriously the idea that hadrons have point-like constituents with a wave function which falls off for large momenta as expected on the basis of non-relativistic intuition for bound states. The leading term in the deep inelastic limit is then given in (5.81) by letting, in accord with the general philosophy expressed by (5.60), $|X\rangle \to |q_f, \tilde{k}\rangle |X'\rangle$ where $|q_f, \tilde{k}\rangle$ denotes a single quark state with flavour index f and 4-momentum $\tilde{k}$.



Parton model for deep inelastic scattering

In this case we may write, neglecting any quark masses,

$$\sum_X \simeq \sum_f \sum_{X'} \frac{1}{(2\pi)^3} \int \mathrm{d}^4\tilde{k}\,\theta(\tilde{k}^0)\delta(\tilde{k}^2) \sum_{q\ \text{spins}}\,. \tag{5.110}$$

If we rewrite (5.52) as sum over quark flavours,

$$J_{\mathrm{h}}^{\mu} = \sum_f Q_f \, \overline{q}_f \gamma^\mu q_f \,, \tag{5.111}$$

then with the assumptions implied by (5.110), summing over both quarks and anti-quarks, in (5.81)

$$W_H^{\nu\mu}(q,P) \sim \sum_f \int \mathrm{d}^4 k \, \mathrm{tr}\Big(W_f^{\nu\mu}(q,k)\Gamma_{H,f}(P,k) + \overline{W}_f^{\nu\mu}(q,k)\overline{\Gamma}_{H,f}(P,k)\Big) \,, \tag{5.112}$$

where $W_f^{\nu\mu}(q,k)$, $\overline{W}_f^{\nu\mu}(q,k)$, denotes the relevant contributions when the virtual photon with momentum q couples to a quark, anti-quark, with flavour f and momentum k, as given by (5.111),

$$W_f^{\nu\mu}(q,k) = \overline{W}_f^{\mu\nu}(q,k) = \tfrac{1}{2} Q_f^{\,2} \, \gamma^\nu \gamma{\cdot}(k+q)\, \gamma^\mu \, \delta((k+q)^2) \,, \tag{5.113}$$

and we define

$$\begin{aligned}
\Gamma_{H,f}(P,k)_{\beta\alpha} &= \sum_{X'} \delta^4(P-k-p_{X'}) \, \langle H, P | \overline{q}_{f\alpha} | X' \rangle \, \langle X' | q_{f\beta} | H, P \rangle \,, \\
\overline{\Gamma}_{H,f}(P,k)_{\beta\alpha} &= \sum_{X'} \delta^4(P-k-p_{X'}) \, \langle H, P | q_{f\beta} | X' \rangle \, \langle X' | \overline{q}_{f\alpha} | H, P \rangle \,,
\end{aligned} \tag{5.114}$$

for α, β Dirac spinor indices. If appropriate then the definition of $\Gamma_{H,f}(P,k)$ and $\overline{\Gamma}_{H,f}(P,k)$ in (5.114) should be averaged over the hadron spins. The expression (5.112) obtained by applying (5.110) for $W_H^{\nu\mu}(q,P)$ tacitly assumes that the quark, or anti-quark, does not interact with the state X' after it couples to the virtual photon and so this is not, by any means, the sole contribution to $W_H^{\nu\mu}(q,P)$. Nevertheless, subject to suitable assumptions, (5.112) is the dominant term in the deep inelastic limit, other contributions being suppressed by inverse powers of Q^2. The critical requirement is that $\Gamma_{H,f}(P,k)$, and also $\overline{\Gamma}_{H,f}(P,k)$, which depend on the invariants $k^2, P{\cdot}k$, fall off sufficiently rapidly so that, assuming light cone variables with (5.101), the limit $q^- \to \infty$, with $q^+, P^\pm$ fixed, can be taken inside the integral. Thus, using (5.103),

$$(k+q)^2 \sim q^-(k^+ + q^+) \sim 2\nu \Big(\frac{k^+}{P^+} - x \Big) \,, \tag{5.115}$$

and hence in (5.113)

$$\delta((k+q)^2) \sim \frac{1}{2\nu} \, \delta\Big(\frac{k^+}{P^+} - x \Big) \,. \tag{5.116}$$

In (5.114) since X' is a state with positive energy, with mass^2 $(P-k)^2$, we must have $k^+/P^+ \le 1$. The Dirac matrices in (5.113) may be simplified with the aid of

$$\gamma^\nu \gamma^\lambda \gamma^\mu = s^{\nu\mu\lambda\kappa} \gamma_\kappa + i \epsilon^{\nu\mu\lambda\kappa} \gamma_\kappa \gamma_5 \,, \quad s^{\nu\mu\lambda\kappa} = g^{\nu\lambda} g^{\mu\kappa} + g^{\nu\kappa} g^{\mu\lambda} - g^{\mu\nu} g^{\lambda\kappa} \,, \tag{5.117}$$

which applied in the present context, since $\gamma{\cdot}(k+q) \sim \frac{1}{2}q^-\gamma^+$, leads to

$$\gamma^j\gamma^+\gamma^i = \gamma^+(\delta^{ji} + i\,\epsilon^{ji}\gamma_5)\,. \tag{5.118}$$

Hence, defining

$$\begin{aligned}
\frac{1}{2}\int \mathrm{d}^4k\,\delta\Big(\frac{k^+}{P^+}-x\Big)\,\mathrm{tr}\Big(\gamma^+\Gamma_{H,f}(P,k)\Big) &= P^+\,q_f(x)\,,\\
\frac{1}{2}\int \mathrm{d}^4k\,\delta\Big(\frac{k^+}{P^+}-x\Big)\,\mathrm{tr}\Big(\gamma^+\overline{\Gamma}_{H,f}(P,k)\Big) &= P^+\,\overline{q}_f(x)\,,
\end{aligned} \tag{5.119}$$

and applying the version of (5.109) appropriate to deep inelastic electron scattering gives

$$F_1(x,Q^2) \sim \tfrac{1}{2}\sum_f Q_f^2\Big(q_f(x)+\overline{q}_f(x)\Big)\,, \tag{5.120}$$

since replacing $\gamma^+ \to \gamma^+\gamma_5$ in (5.119) gives zero due to parity invariance after averaging over spins. The result (5.120) demonstrates that F_1 depends only on the dimensionless variable $x = Q^2/2\nu$ in the deep inelastic limit, which is known as Bjorken scaling. The quark distribution functions $q_f(x)$, $\overline{q}_f(x)$ defined by (5.119) for $x \geq 0$ are positive and may be interpretated as representing a one-dimensional momentum distribution for quarks, anti-quarks inside a hadron H. Using crossing symmetry they may be extended to $x < 0$ since

$$\Gamma_{H,f}(P,k) = -\overline{\Gamma}_{H,f}(P,-k) \quad\Rightarrow\quad q_f(x) = -\overline{q}_f(-x)\,. \tag{5.121}$$

If we apply a similar limit to the 'longitudinal' components of $W_H^{\nu\mu}(q,P)$, as given by (5.112), then using $\gamma^-\gamma^+\gamma^- = 4\gamma^-$ we may see from (5.113) and (5.116) that $W_f^{--}(q,k) = \mathrm{O}(1)$ and comparing with (5.104,5.105) and (5.108),

$$F_L(x,Q^2) \sim 0 \quad\Rightarrow\quad F_2(x,Q^2) \sim 2xF_1(x,Q^2) \sim x\sum_f Q_f^2\Big(q_f(x)+\overline{q}_f(x)\Big)\,. \tag{5.122}$$

The $+-$ and $++$ components of (5.112) are also compatible with the asymptotic vanishing of $F_L(x,Q^2)$, as required by current conservation, using $(\gamma^-)^2 = (\gamma^+)^2 = 0$ and $(k+q)^+ \sim 0$ since $k^+/P^+ \sim x$ as a consequence of the delta function in (5.116).

Applying these results to deep inelastic scattering on a proton target, and restricting to just the u,d,s quarks, which should be valid to a very good approximation, leads, with an evident notation $q_u(x) = u(x)$, $\overline{q}_u(x) = \overline{u}(x)$ etc, to

$$F_{2,\text{proton}}(x,Q^2) \sim x\Big(\tfrac{4}{9}(u(x)+\overline{u}(x)) + \tfrac{1}{9}(d(x)+\overline{d}(x)+s(x)+\overline{s}(x))\Big)\,. \tag{5.123}$$

For a neutron target, by isospin rotation $u \leftrightarrow d$ so that

$$F_{2,\text{neutron}}(x,Q^2) \sim x\Big(\tfrac{4}{9}(d(x)+\overline{d}(x)) + \tfrac{1}{9}(u(x)+\overline{u}(x)+s(x)+\overline{s}(x))\Big)\,. \tag{5.124}$$

For deep inelastic neutrino, or anti-neutrino, scattering for simplicity we set the Cabibbo angle to zero and neglect terms involving charm quarks so that we may restrict the weak currents appearing in (5.95) to just the u, d quark contributions so that

$$J_{\rm h}^{+\mu} = \overline{u}\gamma^\mu(1-\gamma_5)d\,, \qquad J_{\rm h}^{-\mu} = \overline{d}\gamma^\mu(1-\gamma_5)u\,. \tag{5.125}$$

With the same assumptions as led to (5.112)

$$W_H^{+\nu\mu}(q,P) \sim \int \mathrm{d}^4 k\, \mathrm{tr}\Big(W_d^{\nu\mu}(q,k)\Gamma_{H,d}(P,k) + \overline{W}_u^{\nu\mu}(q,k)\overline{\Gamma}_{H,u}(P,k)\Big)\,, \tag{5.126}$$

where now

$$W_d^{\nu\mu}(q,k) = \overline{W}_u^{\mu\nu}(q,k) = \tfrac{1}{2}\,\gamma^\nu(1-\gamma_5)\,\gamma\cdot(k+q)\,\gamma^\mu(1-\gamma_5)\,\delta((k+q)^2)\,. \tag{5.127}$$

