The Standard Model 2, Spontaneous Symmetry Breakdown

(1) A field theory is described in terms of the elements of a complex $N \times N$ matrix $M$ by a Lagrangian

$$\mathcal{L} = \text{tr}(\partial^\mu M^\dagger \partial_\mu M) - \frac{1}{2} \lambda \text{tr}(M^\dagger M M^\dagger M) - k \text{tr}(M^\dagger M),$$

where $\text{tr}$ denotes the matrix trace and $\lambda > 0$. Show that this theory is invariant under the symmetry group $U(N) \times U(N)/U(1)$ for transformations given by $M \to A M B^{-1}$ for $A, B \in U(N)$ and where the $U(1)$ corresponds to $A = B = e^{i \theta} I$ (note that if $H$ is a subgroup of $G$ then $G/H$ is a group if $H$ belongs to the centre of $G$, i.e. $h g = g h$ for all $h \in H, g \in G$). Show that if $k < 0$ spontaneous symmetry breakdown occurs and that in the ground state $M_0^\dagger M_0 = v^2 I$ for some $v$. What is the unbroken symmetry group and how many Goldstone modes are there?

If $\mathcal{L} \to \mathcal{L} + \mathcal{L}'$ where

$$\mathcal{L}' = h(\text{det}(M) + \text{det}(M^\dagger)),
$$

what is the symmetry group and how many Goldstone modes are there now after spontaneous symmetry breakdown? (assume the ground state still satisfies $M_0^\dagger M_0 = v^2 I$)

[Note $U(N) = SU(N) \times U(1)/Z_N$ where $Z_N$ is the finite group corresponding to the complex numbers $e^{2 \pi i k/N}, k = 0, \ldots, N - 1$, under multiplication.]

(2) A field theory has 5 real scalar fields $\phi_{a}$ which are expressed in terms of a symmetric traceless $3 \times 3$ matrix $\Phi = \sum_{a} \phi_{a} t_{a}$ where $t_{a}$ are a basis of symmetric traceless matrices with $\text{tr}(t_{a} t_{b}) = \delta_{ab}$, where $\text{tr}$ denotes the trace. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \text{tr}(\partial^\mu \Phi \partial_\mu \Phi) - V(\Phi), \quad V(\Phi) = g \left( \frac{1}{4} \text{tr}(\Phi^4) + \frac{1}{3} b \text{tr}(\Phi^3) + \frac{1}{2} c \text{tr}(\Phi^2) \right),$$

where $g > 0$. Show that this theory has an $SO(3)$ symmetry. Let $\mathcal{M}_0 = \{ \Phi_0 : V(\Phi_0) = V_{\text{min}} \}$. Assume $SO(3)$ acts transitively on $\mathcal{M}_0$, i.e. all points in $\mathcal{M}_0$ can be linked by an $SO(3)$ transformation. Show that then all $\Phi_0 \in \mathcal{M}_0$ have the same eigenvalues, which add up to zero, and that we may choose $\Phi_0$ so that it is diagonal. Describe how the eigenvalues of $\Phi_0$ determine the unbroken subgroup of $SO(3)$.

For this theory show that $\mathcal{M}_0$ is determined by the equation

$$\Phi_0^3 + b \Phi_0^2 + c \Phi_0 = \mu I, \quad 3\mu = \text{tr}(\Phi_0^3) + b \text{tr}(\Phi_0^2).$$

($\mu$ may be regarded as a Lagrange multiplier for the condition $\text{tr}(\Phi) = 0$ when varying $V(\Phi)$).

Verify that there is a potential solution in which the unbroken subgroup is $SO(2)$ if $b^2 > 12c$ (note that in this case $\Phi_0$ may be given in terms of a single eigenvalue).

For $3 \times 3$ traceless matrices $\text{tr}(M^4) = \frac{1}{2} \left( \text{tr}(M^2) \right)^2$. Show that if $b = 0$ the initial symmetry is in fact $SO(5)$ and that $V_{\text{min}} = \frac{1}{8} g c^2$ with an unbroken group $SO(4)$.

How do the results on possible unbroken symmetry groups generalise to the analogous theory with $SO(N)$ symmetry defined in terms of $N \times N$ symmetric traceless matrices?