For the transverse components inside the integral in (5.126) we may write from (5.118)

$$\gamma^j(1-\gamma_5)\gamma\cdot(k+q)\,\gamma^i(1-\gamma_5) \sim q^-\gamma^+(\delta^{ji} - i\,\epsilon^{ji})(1-\gamma_5)\,, \tag{5.128}$$

and hence from (5.109) with identical notation as previously, since also the longitudinal components give $F_L^+(x,Q^2) \sim 0$,

$$\frac{1}{2x}\,F_2^+(x,Q^2) \sim F_1^+(x,Q^2) \sim d(x) + \overline{u}(x)\,, \quad F_3^+(x,Q^2) \sim 2(d(x) - \overline{u}(x))\,. \tag{5.129}$$

Similarly

$$\frac{1}{2x}\,F_2^-(x,Q^2) \sim F_1^-(x,Q^2) \sim u(x) + \overline{d}(x)\,, \quad F_3^-(x,Q^2) \sim 2(u(x) - \overline{d}(x))\,. \tag{5.130}$$

If we apply (5.129) in the result (5.97) for neutrino scattering we find

$$\frac{\mathrm{d}\sigma_{\nu H}}{\mathrm{d}x\mathrm{d}y} = \frac{G_F^2}{\pi}\,2ME\,x\Big(d(x) + (1-y)^2\overline{u}(x)\Big)\,, \tag{5.131}$$

while for anti-neutrino scattering

$$\frac{\mathrm{d}\sigma_{\overline{\nu} H}}{\mathrm{d}x\mathrm{d}y} = \frac{G_F^2}{\pi}\,2ME\,x\Big(\overline{d}(x) + (1-y)^2 u(x)\Big)\,. \tag{5.132}$$

To the extent that $Q^2 \ll M_W^2$ the total cross sections formed by integrating over x, y rise linearly with the energy E. On a nuclear target N with equal numbers of protons and protons then $u = d$, $\overline{u} = \overline{d}$ and $\sigma_{\mathrm{tot},\overline{\nu}N}/\sigma_{\mathrm{tot},\nu N} \geq \frac{1}{3}$, with equality if $\overline{u} = \overline{d} = 0$.

The parton model leads to various relations. For instance, from (5.123,5.124) and (5.129) for proton, neutron targets in the Bjorken scaling limit

$$F_{2,\mathrm{proton}}(x) + F_{2,\mathrm{neutron}}(x) \geq \tfrac{5}{18}\Big(F^{+}_{2,\mathrm{proton}}(x) + F^{+}_{2,\mathrm{neutron}}(x)\Big)\,, \tag{5.133}$$

with equality if $s = \overline{s} = 0$ (this is nearly satisfied and is a test of fractionally charged quarks), and also the exact relation

$$F_{2,\mathrm{proton}}(x) - F_{2,\mathrm{neutron}}(x) = \tfrac{1}{6}x(F^{+}_{3,\mathrm{neutron}}(x) - F^{+}_{3,\mathrm{proton}}(x)\Big)\,. \tag{5.134}$$

3.4 Sum Rules

The quark distribution functions defined in (5.119) obey important sum rules. With the definition (5.114) and (5.121) it is straightforward to see that

$$\frac{1}{2}\int_{k^+>0}\mathrm{d}^4k\,\mathrm{tr}\Big(\gamma^\mu(\Gamma_{H,f}(P,k) - \overline{\Gamma}_{H,f}(P,k))\Big) = \tfrac{1}{2}\langle H,P|\overline{q}_f\gamma^\mu q_f|H,P\rangle = P^\mu N_{H,f}\,, \tag{5.135}$$

where $N_{H,f}$ is the net number of f quarks in the hadron H. If $Q_f = \int \mathrm{d}^3x\,\overline{q}_f\gamma^0 q_f$ then $Q_f|H,P\rangle = N_{H,f}|H,P\rangle$. Applying (5.135) for $\mu = +$ with (5.119) gives

$$\int_0^1 \mathrm{d}x\,\Big(q_f(x) - \overline{q}_f(x)\Big) = N_{H,f}\,. \tag{5.136}$$

This gives rise to various sum rules for the measured structure functions in the Bjorken scaling limit. For instance the Gross-Llewellyn-Smith sum rule is

$$\int_0^1 \mathrm{d}x\,\Big(F^{+}_{3,\mathrm{proton}}(x,Q^2) + F^{-}_{3,\mathrm{proton}}(x,Q^2)\Big) \underset{Q^2\to\infty}{\longrightarrow} 2\int_0^1 \mathrm{d}x\,\Big(u(x) - \overline{u}(x) + d(x) - \overline{d}(x)\Big) = 6\,. \tag{5.137}$$

Since $F^{-}_{2,\mathrm{proton}}(x,Q^2) - F^{+}_{2,\mathrm{proton}}(x,Q^2) \sim 2x(u(x) + \overline{d}(x) - d(x) - \overline{u}(x))$ we may derive also the Adler sum rule,

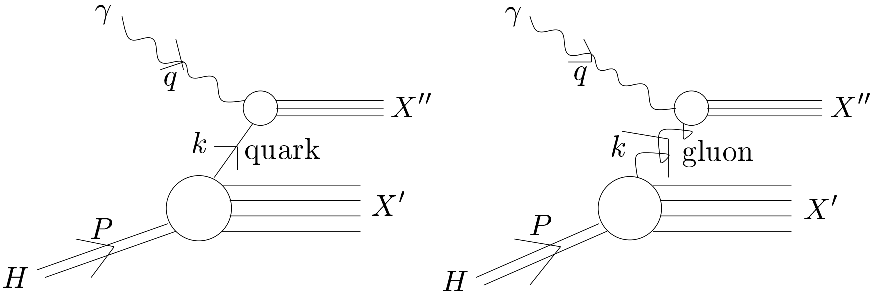
$$\int_0^1 \mathrm{d}x\,\frac{1}{x}\Big(F^{-}_{2,\mathrm{proton}}(x,Q^2) - F^{+}_{2,\mathrm{proton}}(x,Q^2)\Big) = 2\,. \tag{5.138}$$

This is in fact valid for all Q^2, not just as $Q^2 \to \infty$.

3.5 QCD Corrections

In the derivation of the parton model the quark interacts with the virtual γ, or virtual W, for large Q^2 with a pointlike coupling, not including any corrections due to strong interactions. In a field theory approach the quark fields in the currents (5.111) or (5.125) are treated as if they were effectively free, disregarding QCD effects. This is ultimately justified by asymptotic freedom but a detailed analysis shows that there are calculable corrections to Bjorken scaling. The application of asymptotic freedom to deep inelastic scattering is not immediately straightforward since not all momenta are becoming large, the target hadron momentum P satisfies $P^2 = M^2$ which is fixed and the hadron wave function, which determines the quark structure functions in the parton model, intrinsically depend on low energy scales (these determine the fall off of $\Gamma(P,k)$ and $\overline{\Gamma}(P,k)$ for large $-k^2$). It is necessary to introduce a further factorisation assumption, which can be derived to all orders in the perturbation expansion, in order to justify using the ideas of asymptotic freedom.

To simplify the discussion we drop spinor and vector indices in a schematic treatment which can be extended without difficulty to realistic cases. We analyse a generic structure function $F(x,Q^2)$, such as might be measured in deep inelastic scattering. The dominant contributions for $Q^2 \to \infty$ arise from the elementary particles of perturbative QCD, quarks and gluons, but QCD corrections are no longer ignored and $F(x,Q^2)$ cannot any more be represented in terms of solely pointlike couplings to the quarks, as in (5.112) and (5.126). Instead we assume that the pointlike vertex is replaced by $C_i(q,k)$, where $i = q_f, \overline{q}_f, G$ for a quark, anti-quark, gluon with 4-momentum k coupling to a current J carrying 4-momentum q, $q^2 = -Q^2$, and which includes all QCD corrections.



Deep Inelastic Scattering in QCD, quarks and gluons

In the relevant limit $Q^2 = -q^2 \to \infty$, $x = Q^2/2\nu$ ($\nu = P\cdot q$), fixed, $F(x,Q^2)$ is assumed to have the form of a sum over contributions for different $i = q_f, \overline{q}_f, G$,

$$F(x,Q^2) \sim \sum_{i=q_f,\overline{q}_f,G} \int \mathrm{d}^4k \, C_i(q,k) \, \Gamma_i(P,k) \,. \tag{5.139}$$

Replacing (5.110) in this case

$$\sum_X = \sum_{i=q_f,\overline{q}_f,G} \sum_{X'} \sum_{X''} , \tag{5.140}$$

and $C_i(q,k)$ represents the sum over states X'' for $J(q) + i(k) \to X''$. Since $C_i(q,k)$ is taken to be a Lorentz scalar it can depend on $k\cdot q, Q^2$ and also k^2. Assuming also that $C_i(q,k)$ is dimensionless then if we consider the limit $k^2 \to 0$

and neglect any quark masses we may write

$$C_i(q,k)\Big|_{k^2=0} = C_i\Big(\frac{Q^2}{2k{\cdot}q},\frac{Q^2}{\mu^2};\alpha_s\Big)\,, \tag{5.141}$$

where setting $k^2 = 0$ is possible without introducing infra red singularities if we introduce an arbitrary renormalisation mass scale μ. In (5.141) we have also displayed the QCD coupling α_s, as in (5.39), since C_i may be calculated perturbatively. To zeroth order X'' is just a single quark or gluon, $(k+q)^2 = 0$, and there is no dependence on μ so that we may take

$$C_{q_f}\Big(x,\frac{Q^2}{\mu^2};0\Big) = J_f\delta(1-x)\,,\quad C_{\overline{q}_f}\Big(x,\frac{Q^2}{\mu^2};0\Big) = \bar{J}_f\delta(1-x)\,, \tag{5.142}$$

where $J_f, \bar{J}_f$ are given by the coupling of the the current J to the quark, anti-quark with flavour f. When $q^- \to \infty$ for fixed k, using (5.103),

$$\frac{Q^2}{2k{\cdot}q} \sim -\frac{q^+}{k^+} \sim \frac{x}{y}\,, \qquad y = \frac{k^+}{P^+}\,. \tag{5.143}$$

In general X'' is a positive energy state, with mass^2 $(k+q)^2 > 0$, and hence in (5.141) for C_i to be non zero we must have

$$(k+q)^+ \geq 0 \quad \Rightarrow \quad y = \frac{k^+}{P^+} \geq x\,. \tag{5.144}$$

If we consider the deep inelastic limit by taking $q^- \to \infty$, with q, P constrained by (5.101), in (5.139) then, since in the integral there is a fall off for large $-k^2$, we can take from (5.141)

$$C_i(q,k) \sim C_i\Big(\frac{Q^2}{2k{\cdot}q},\frac{Q^2}{\mu^2};\alpha_s\Big)\,. \tag{5.145}$$

Hence (5.139) reduces to a single variable integral

$$F(x,Q^2) \sim \sum_{i=q_f,\overline{q}_f,G}\int_x^1 \frac{\mathrm{d}y}{y}\, C_i\Big(\frac{x}{y},\frac{Q^2}{\mu^2};\alpha_s\Big)\, f_i(y,\mu^2)\,, \tag{5.146}$$

where

$$f_i(y,\mu^2) = y\int \mathrm{d}^4k\,\delta\Big(\frac{k^+}{P^+}-y\Big)\,\Gamma_i(P,k)\,, \tag{5.147}$$

which may be decomposed in terms of quark, anti-quark and gluon contributions by

$$f_i(y,\mu^2) = \Big(q_f(y,\mu^2),\overline{q}_f(y,\mu^2),G(y,\mu^2)\Big)\,,\quad i = q_f,\overline{q}_f,G\,. \tag{5.148}$$

In obtaining (5.139) and hence (5.146) the dependence on the large momentum q, and also the particular current J, has been factorised from the details of the hadron wave function contained implicitly in $\Gamma_i(P,k)$ and $f_i(y,\mu^2)$, which is non zero if $y \le 1$. This factorisation, which allows use of the limit (5.145), is only possible at the expense of introducing a dependence on the renormalisation scale μ, as shown explicitly in (5.141) and (5.147). If we use just the lowest order result (5.142) for C_q then we recover the naive parton model result which entails Bjorken scaling,

$$\int_x^1 \frac{\mathrm{d}y}{y}\,\delta\Big(1-\frac{x}{y}\Big)\,q_f(y) = q_f(x) \quad \Rightarrow \quad F(x,Q^2) \sim \sum_f \Big(J_f q_f(x) + \bar J_f \overline{q}_f(x)\Big)\,, \tag{5.149}$$

since q_f must also be supposed to be then independent of μ.

It is important to recognise that $F(x,Q^2)$ as a potentially measurable physical quantity must be independent of μ. In general for vectors A_i, B_i

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\Big(A_i B_i\Big) = 0 \quad \Rightarrow \quad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}A_i = -A_j P_{ji}\,, \quad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}B_i = P_{ij}B_j\,. \tag{5.150}$$

The integral convolution in (5.146) can be regarded similarly as a form of matrix multiplication for two μ-dependent factors. The analogous version of the equations for A, B in (5.150) become integral relations

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}C_i\Big(x,\frac{Q^2}{\mu^2};\alpha_s\Big) \;=\; -\sum_{j=q_f,\overline{q}_f,G}\int_x^1 \frac{\mathrm{d}y}{y}\,C_j\Big(y,\frac{Q^2}{\mu^2};\alpha_s\Big)\,P_{ji}\Big(\frac{x}{y};\alpha_s\Big)\,, \tag{5.151}$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}f_i(y,\mu^2) \;=\; \sum_{j=q_f,\overline{q}_f,G}\int_y^1 \frac{\mathrm{d}z}{z}\,P_{ij}\Big(\frac{y}{z};\alpha_s\Big)\,f_j(z,\mu^2)\,, \tag{5.152}$$

where $P_{ij}(y;\alpha_s)$ is independent of Q^2, the particular current J and the hadron H, and may be determined as an expansion in α_s from (5.151). In general all components of $P_{ij}(y;\alpha_s)$ are non zero. The equations (5.151,5.152), referred to as the Altarelli-Parisi equations, are an extension of the renormalisation group equations to this case. In (5.151)

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\,, \tag{5.153}$$

or equivalently in (5.151,5.152) we should take $\alpha_s \to \alpha_s(\mu^2)$ the running coupling, which is explicitly given by (5.41) to lowest order. Since μ is arbitrary we may set $\mu^2 = Q^2$ so that (5.146) becomes

$$F(x,Q^2) \sim \sum_{i=q_f,\overline{q}_f,G}\int_x^1 \frac{\mathrm{d}y}{y}\,C_i\Big(\frac{x}{y},1;\alpha_s(Q^2)\Big)\,f_i(y,Q^2)\,, \tag{5.154}$$

where from (5.152)

$$Q\frac{\mathrm{d}}{\mathrm{d}Q}f_i(y,Q^2) = \sum_{j=q_f,\overline{q}_f,G}\int_y^1 \frac{\mathrm{d}z}{z}\,P_{ij}\Big(\frac{y}{z};\alpha_s(Q^2)\Big)\,f_j(z,Q^2)\,. \tag{5.155}$$

The results (5.154) and (5.155) then provide the justification for the claim that asymptotic freedom allows the Q^2 dependence of $F(x,Q^2)$ to be calculated perturbatively in the deep inelastic limit.

The integrals in (5.154) and (5.155) can be disentangled by using moments. If $f(x)$ is defined on $0 \le x \le 1$ then its moments f^N are defined by

$$\int_0^1 \mathrm{d}x\, x^{N-1} f(x) = f^N\,, \quad N = 0,1,2,\dots\,. \tag{5.156}$$

For $f(x) = \delta(1-x)$ then $f^N = 1$ for all N. If $g(x)$ is similarly defined, so that $f(x), g(x) \to f^N, g^N$, then, similarly to the usual convolution theorem, the essential integral becomes a product of moments,

$$\int_0^1 \mathrm{d}x\, x^{N-1} \int_x^1 \frac{\mathrm{d}y}{y}\, f(x/y)\, g(y) = \int_0^1 \frac{\mathrm{d}y}{y}\, g(y) \int_0^y \mathrm{d}x\, x^{N-1}\, f(x/y) = f^N g^N\,. \tag{5.157}$$

If we therefore let

$$F(x,Q^2) \to M^N(Q^2)\,, \quad f_i(x,Q^2) \to f_i^N(Q^2)\,, \quad P_{ij}(x;\alpha_s(Q^2)) \to P_{ij}^N(\alpha_s(Q^2))\,, \tag{5.158}$$

then (5.154) and (5.155) are equivalent to

$$\begin{aligned} M^N(Q^2) &\sim \sum_{i=q_f,\overline{q}_f,G} C_i^N(\alpha_s(Q^2)) O_i^N(Q^2)\,, && (5.159)\\ Q\frac{\mathrm{d}}{\mathrm{d}Q} O_i^N(Q^2) &= \sum_{j=q_f,\overline{q}_f,G} P_{ij}^N(\alpha_s(Q^2)) O_j^N(Q^2)\,. && (5.160) \end{aligned}$$

(5.160) is basically straightforward to solve for $O_i^N(Q^2)$ in terms of $O_i^N(Q_0^2)$.

The matrix structure in (5.155) or (5.160) may be simplified by making use of symmetries. For $N_{\rm fl}$ flavours, since all quark masses are neglected, the symmetry group $SU(N_{\rm fl})$ for quark flavours may be assumed to restrict C_i and P_{ij}. C_G is an $SU(N_{\rm fl})$ singlet while $C_{q_f}, C_{\overline{q}_f}$ both belong to the product representation $N_{\rm fl} \times \overline{N}_{\rm fl}$ which can be decomposed into just the singlet and adjoint, of dimension $N_{\rm fl}^2 - 1$, irreducible representations. Furthermore under charge conjugation C_G is invariant while $C_{q_f} \leftrightarrow C_{\overline{q}_f}$. Applying these symmetry conditions gives for P_{ij} the general structure

$$\begin{aligned} &P_{q_f G} = P_{\overline{q}_f G} = P_{qG}\,, \qquad P_{G q_f} = P_{G\overline{q}_f} = P_{Gq}\,,\\ &P_{q_f q_{f'}} = P_{\overline{q}_f \overline{q}_{f'}} = P_{qq}^{NS}\,\delta_{ff'} + \frac{1}{N_{\rm fl}}\Big(P_{qq}^{S} - P_{qq}^{NS}\Big)\,,\\ &P_{q_f \overline{q}_{f'}} = P_{\overline{q}_f q_{f'}} = P_{q\overline{q}}^{NS}\,\delta_{ff'} + \frac{1}{N_{\rm fl}}\Big(P_{q\overline{q}}^{S} - P_{q\overline{q}}^{NS}\Big)\,. \end{aligned} \tag{5.161}$$

If we consider a structure function $F_{\pm}^{NS}(x,Q^2)$ which involves non singlet quantum numbers and $\pm$ charge conjugation then, including QCD corrections

$C_{q_f}, C_{\overline{q}_f}$ have the form

$$C_{q_f}\Big(x,\frac{Q^2}{\mu^2};\alpha_s\Big) = \pm C_{\overline{q}_f}\Big(x,\frac{Q^2}{\mu^2};\alpha_s\Big) = J_f\, C_{\pm}^{NS}\Big(x,\frac{Q^2}{\mu^2};\alpha_s\Big)\,, \quad \sum_f J_f = 0\,, \tag{5.162}$$

and (5.154) reduces for this case to

$$F_{\pm}^{NS}(x,Q^2) \sim \int_x^1 \frac{\mathrm{d}y}{y}\, C_{\pm}^{NS}\Big(\frac{x}{y},1;\alpha_s(Q^2)\Big)\, q_{\pm}^{NS}(y,Q^2)\,, \tag{5.163}$$

where

$$q_{\pm}^{NS}(y,Q^2) = \sum_f J_f\Big(q_f(x,Q^2) \pm \overline{q}_f(x,Q^2)\Big)\,. \tag{5.164}$$

The general result (5.155) now becomes a one-component Q^2 evolution equation for $q_{\pm}^{NS}$,

$$Q\frac{\mathrm{d}}{\mathrm{d}Q}\, q_{\pm}^{NS}(y,Q^2) = \int_y^1 \frac{\mathrm{d}z}{z}\, P_{\pm}^{NS}\Big(\frac{y}{z};\alpha_s(Q^2)\Big)\, q_{\pm}^{NS}(z,Q^2)\,, \tag{5.165}$$

with, from (5.161), $P_{\pm}^{NS} = P_{qq}^{NS} \pm P_{q\overline{q}}^{NS}$. A similar equation may be derived for the singlet case for $q_-^S = \sum_f (q_f - \overline{q}_f)$ while for $q_+^S = \sum_f(q_f+\overline{q}_f)$ there is a two component coupled equation involving the gluon distribution function G as well,

$$Q\frac{\mathrm{d}}{\mathrm{d}Q}\begin{pmatrix} q_+^S(y,Q^2) \\ G(y,Q^2)\end{pmatrix} = \int_y^1 \frac{\mathrm{d}z}{z}\begin{pmatrix} P_+^S(\frac{y}{z};\alpha_s(Q^2)) & 2N_{\mathrm{fl}}P_{qG}(\frac{y}{z};\alpha_s(Q^2)) \\ P_{Gq}(\frac{y}{z};\alpha_s(Q^2)) & P_{GG}(\frac{y}{z};\alpha_s(Q^2))\end{pmatrix}\begin{pmatrix} q_+^S(z,Q^2) \\ G(z,Q^2)\end{pmatrix}, \tag{5.166}$$

where $P_+^S = P_{qq}^S + P_{q\overline{q}}^S$.

To first order in α_s calculations of C_i give expressions of the same form for $i = q_f, \overline{q}_f$ so that, assuming (5.142), $C_{q_f} = J_f C_q$, $C_{\overline{q}_f} = \bar{J}_f C_q$ where

$$C_q\Big(x,\frac{Q^2}{\mu^2};\alpha_s\Big) = \delta(1-x) + \frac{\alpha_s}{2\pi}\, p_q(x)\, \ln\frac{Q^2}{\mu^2}\,. \tag{5.167}$$

and hence from (5.163)

$$F_{\pm}^{NS}(x,Q^2) \sim q_{\pm}^{NS}(x,Q^2) + \mathrm{O}(\alpha_s(Q^2))\,. \tag{5.168}$$

From (5.167) we may determine $P_{\pm}^{NS}$ for use in (5.165) giving

$$P_{\pm}^{NS}\Big(y;\alpha_s(Q^2)\Big) = \frac{\alpha_s(Q^2)}{\pi}\, p_q(y)\,. \tag{5.169}$$

If we define moments of $p_q(y)$,

$$\int_0^1 \mathrm{d}x\, x^{N-1} p_q(x) = -\tfrac{1}{4}\gamma_q^N\,, \tag{5.170}$$

then instead of (5.160), if $q_{\pm}^{NS}(x,Q^2) \to O^N(Q^2)$,

$$Q\frac{\mathrm{d}}{\mathrm{d}Q}\,O^N(Q^2) = -\frac{\alpha_s(Q^2)}{4\pi}\,\gamma_q^N\,O^N(Q^2)\,, \qquad \alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln\frac{Q^2}{\Lambda^2}}\,, \tag{5.171}$$

where we exhibit from (5.41) the lowest order expression for the running coupling $\alpha_s(Q^2)$. The solution of (5.171) is then

$$O^N(Q^2) = \left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{\frac{\gamma_q^N}{2\beta_0}} O^N(Q_0^2)\,. \tag{5.172}$$

This result illustrates how QCD gives rise to calculable corrections to Bjorken scaling, as $Q^2 \to \infty$ there remains a dependence on $\ln Q^2$. $\gamma_q^1 = 0$ so that there is no such factor for the $N=1$ moment but sum rules for the number of quarks of different flavours, such as the Gross-Llewellyn-Smith sum rule in (5.137), have corrections on the right hand side proportional to $\alpha_s(Q^2)$.

Part VI

QCD, low energy aspects

1 QCD as the theory of Strong Interactions

The remarkable, and hitherto unprecedented, feature of QCD is that the physical states bear no direct relation to the quarks and gluons which are present in the standard perturbative treatment. This is a reflection of the dynamical assumption of confinement so that all physical states are singlets under the colour gauge group. For a gauge group $SU(3)_{\text{colour}}$ and assuming that the quarks q are all triplets belonging to the 3 representation, while anti-quarks belong to the conjugate 3^* representation, the simplest states are then $q\overline{q}$ mesons as well as qqq baryons, although states with a more complicated quark content as well as pure gluon states are possible. In order to understand what hadron states may occur and also their interactions it is essential to take account of all symmetries, exact and approximate, of QCD and their implications. A crucial aspect of QCD as it is manifested in the real world is that there is an approximate chiral symmetry such that the ground state or vacuum is not invariant in the limit of exact symmetry. Such a spontaneously broken symmetry leads to Goldstone bosons which correspond to pions, and also to the other pseudoscalar mesons, which are much lighter than other hadrons becoming massless in the symmetry limit. The essential implication of spontaneously broken chiral symmetry is that the low energy interactions of pions can be described by a relatively simple effective lagrangian with only a few parameters which are to be determined by experiment (and might in principle be calculated from QCD).

1.1 Symmetries of QCD

The initial Lagrangian for QCD is simply

$$\mathcal{L}_{\text{QCD}} = -\tfrac{1}{4}F^{\mu\nu}\cdot F_{\mu\nu} + \sum_f \overline{q}_f(i\gamma^\mu D_\mu - m_f)q_f \,, \tag{6.1}$$

with f a label for the different flavours of quarks and where colour indices, as well as spinor indices, are suppressed. For the most part we restrict our attention to the two lightest u and d quarks and write

$$q = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad \overline{q} = (\,\overline{u} \quad \overline{d}\,) \,. \tag{6.2}$$

If $m_u = m_d = m$ in (6.1) then the relevant part of the Lagrangian is

$$\mathcal{L}_{\text{u,d}} = \overline{q}(i\gamma^\mu D_\mu - m\,1)q \,, \tag{6.3}$$

which is invariant under

$$q \to Uq\,, \quad \overline{q} \to \overline{q}U^\dagger \;\; \text{if} \;\; U^\dagger U = 1 \quad \Rightarrow \quad U \in U(2)\,, \tag{6.4}$$

with $U(2)$ the group of unitary 2×2 matrices. The group $U(2)$ may be decomposed as

$$U(2) \simeq SU(2)_I \times U(1)_V/Z_2\,, \qquad Z_2 = \{1, -1\}\,, \tag{6.5}$$

and associated with such continuous symmetries there are the conserved colour singlet currents and associated charges

$$j^\mu_{I,i} = \overline{q}\gamma^\mu \tfrac{1}{2}\tau_i q\,, \quad I_i = \int \mathrm{d}^3x\, j^0_{I,i}\,,\; i = 1,2,3\,, \qquad j^\mu = \overline{q}\gamma^\mu q\,, \quad Q_V = \int \mathrm{d}^3 x\, j^0\,. \tag{6.6}$$

The operators $\mathbf{I}$ correspond to the generators of $SU(2)_I$ isospin symmetry while Q_V counts the net number of u, d quarks and is the generator of a $U(1)_V$ symmetry. If we include to s quark as well and assume $m_u = m_d = m_s$ then $SU(2)_I \to SU(3)_F$ which is in fact realised as an approximate symmetry group of strong interactions, so that particle states of given spin form multiplets which correspond to representations of $SU(3)$. For any number of quarks with arbitrary masses m_f there is a $U(1)_V$ symmetry corresponding to $q_f \to e^{i\alpha}q_f$, $\overline{q}_f \to e^{-i\alpha}\overline{q}_f$ for every flavour. The associated current is then $j^\mu = \sum_f \overline{q}_f\gamma^\mu q_f$ and we may then identify the singlet charge $Q_V = 3B$ where B is the baryon number, which is necessarily conserved in QCD.

If $m_u = m_d = 0$ then (6.3) becomes

$$\mathcal{L}_{\mathrm{u,d}} = \overline{q_R}\, i\gamma^\mu D_\mu q_R + \overline{q_L}\, i\gamma^\mu D_\mu q_L\,, \tag{6.7}$$

which has an additional so called chiral symmetry due to the fact that the right handed and left handed chiral projections may be transformed independently,

$$q_R \to A q_R\,, \qquad q_L \to B q_L\,, \qquad A^\dagger A = B^\dagger B = 1\,, \tag{6.8}$$

so that the symmetry group becomes

$$(A, B) \in U(2)_R \times U(2)_L\,. \tag{6.9}$$

This contains $SU(2)_I$ as a so called diagonal subgroup composed of elements (U, U), i.e $A = B = U$, with also $\det U = 1$. Initially we restrict our attention to the group $SU(2)_R \times SU(2)_L$ when $\det A = \det B = 1$ and we can write $A = e^{\frac{1}{2}i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}$, $B = e^{\frac{1}{2}i\boldsymbol{\beta}\cdot\boldsymbol{\tau}}$ with $\boldsymbol{\tau}$ the usual Pauli matrices. The corresponding conserved currents and associated charges are then

$$j^\mu_{R,i} = \overline{q_R}\gamma^\mu q_R\,, \quad Q_{R,i} = \int \mathrm{d}^3x\, j^0_{R,i}\,, \qquad j^\mu_{L,i} = \overline{q_L}\gamma^\mu q_L\,, \quad Q_{L,i} = \int \mathrm{d}^3x\, j^0_{L,i}\,, \tag{6.10}$$

which satisfy the algebra for the generators of $SU(2)_R \times SU(2)_L$,

$$[Q_{R,i}, Q_{R,j}] = i\epsilon_{ijk} Q_{R,k}\,, \qquad [Q_{L,i}, Q_{L,j}] = i\epsilon_{ijk} Q_{L,k}\,, \qquad [Q_{R,i}, Q_{L,j}] = 0\,. \tag{6.11}$$

The isospin charges, as given in (6.6), are now

$$I_i = Q_{R,i} + Q_{L,i}\,, \tag{6.12}$$

and the associated axial charge is

$$Q_{5,i} = Q_{R,i} - Q_{L,i} = \int \mathrm{d}^3x\, j^0_{5,i}\,, \qquad j^\mu_{5,i} = \overline{q}\gamma^\mu\gamma_5 \tfrac{1}{2}\tau_i q\,. \tag{6.13}$$

The algebra in (6.11) then becomes

$$[I_i, I_j] = i\epsilon_{ijk} I_k\,, \qquad [I_i, Q_{5,j}] = i\epsilon_{ijk} Q_{5,k}\,, \qquad [Q_{5,i}, Q_{5,j}] = i\epsilon_{ijk} I_k\,. \tag{6.14}$$

Of course this includes the standard algebra for the isospin charges and also shows that $Q_{5,i}$ transforms as an isovector. The action on the u, d quark fields is given by

$$[I_i, q] = -\tfrac{1}{2}\tau_i q\,, \quad [I_i, \overline{q}] = \overline{q}\tfrac{1}{2}\tau_i\,, \quad [Q_{5,i}, q] = -\tfrac{1}{2}\tau_i\gamma_5 q\,, \quad [Q_{5,i}, \overline{q}] = -\overline{q}\gamma_5\tfrac{1}{2}\tau_i\,. \tag{6.15}$$

For subsequent application it is useful to define the scalar and pseudoscalar fields

$$S = \overline{q}q\,, \qquad \mathbf{P} = \overline{q}\boldsymbol{\tau} i\gamma_5 q\,. \tag{6.16}$$

It is easy to see by considering their commutators with $\mathbf{I}$ that S is an isoscalar while $\mathbf{P}$ is an isovector. Under commutation with the axial charges we have

$$[Q_{5,i}, S] = iP_i\,, \qquad [Q_{5,i}, P_j] = -\tfrac{1}{2}\overline{q}\{\tau_i, \tau_j\}q = -i\delta_{ij} S\,. \tag{6.17}$$

In consequence $S, \mathbf{P}$ form a four dimensional representation of $SU(2)_R \times SU(2)_L$.

Although $SU(2)_R \times SU(2)_L$ is a symmetry of the QCD Lagrangian it is not apparent, unlike $SU(2)_I$ isospin, in the multiplets of nearly degenerate particles found in nature. If it were realised in a conventional manner the action of the axial charges $Q_{5,i}$ would require there to be particles of opposite parity, although the same spin, in the multiplets contrary to experiment. It is now clear that the symmetry is realised in a spontaneously broken fashion so that although, in the symmetry limit, the vacuum is invariant under isospin or

$$I_i|0\rangle = 0\,, \tag{6.18}$$

it does not form a singlet under $SU(2)_R \times SU(2)_L$, annihilated by $Q_{5,i}$ as well. Thus we have

$$\langle 0|S|0\rangle \neq 0\,, \tag{6.19}$$

reflecting the breaking $SU(2)_R \times SU(2)_L \to SU(2)_I$ by the ground state. As shown later this leads to three pseudoscalar Goldstone bosons which may be interpreted as the three pions.

2 $U(1)_A$ Symmetry and θ parameter in QCD

Before discussing further the consequences of the spontaneous breaking of chiral $SU(2)_R \times SU(2)_L$ symmetry we consider the axial current whose corresponding charge would appear to generate a $U(1)_A$ symmetry whenever one or more quarks are massless. If $m_u = m_d = 0$ then in addition to the currents given in (6.6) and (6.13) there is the singlet current

$$j_5^\mu = \overline{q}\gamma^\mu\gamma_5 q\,, \tag{6.20}$$

which is formally conserved. The apparent symmetry group, as shown by (6.8) and (6.9), is then $U(2)_R \times U(2)_L \simeq SU(2)_R \times SU(2)_L \times U(1)_V \times U(1)_A$. In general if any quark is massless then there is an axial current which is formally conserved and generates a $U(1)_A$ symmetry associated with the massless quark field being multiplied by $e^{i\beta\gamma_5}$. As a conventional symmetry with an invariant vacuum state the $U(1)_A$ group is unacceptable since it again leads to particle multiplets with opposite parity. It is also unacceptable even if the symmetry is just spontaneously broken by the vacuum. For $m_u = m_d = 0$ there would then be four massless Goldstone bosons whereas experimentally there are three very light pions but no corresponding fourth $I = 0$ pseudoscalar particle.

The situation is saved by the conservation of the singlet axial current j_5^μ being anomalous. By one loop calculations, maintaining carefully $SU(3)_{\text{colour}}$ gauge invariance which is necessary for the consistency and renormalisability of QCD, we have for a general singlet axial current formed from n_F flavours of massless quarks,

$$\partial_\mu j_5^\mu = n_F\,\frac{\hbar g^2}{32\pi^2}\,\epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}{\cdot}F_{\sigma\rho}\,. \tag{6.21}$$

Here g is the QCD coupling and in the present case of considering massless u, d quarks $n_F = 2$. With appropriate careful definitions of both the left and right hand sides there are no higher order corrections.

If, for the quark fields appearing in the axial current in (6.20), we let $q \to e^{i\beta\gamma_5}q$, $\overline{q} \to \overline{q}e^{i\beta\gamma_5}$ then taking $\beta(x)$ to be infinitesimal and x-dependent then the change in the QCD action, by a variant of Noether's theorem is

$$\begin{aligned} \delta S_{\text{QCD}} &= -\int \mathrm{d}^4x\,\partial_\mu\beta\, j_5^\mu \\ &= \int \mathrm{d}^4x\,\beta\partial_\mu j_5^\mu = 2n_F\beta\hbar\,\mathbf{Q}\,, \end{aligned} \tag{6.22}$$

where in the second line after integrating by parts we have taken β to be constant and used the anomalous conservation equation (6.21) where

$$\mathbf{Q} = \frac{g^2}{64\pi^2}\int \mathrm{d}^4x\,\epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}{\cdot}F_{\sigma\rho}\,. \tag{6.23}$$

As defined in (6.23) $\mathbf{Q}$ is a topological invariant since it is invariant under smooth changes in the gauge fields. To show this we may consider a variation δA_μ which gives for the field strength $\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu$, where $D_\mu \delta A_\nu = \partial_\mu A_\nu + g A_\mu \times A_\nu$ is the covariant derivative for the adjoint gauge fields, and then

$$\delta(\epsilon^{\mu\nu\sigma\rho} F_{\mu\nu}{\cdot}F_{\sigma\rho}) = 4\delta(\epsilon^{\mu\nu\sigma\rho} D_\mu \delta A_\nu {\cdot} F_{\sigma\rho} = \partial_\mu (4\epsilon^{\mu\nu\sigma\rho} A_\nu {\cdot} F_{\sigma\rho})\,, \tag{6.24}$$

using the Bianchi identity $D_{[\mu} F_{\sigma\rho]} = 0$ or $\epsilon^{\mu\nu\sigma\rho} D_\mu F_{\sigma\rho} = 0$. Hence $\delta\mathbf{Q} = 0$ since the variation of the integrand is a total derivative and with suitable boundary conditions any surface terms vanish. By integrating the variation in (6.24) we may show that

$$\epsilon^{\mu\nu\sigma\rho} F_{\mu\nu}{\cdot}F_{\sigma\rho} = \partial_\mu K^\mu\,, \qquad K^\mu = 4\epsilon^{\mu\nu\sigma\rho}(A_\nu{\cdot}\partial_\sigma A_\rho + \tfrac{1}{3} g A_\nu{\cdot}A_\sigma \times A_\rho)\,. \tag{6.25}$$

Nevertheless this does not show that $\mathbf{Q}$ is zero since although the integral in (6.23) may be reduced to a surface term we may assume that on the surface $|x| \to \infty$ the gauge field becomes a pure gauge, so that $gA_\mu \sim h^{-1}\partial_\mu h$, with $h(x) \in SU(3)_{\text{colour}}$ for QCD, and hence $F_{\mu\nu} \to 0$ and $K^\mu \sim -\frac{2}{3} g A_\nu{\cdot}A_\sigma \times A_\rho$. With the normalisation in (6.23) and smooth gauge field configurations $\mathbf{Q}$ takes integer values or

$$\mathbf{Q} = k\,, \qquad k \in \mathbb{Z}\,, \tag{6.26}$$

with k depending on the topology of $h(x)$ on the surface at infinity. There are smooth fields called instantons which give all possible values of k. The existence of such field configurations which give non zero $\mathbf{Q}$ justifies the solution to the $U(1)_A$ problem, which was the presence of four rather than three massless bosons in the chiral symmetry limit of zero m_u, m_d, through the anomalous conservation equation (6.21).

In general the QCD action can therefore be modified by an extra term proportional to Q

$$S_{\text{QCD}} \to S_{\text{QCD}} + \hbar\theta\,\mathbf{Q}\,. \tag{6.27}$$

Under the $U(1)_A$ chiral transformation as in (6.22) we therefore have

$$\theta \to \theta + 2n_F\beta\,. \tag{6.28}$$

In consequence for any massless quark θ can be transformed to zero by a suitable $U(1)_A$ transformation. However if the quark fields have a general mass term of the form

$$\mathcal{L}_m = -\overline{q_L}\mathcal{M} q_R - \overline{q_R}\mathcal{M}^\dagger q_L\,, \tag{6.29}$$

with $\mathcal{M}$ a complex $n_F \times n_F$ times matrix, then if $q \to e^{i\beta\gamma_5} q$ we must require, as well as (6.28),

$$\mathcal{M} \to e^{-2i\beta}\mathcal{M}\,. \tag{6.30}$$

Hence

$$\theta + \arg\det\mathcal{M} \tag{6.31}$$

is invariant. The extra term in (6.27) cannot be then set to zero, and will in general be generated when the quark mass matrix $\mathcal{M}$ is diagonalised in terms of real positive quark masses m_f. This θ term, if non zero, violates both P and T and experimentally it is necessary that θ be very small. Although there are some theoretical explanations involving additional particles called axions for this they are not wholly plausible and have not been tested in terms of other predictions.

3 Pions as Goldstone Bosons

In any quantum field theory when a continuous symmetry group G is spontaneously broken to a subgroup H by the vaccuum state there are $\dim G - \dim H$ massless Goldstone bosons. In the present context we take, for QCD, $G = SU(2)_R \times SU(2)_L$ and $H = SU(2)_I$ so that the general theorem requires three massless bosons. In the real world the u and d quarks are very light, with masses a few MeV, $\ll \Lambda$ the QCD scale. In this case $G = SU(2)_R \times SU(2)_L$ is not quite exact but there should remain three nearly massless bosons which are identified with the pions. With more approximation we may consider the light u, d, s quarks and take $G = SU(3)_R \times SU(3)_L$ and $H = SU(2)_F$ and the eight pseudoscalar mesons are interpreted as the required nearly massless Goldstone bosons.

Initially at least it is simplest to consider exact chiral symmetry with massless u, d quarks. The axial current $j^\mu_{5,i}$ is then conserved and we may write the second equation in (6.17) in the form

$$[Q_{5,i}, P_j(0)] = \int_V \mathrm{d}^3x\, [j^0_{5,i}(x), P_j(0)] = -i\delta_{ij} S(0)\,, \tag{6.32}$$

where V may be restricted to some finite volume since, for large $|\mathbf{x}|$, the commutator $[j^0_{5,i}(x), P_j(0)] = 0$ since then $x^2 < 0$ and also, as a result of the conservation of $j^\mu_{5,i}$, the left hand side is independent of x^0. Alternatively, neglecting possible terms involving derivatives of $\delta^3(\mathbf{x})$ we must have the local equation,

$$[j^\mu_{5,i}(x), P_j(0)]|_{x^0=0} = -i\delta_{ij}\delta^3(\mathbf{x})\, S(0)\,. \tag{6.33}$$

If we now assume, following (6.19),

$$\langle 0|S|0\rangle = -v \neq 0\,, \tag{6.34}$$

By virtue of the general proof of the Goldstone theorem there must exist three zero mass bosons $|\pi_i(p)\rangle$, $p^2 = 0$, such that

$$\langle 0|j^\mu_{5,i}(0)|\pi_k(p)\rangle = i\delta_{ik} F_\pi p^\mu\,, \qquad \langle \pi_k(p)|P_j(0)|0\rangle = \delta_{kj} Z_\pi\,, \tag{6.35}$$

with

$$F_\pi Z_\pi = v\,. \tag{6.36}$$

Since $j^\mu_{5,i}$ is an axial current then $|\pi_k(p)\rangle$ must correspond to a pseudoscalar particle. Thus the zero mass bosons have exactly the properties of pions and the usual charged particle states are given by

$$|\pi^\pm(p)\rangle = \frac{1}{\sqrt{2}}\Big(|\pi_1(p)\rangle \pm i|\pi_2(p)\rangle\Big)\,, \qquad |\pi^0(p)\rangle = |\pi_3(p)\rangle\,. \tag{6.37}$$

The coefficient F_π appearing in (6.35) is exactly the quantity which determines the decay rate $\pi^\pm \to \mu\nu$ so that it is measured to be $F_\pi = 92$ MeV.

If $m_u, m_d \neq 0$ then $SU(2)_R \times SU(2)_L$ is non longer an exact symmetry and the pion Goldstone bosons need no longer be massless but we may determine m_π^2 to first order in m_u, m_d. From (5.2) and (6.13), with $q, \overline{q}$ as in (6.2), we have

$$\partial_\mu j^\mu_{5,i} = \tfrac{1}{2}\overline{q}\{\tau_i, \mathcal{M}\}i\gamma_5 q = \tfrac{1}{2}(m_u + m_d)\overline{q}\tau_i i\gamma_5 q + \tfrac{1}{2}(m_u - m_d)\delta_{i3}\,\overline{q}i\gamma_5 q\,, \tag{6.38}$$

where

$$\mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \tfrac{1}{2}(m_u + m_d)1 + \tfrac{1}{2}(m_u - m_d)\tau_3\,. \tag{6.39}$$

Taking the divergence of the axial current matrix element in (6.35) now gives, using the definitions in (6.16),

$$\langle 0|\partial_\mu j^\mu_{5,i}(0)|\pi_k(p)\rangle = \delta_{ik}F_\pi m_\pi^2 = \tfrac{1}{2}(m_u + m_d)\langle 0|P_i(0)|\pi_k(p)\rangle = \tfrac{1}{2}(m_u + m_d)\delta_{ik}Z_\pi\,, \tag{6.40}$$

where in $\langle 0|P_i(0)|\pi_k(p)\rangle$ the pion state may be identified with that of the massless theory and we have used (6.35) as well as $\langle 0|\overline{q}i\gamma_5 q|\pi_k(p)\rangle = 0$. Hence, using (6.36), we have

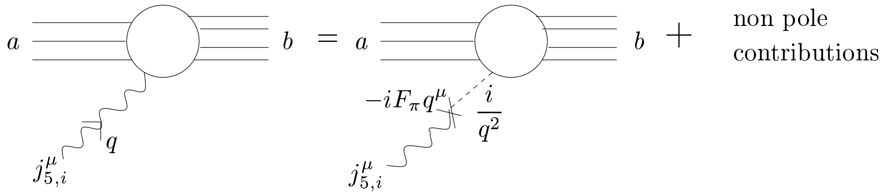
$$m_\pi^2 = \tfrac{1}{2}(m_u + m_d)\frac{v}{F_\pi^2}\,. \tag{6.41}$$

Even though m_u/m_d may not be close to 1 the pion masses for $\pi^\pm, \pi^0$ are the same so long as m_u, m_d are small.

Treating the pion as a Goldstone boson for spontaneously broken chiral symmetry leads to predictions for low energy processes involving pions. As an illustration we may consider states $|a\rangle, |b\rangle$ and then, with $p_b - p_a = q$, we have for general q^2

$$\langle b|j^\mu_{5,i}|a\rangle = \langle b|a, \pi_i(q)\rangle\,\frac{q^\mu}{q^2}F_\pi + N^\mu_{b,ai}\,. \tag{6.42}$$

The first term on the right hand side, involving a pole at $q^2 = 0$ which arises from the pion propagator i/q^2, represents the contribution of the pion produced from the vacuum, with amplitude F_π, by the axial current, since according to (6.35) we have $\langle \pi_j(q)|j^\mu_{5,i}|0\rangle = -i\delta_{ji}F_\pi q^\mu$.



Pion pole contribution to axial current matrix elements

In the chiral symmetry limit the axial current is conserved so that $q_\mu\langle b|j^\mu_{5,i}|a\rangle = 0$. Hence

$$\langle b|a,\pi_i(q)\rangle = -\frac{1}{F_\pi}\, q_\mu N^\mu_{b,ai}\,, \tag{6.43}$$

which ensures that the amplitude for $a+\pi \to b$ vanishes in the pion low energy limit, $q \to 0$, unless there are some singularities present in $p_\mu N^\mu_{b,ai}$. These can only arise due to known couplings of pions to the external particles.

3.1 Goldberger-Treiman Relation

An important result, which connects F_π with the pion-nucleon coupling constant, is called the Golberger-Treiman relation and was derived well before its connection with spontaneously broken chiral symmetry was understood. It is obtained by considering the matrix element of the $\Delta Q = 1$ axial current between a neutron and a proton state. Using parity this has the general form (there could also be a term $\sigma^{\mu\nu}\gamma_5 q_\nu g_T(q^2)$, $\sigma^{\mu\nu} = \frac{1}{4}i[\gamma^\mu,\gamma^\nu]$, on the right hand side but there are arguments why this should be zero and it is irrelevant in the following discussion)

$$\begin{aligned}&\langle p(p's')|(j^\mu_{5,1}(0)+ij^\mu_{5,2}(0))|n(ps)\rangle\\ &\qquad = \overline{u}_p(p's')\Big(\gamma^\mu\gamma_5 g_A(q^2)+\gamma_5 q^\mu g_P(q^2)\Big)u_n(ps)\,, \quad q = p'-p\,.\end{aligned} \tag{6.44}$$

This matrix element is part of the amplitude for the β-decay of a neutron, for which $q^2 \approx 0$, and the experimental decay rate determines

$$g_A = g_A(0) = 1.27\,. \tag{6.45}$$

If we assume chiral symmetry then imposing conservation of the axial current requires

$$q_\mu\langle p(p's')|(j^\mu_{5,1}(0)+ij^\mu_{5,2}(0))|n(ps)\rangle = 0\,, \tag{6.46}$$

which leads to, assuming $M_n = M_p = M$,

$$2M g_A(q^2) + q^2 g_P(q^2) = 0\,. \tag{6.47}$$

Given that $g_A(0) \neq 0$ it follows that $g_P(q^2)$ must contain a pole at $q^2 = 0$ which is a reflection of the contribution of the zero mass pion. The residue of the pion pole, apart from a factor $\langle\pi^+(q)|(j^\mu_{5,1}(0)+ij^\mu_{5,2}(0))|0\rangle = -\sqrt{2}iF_\pi q^\mu$, can be calculated in terms of the coupling constant for $\pi^+ + n \to p$. The precise definition of the pion nucleon coupling constant, which can be measured through pion nucleon scattering, can be summarised in terms of an interaction lagrangian for nucleon spinor and pion fields given by

$$\mathcal{L}_I = g_{\pi NN}\overline{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N = \sqrt{2}g_{\pi NN}\overline{p}i\gamma_5 n\pi^+ + \dots\,, \qquad N = \begin{pmatrix}p\\ n\end{pmatrix}. \tag{6.48}$$

Using this the pion pole contribution to $g_P(q^2)$ may be directly calculated

$$g_P(q^2)\Big|_{\text{pion pole}} = -2g_{\pi NN}\frac{1}{q^2}F_\pi \, , \tag{6.49}$$

and although this is need not be the complete form for $g_P(q^2)$ the remaining parts are non singular at $q^2 = 0$. Combining (6.47) and (6.49) with (6.45), in a similar fashion to the result (6.43), finally gives the Goldberger-Treiman relation

$$g_A M = g_{\pi NN} F_\pi \, , \tag{6.50}$$

which is an exact relation in the chiral symmetry limit, with the pion as a goldstone boson. Taking $g_{\pi NN} = 12.7$, $F_\pi = 92$ MeV and $M = 940$ MeV as well as (6.45) in the real world it is accurate to 2-3%.

4 Effective Lagrangians

The most efficient and, in terms of current understanding of quantum field theory, natural method for deriving the consequences of spontaneously broken chiral symmetry is in terms of an effective Lagrangian which determines an effective quantum field theory which is an approximation to a more fundamental theory, valid for a certain range of energies, from which physical results may be more readily be calculated. Although QCD may be regarded as a fundamental theory it does not directly describe the appropriate physical degrees of freedom at low energies. The idea of an effective field theory is to construct a field theory in terms of the appropriate degrees of freedom at low energies which for QCD are the pion fields. Using the symmetries of the underlying fundamental theory (or experiment if that is unknown) the effective field theory should be determined in terms of just a few parameters or couplings. A quantum field theory allows the constraints of locality, unitarity and Lorentz invariance to be easily imposed. Since an effective field theory is constructed to apply to only a restricted energy range there is always, at least implicitly, a cut off so they need not be renormalisable but, as will become more apparent in the case of chiral lagrangians for pions, they still make sense in terms of an expansion in the energy of the processes described. From this point of view the original Fermi theory of weak interactions is an effective field theory where calculations to first order in G_F are sufficient at low energies. In this case the characteristic energy or mass scale is set by m_W, m_Z which determines the cut off on the validity of the Fermi theory. For QCD the natural energy scale is O(1) GeV, so that the low energy effective theory is appropriate for energies which are small in comparison with this.

Considering again just the case of massless u, d quarks so that QCD is invariant under the chiral symmetry $G = SU(2)_R \times SU(2)_L$ then an arbitrary spontaneous symmetry breakdown by the vacuum may be parameterised by

$$\langle 0|\overline{q_{L\bar{f}}}q_{Rf}|0\rangle = -\mathring{V}_{f\bar{f}} \, , \tag{6.51}$$

with notation as in (6.2). By virtue of the action of the chiral symmetry group in (6.8) any vacua such that

$$\mathring{V}' = A\mathring{V}B^{-1}\,, \qquad (A,B) \in SU(2)_R \times SU(2)_L\,, \tag{6.52}$$

define equivalent theories and the unbroken symmetry group H is defined by

$$\mathring{V} = A\mathring{V}B^{-1}\,, \qquad (A,B) \in H\,. \tag{6.53}$$

The set of equivalent vacua $\mathcal{V}_0$ is then identified with the coset G/H or

$$\mathcal{V}_0 = \{\mathring{V} : \mathring{V} \sim A\mathring{V}B^{-1}\} \simeq SU(2)_R \times SU(2)_L/H\,. \tag{6.54}$$

In general we have

$$\mathring{V} \sim \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \qquad v_1, v_2 \text{ real}\,,\ v_1, v_2 > 0\,, \tag{6.55}$$

so that $\mathcal{V}_0$ is specified by v_1, v_2. In order to ensure that $H = SU(2)_I$ we must require

$$v_1 = v_2 = v\,, \tag{6.56}$$

and assuming this then for the general case $\mathring{V} = v\mathring{U}$, $\mathring{U} \in SU(2)$. At the particular point $\mathring{U} = 1$ (6.51) is identical with (6.34).

For any $\mathring{V} \in \mathcal{V}_0$ there is an associated quantum field theory with a unique vacuum state $|0\rangle$ such that (6.51) holds. Any theories such that the $\mathring{V}$s belong to the same coset are physically equivalent. The particle states in the quantum field theory are obtained by the action of field operators on the vacuum. Classically if the fluctuations of the field are restricted to ground states associated with points in the coset G/H then in the long wavelength limit the energy tends to zero. When these are quantised they correspond to the massless Goldstone bosons. The full implications of spontaneously broken symmetry for the interactions of Goldstone bosons are obtained by considering a low energy effective theory with fields belonging to G/H which is invariant under G. When the fields are constant they represent a point on the vacuum manifold $\mathcal{V}_0$ and the particles present after quantisation are the Goldstone bosons.

For the case of QCD when $G = SU(2)_R \times SU(2)_K$ and $H = SU(2)_I$, so that the coset of equivalent vacua $\mathcal{V}_0$ is specified just by v, the elements of the coset may be identified with the unitary matrices $\mathring{U}$. The effective low energy field theory is then described in terms of pion fields $\boldsymbol{\pi}(x)$ so that $U(\boldsymbol{\pi}) \in SU(2)$ represents a parameterisation of an arbitrary $SU(2)$ matrix (or equivalently of the three dimensional sphere S^3). It is convenient to set $U(\mathbf{0}) = 1$ and one particular choice is,

$$U(\boldsymbol{\pi}) = e^{i\boldsymbol{\pi}\cdot\boldsymbol{\tau}/F}\,, \tag{6.57}$$

where, since the pion field $\boldsymbol{\pi}$ is conventionally assumed to have the dimensions of energy, F is a suitable energy scale. In any parameterisation then for small $\boldsymbol{\pi}$ we require $U(\boldsymbol{\pi}) \approx 1 + i\boldsymbol{\pi}\cdot\boldsymbol{\tau}/F$. For any $(A, B) \in SU(2)_R \times SU(2)_L$ we may define a nonlinear realisation acting on the pion fields by

$$U(\boldsymbol{\pi}) \to U(\boldsymbol{\pi}') = AU(\boldsymbol{\pi})B^{-1} \quad \Rightarrow \quad \boldsymbol{\pi} \underset{(A,B)}{\longrightarrow} \boldsymbol{\pi}' . \tag{6.58}$$

The construction of a low energy effective lagrangian for QCD becomes straightforward by identifying $U(\boldsymbol{\pi})$ as the relevant degrees of freedom at low energies and the consequences of chiral $SU(2)_R \times SU(2)_L$ symmetry are realised by imposing invariance under (6.58). It is easy to see that there is no possibility of constructing an invariant from $U(\boldsymbol{\pi})$ with no derivatives but that there is a unique Lorentz invariant form with two derivatives

$$\mathcal{L}_\pi = \tfrac{1}{4}F^2 \mathrm{tr}(\partial^\mu U(\boldsymbol{\pi})^\dagger \partial_\mu U(\boldsymbol{\pi})) = \tfrac{1}{2}g_{ij}(\boldsymbol{\pi})\partial^\mu \pi_i \partial_\mu \pi_j \, , \tag{6.59}$$

where the normalisation has been chosen to ensure that the $\mathrm{O}(\boldsymbol{\pi}^2)$ term in $\mathcal{L}_\pi$ has the conventional form for free massless fields, so that $g_{ij}(\mathbf{0}) = \delta_{ij}$. $\mathcal{L}_\pi$ may also be regarded as defining a field theory for fields $\boldsymbol{\pi} \in S^3$ with a metric $g_{ij}(\boldsymbol{\pi})$.

The Lagrangian $\mathcal{L}_\pi$ given by (6.59) should be regarded as the leading term in an expansion in derivatives, at the next order there are three possible terms with four derivatives. However (6.59) is sufficient to determine pionic amplitudes to $\mathrm{O}(E^2)$, where E is a typical pion energy, $E \ll 1$ GeV. Schematically $\mathcal{L}_\pi$ has the form $\mathcal{L}_\pi = \sum_n (\partial\pi)^2(\pi/F)^n$. For any Feynman diagram each vertex, which involves two derivatives contributes terms of $\mathrm{O}(E^2)$ whilst each internal line involving a massless propagator gives a contribution which is $\mathrm{O}(E^{-2})$ and each loop integral $\mathrm{d}^4\ell$ over a loop momentum ℓ generates a potential $\mathrm{O}(E^4)$ factor. For a Feynman diagram the overall contribution for V vertices, I internal lines and L loops is then

$$E^{2V-2I+4L} = E^{2+2L} \, , \quad \text{using} \;\; V + L - I = 1 \, . \tag{6.60}$$

Considering only tree Feynman diagrams, $L = 0$, is then sufficient to $\mathrm{O}(E^2)$. One loop diagrams, along with the four derivative terms at the next order in an expansion of the effective action, are necessary to consider $\mathrm{O}(E^4)$. In any loop integration a cut off is necessary to ensure finiteness. The effective theory described by (6.59) is a non renormalisable quantum field theory but this does not matter when the theory is restricted to be applicable only at low energies. At one loop, there may be divergent terms which are quadratic or quartic in the cut off but these can be absorbed in a modification of the coefficient F appearing in (6.59).

To understand the low energy theory further we consider and infinitesimal $SU(2)_R \times SU(2)_L$ transformation given by

$$A = 1 + \tfrac{1}{2}i(\delta\boldsymbol{\alpha} + \delta\boldsymbol{\beta})\cdot\boldsymbol{\tau} \, , \qquad B = 1 + \tfrac{1}{2}i(\delta\boldsymbol{\alpha} - \delta\boldsymbol{\beta})\cdot\boldsymbol{\tau} \, , \tag{6.61}$$

so that $\delta\boldsymbol{\alpha}$ corresponds to an infinitesimal $SU(2)_I$ transformation. In QCD, where the quark fields transform as in (6.2), then if $\delta\boldsymbol{\alpha}, \delta\boldsymbol{\beta}$ are taken to be x-dependent we may define the isopin and associated axial currents by

$$\delta S_{\text{QCD}} = -\int \mathrm{d}^4 x \, \Big(\partial_\mu \delta\alpha_i \, j^\mu_{I,i} + \partial_\mu \delta\beta_i \, j^\mu_{5,i} \Big) \, . \tag{6.62}$$

In the corresponding low energy theory applying (6.61) in (6.58) gives

$$\delta U = \tfrac{1}{2} i \Big(\delta\boldsymbol{\alpha} \cdot [\boldsymbol{\tau}, U] + \delta\boldsymbol{\beta} \cdot \{\boldsymbol{\tau}, U\} \Big) \, , \tag{6.63}$$

and, when $\delta\boldsymbol{\alpha}, \delta\boldsymbol{\beta}$ are x-dependent, we may define vector, axial currents by

$$\delta S_\pi = -\int \mathrm{d}^4 x \, \Big(\partial_\mu \delta\boldsymbol{\alpha} \cdot \mathbf{V}^\mu + \partial_\mu \delta\boldsymbol{\beta} \cdot \mathbf{A}^\mu \Big) \, , \qquad S_\pi = \int \mathrm{d}^4 x \, \mathcal{L}_\pi \, . \tag{6.64}$$

The currents V^μ_i, A^μ_i then directly correspond to the quark currents $j^\mu_{I,i}, j^\mu_{5,i}$ in the low energy effective theory. By using (6.61) with (6.59) we may calculate their explicit form,

$$\begin{aligned} \mathbf{V}^\mu &= \tfrac{1}{4} i F^2 \, \mathrm{tr}\Big(\boldsymbol{\tau} (\partial^\mu U \, U^\dagger + \partial^\mu U^\dagger \, U) \Big) = \boldsymbol{\pi} \times \partial^\mu \boldsymbol{\pi} + \dots \, , \\ \mathbf{A}^\mu &= \tfrac{1}{4} i F^2 \, \mathrm{tr}\Big(\boldsymbol{\tau} (\partial^\mu U \, U^\dagger - \partial^\mu U^\dagger \, U) \Big) = -F \partial^\mu \boldsymbol{\pi} + \dots \, , \end{aligned} \tag{6.65}$$

where we have displayed the lowest order terms in an expansion in the pion fields. It is crucial to note that $\mathbf{A}^\mu$ is has a term which is linear in the pion fields unlike $\mathbf{V}^\mu$ for which the leading term is quadratic. In the quantum field theory then in a perturbative treatment we expand about a theory of free massless pions where the pion states are $|\pi_i(p)\rangle$, $i = 1, 2, 3$, $p^2 = 0$, and the pion field $\boldsymbol{\pi}(x)$ satisfies

$$\langle 0 | \pi_i(x) | \pi_j(p) \rangle = \delta_{ij} e^{-ip \cdot x} \, . \tag{6.66}$$

With the axial current given by (6.65) then, neglecting any loops, only the linear term in the pion fields contributes to the corresponding matrix element giving

$$\langle 0 | A^\mu_i(0) | \pi_j(p) \rangle = i F \delta_{ij} p^\mu \, . \tag{6.67}$$

Comparing with (6.35) we must therefore take

$$F = F_\pi \tag{6.68}$$

so that the single parameter in the lowest order term in a derivative expansion of the effective action, given by (6.59), is determined.

Besides the currents given by (6.65) the effective theory posesses an additional topological current

$$V^\mu = \frac{1}{24\pi^2} \, \epsilon^{\mu\alpha\beta\gamma} \mathrm{tr}(U^{-1} \partial_\alpha U \, U^{-1} \partial_\beta U \, U^{-1} \partial_\gamma U) \, . \tag{6.69}$$

This may be seen to be conserved identically, $\partial_\mu V^\mu = 0$, independent of the equations of motion. Under parity, since the pion is pseudoscalar, we should take

$$U(\boldsymbol{\pi}(x)) \underset{P}{\longrightarrow} U(\boldsymbol{\pi}(x_P))^{-1}\,, \qquad x_P = (x^0, -\mathbf{x})\,, \tag{6.70}$$

so that V^μ is a vector current, $V^\mu(x) \underset{P}{\longrightarrow} (V^0(x_P), -\mathbf{V}(x_P))$. This current may be identified with that associated with the $U(1)_V$ symmetry in QCD leading to the conservation of baryon number. The coefficient in (6.69) is chosen so that the associated charge takes integer values for fields obeying suitable boundary conditions.

5 Electromagnetic Interactions, $\pi^0 \to \gamma\gamma$ decay

The simple effective lagrangian for low energy pion amplitudes can be extended in various ways. An important and non trivial task is to extend it to include electromagnetic interactions involving the photon field A_μ. For pions we have $Q = I_3$ so that the gauge group $U(1)_Q$ is identified with the $U(1)$ subgroup of $SU(2)_I$ generated by I_3. Gauge transformations are, from (6.58), then $U \to e^{\frac{1}{2}i\lambda\tau_3} U e^{-\frac{1}{2}i\lambda\tau_3}$ as well $A_\mu \to A_\mu + \partial_\mu \lambda / e$. As usual the effective lagrangian such as (6.59) is made gauge invariant by replacing derivatives by covariant derivatives,

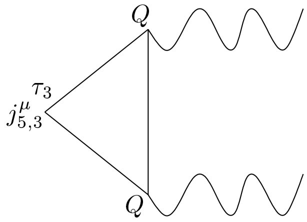
$$\partial_\mu U \longrightarrow D_\mu U = \partial_\mu U - \tfrac{1}{2} ie A_\mu [\tau_3, U]\,. \tag{6.71}$$

However the interactions obtained in this way do not describe the decay $\pi^0 \to \gamma\gamma$, which is the dominant decay of the π^0. By parity the simplest amplitude for this process must involve an ϵ-tensor which cannot arise from any lagrangian obtained through the replacement (6.71).

The solution is another consequence of anomalies in the conservation of axial currents constructed from fermions in quantum field theory. With the quarks coupled to the electromagnetic field the current $j^\mu_{5,3}$, defined in (6.13), is no longer conserved when $m_u = m_d = 0$ but

$$\partial_\mu j^\mu_{5,3} = \frac{e^2}{32\pi^2}\, S\, \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}\,, \qquad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha\,. \tag{6.72}$$

To derive this it is sufficient to consider just the triangle diagram formed from a quark loop coupled to two photons through the usual electromagnetic current $\overline{q} Q \gamma^\mu q$ and also $j^\mu_{5,3} = \overline{q} \tfrac{1}{2} \tau_3 \gamma^\mu \gamma_5 q$,



Triangle Graph Giving the Axial Anomaly

The coefficient S in (6.72) is given by

$$S = \operatorname{tr}(\tau_3 Q^2)\,, \tag{6.73}$$

and it can be shown that it is not affected by QCD corrections. For the case here of just u, d quarks

$$Q^2 = \begin{pmatrix} \frac{4}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \quad \Rightarrow \quad S = 3 \times \tfrac{1}{3} = 1\,, \tag{6.74}$$

where the factor 3 comes from three colours.

Corresponding to a chiral transformation as in (6.8) with $A = B^{-1} = e^{\frac{1}{2}i\tau_3\beta}$ we have, in a similar fashion to (6.22),

$$\delta S_{\mathrm{QCD}} = -\int \mathrm{d}^4x\, \partial_\mu \beta\, j^\mu_{5,3} = \frac{e^2}{32\pi^2}\, S \int \mathrm{d}^4x\, \beta\, \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}\,. \tag{6.75}$$

The anomaly may then be incorporated in the low energy effective theory by modifying the action so that it satisfies

$$\delta S_\pi = \frac{e^2}{32\pi^2}\, S \int \mathrm{d}^4x\, \beta\, \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}\,, \tag{6.76}$$

when

$$\delta U(\boldsymbol{\pi}) = i\beta\{\tfrac{1}{2}\tau_3, U(\boldsymbol{\pi})\}\,. \tag{6.77}$$

The desired result may be achieved by adding to the low energy effective theory and additional anomaly piece

$$\mathcal{L}_{\mathrm{anomaly}} = F(\boldsymbol{\pi})\, \frac{e^2}{32\pi^2}\, S\, \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}\,, \tag{6.78}$$

where under (6.77) we require

$$\delta F(\boldsymbol{\pi}) = \beta\,. \tag{6.79}$$

For simplicity we consider only the lowest order contribution to $F(\boldsymbol{\pi})$. Using $U(\boldsymbol{\pi}) \approx 1 + i\boldsymbol{\pi}{\cdot}\boldsymbol{\tau}/F$ then (6.77) gives $\delta\pi_i = \beta F \delta_{i3} + \ldots$ so that we may take

$$F(\boldsymbol{\pi}) = \frac{1}{F}\, \pi_3 + \ldots\,. \tag{6.80}$$

For $\pi^0 \to \gamma\gamma$ decay the amplitude to $\mathrm{O}(e^2)$ may then be taken as

$$\mathcal{M} = \langle \gamma(q_1\varepsilon_1)\gamma(q_2\varepsilon_2)|\mathcal{L}_{\mathrm{anomaly}}(0)|\pi^0(p)\rangle\,, \tag{6.81}$$

where $\varepsilon_1, \varepsilon_2$ are the polarisation vectors of the decay photons and

$$q_1^2 = q_2^2 = 0\,, \qquad p^2 = m_\pi^2\,, \qquad p = q_1 + q_2\,. \tag{6.82}$$

Using (6.78) and (6.80) with (6.68) we may obtain

$$\mathcal{M} = 8\,\frac{1}{F_\pi}\,\frac{e^2}{32\pi^2}\,S\,\epsilon^{\alpha\beta\gamma\delta}\varepsilon^*_{1\alpha}q_{1\beta}\varepsilon^*_{2\gamma}q_{2\delta}\,. \tag{6.83}$$

The decay rate is then given by, with an extra factor $\frac{1}{2}$ since the two final photons are identical bosons,

$$\Gamma_{\pi^0\to\gamma\gamma} = \frac{1}{2}\,\frac{1}{2m_\pi}\sum_{q_1\varepsilon_1,q_2\varepsilon_2}\delta^4(p-q_1-q_2)|\mathcal{M}|^2\,, \qquad \sum_{q_1,q_2} = \int\frac{\mathrm{d}^3q_1}{(2\pi)^3 2q_1^0}\frac{\mathrm{d}^3q_2}{(2\pi)^3 2q_2^0}\,. \tag{6.84}$$

Since, summing over the photon spins we may take $\sum_\varepsilon \varepsilon_\mu\varepsilon^*_\nu \to -g_{\mu\nu}$ and in contracting ϵ-tensors $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\gamma\delta} = -2(\delta^\alpha{}_\gamma\delta^\beta{}_\delta - \delta^\alpha{}_\delta\delta^\beta{}_\gamma)$, we have, using also (6.82),

$$\sum_{\varepsilon_1,\varepsilon_2}|\epsilon^{\alpha\beta\gamma\delta}\varepsilon^*_{1\alpha}q_{1\beta}\varepsilon^*_{2\gamma}q_{2\delta}|^2 = 2(q_1{\cdot}q_2)^2\,, \tag{6.85}$$

so that, since $2q_1{\cdot}q_2 = m_\pi^2$,

$$\sum_{\varepsilon_1,\varepsilon_2}|\mathcal{M}|^2 = \frac{\alpha^2 m_\pi^4}{2\pi^2 F_\pi^2}\,S^2\,, \qquad \alpha = \frac{e^2}{4\pi}\,. \tag{6.86}$$

The remaining phase space integration is easy since

$$\int\frac{\mathrm{d}^3q_1}{|\mathbf{q}_1|}\frac{\mathrm{d}^3q_2}{|\mathbf{q}_2|}\,\delta^4(p-q_1-q_2) = 2\pi\,, \tag{6.87}$$

and hence finally

$$\Gamma_{\pi^0\to\gamma\gamma} = \frac{\alpha^2 m_\pi^3}{64\pi^3 F_\pi^2}\,S^2\,. \tag{6.88}$$

This is in agreement with experiment for $S = 1$.