(3) Consider a $SU(2)$ gauge theory coupled to a two component complex scalar field $\phi$ acting on which the $SU(2)$ generators are represented by $\frac{1}{2} \mathbf{\tau}$, for $\mathbf{\tau}$ the usual Pauli matrices,

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} + (D_\mu \phi)^\dagger D_\mu \phi - \frac{1}{2} \lambda (\phi^\dagger \phi - \frac{1}{2} v^2)^2,$$
Verify that impose the conditions Re scattering amplitude $M$ 3-direction. Explain why transformations, by choosing $\phi$ Show that in the classical ground state the potential may be minimised, up to a freedom of gauge $v$ Why are theories with different values of $\phi$ Let $g$, $s$ = $(4)$ A triplet gauge field $A_\mu$ is coupled to a real triplet field $\phi$ with the Lagrangian, $L = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} (D^\mu \phi) D_\mu \phi - \frac{1}{8} \lambda (\phi^2 - v^2)^2$ , $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu$ , $D_\mu \phi = \partial_\mu \phi - ig A_\mu \phi$. Show that this theory is invariant under $SU(2)$ gauge transformations but that this is broken by the ground state to $U(1)$. Rewrite the theory in terms of physical fields and determine their masses and couplings. For a complex triplet field $\phi$ suppose the Lagrangian is $L = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + (D^\mu \phi)^* D_\mu \phi + \frac{1}{2} g^2 (\phi^* \times \phi)^2$. Show that in the classical ground state the potential may be minimised, up to a freedom of gauge transformations, by choosing $\phi_0 = \frac{1}{v} v e_3$ for any complex $v$ where $e_3$ is the unit vector in the 3-direction. Explain why $v \sim -v$ under residual gauge transformations. Why is it possible to impose the conditions $\text{Re} (v^* \phi \cdot e_1) = \text{Re} (v^* \phi \cdot e_2) = 0$? Determine the masses of the physical fields. Why are theories with different values of $v^2$ inequivalent? (4) A gauge theory for the group $G$ is described by the Lagrangian, $L = -\frac{1}{4} F^{\mu \nu \alpha \sigma} F_{\mu \nu \alpha \sigma} \frac{1}{2} (D^\mu \phi) D_\mu \phi - V(\phi)$, $F_{\mu \nu \alpha \sigma} = \partial_\mu A_\nu - \partial_\nu A_\mu + g c_{abc} A_\mu \theta_a \theta_b \theta_c$, $D_\mu \phi = \partial_\mu \phi + g A_\mu \theta_a \phi$, with $a = 1, \ldots \dim G$ and $\theta_a$ matrices representing the Lie algebra of $G$, $[\theta_a, \theta_b] = c_{abc} \theta_c$ and $c_{abc}$ is completely antisymmetric. Assuming $V'(\phi) \theta_a \phi = 0$ and $\phi' (\theta_a \phi) = -(\theta_a \phi') \phi$ show that $L$ is invariant under $G$ gauge transformations. Suppose $V(\phi)$ is minimised at $\phi = \phi_0$ and that we add a gauge fixing term of the form $L_{\text{fix.}} = -\frac{1}{2} \left( (\partial^\mu A_{\mu a} - g (\theta_a \phi_0) \phi) (\partial^\nu A_{\nu a} - g (\theta_a \phi_0) \phi) \right)$. If $\phi = \phi_0 + f$ derive the decoupled linearised equations of motion for the vector, scalar fields, $\partial^2 A_{\mu a} + g^2 (\theta_a \phi_0) (\theta_b \phi_0) A_{\mu b} = 0$ , $\partial^2 f + M \cdot f + g^2 (\theta_a \phi_0) (\theta_b \phi_0) f = 0$ , where $M$ is a matrix determined by the second derivatives of $V(\phi)$ at $\phi = \phi_0$. Show that the mass eigenstates form multiplets of the unbroken gauge group $H$, for which the corresponding gauge fields are massless (it is sufficient to show that the mass matrices appearing in the linear field equations commute with the generators of $H$ in the appropriate representation). (6) Let $L = \partial^\mu \phi^* \partial_\mu \phi - \frac{1}{2} g (\phi^* \phi - \frac{1}{2} v^2)^2$ be the Lagrangian for a complex scalar field $\phi$. Writing $\phi = \frac{1}{\sqrt{2}} (v + f + i \alpha)$ show that the $\alpha$ field is massless whereas the $f$ field has a mass $\sqrt{v^2}$. Consider the scattering amplitude $\mathcal{M}$ for $\alpha$ particle scattering which is defined by $\langle \alpha(p_3) \alpha(p_4) | T | \alpha(p_1) \alpha(p_2) \rangle = (2\pi)^{4} \delta^{4}(p_3 + p_4 - p_1 - p_2) \mathcal{M}$ where $S = 1 - iT$. Neglecting any Feynman diagrams with loops, show that $\mathcal{M} = g^2 v^2 \left( \frac{1}{s - g v^2} + \frac{1}{t - g v^2} + \frac{1}{u - g v^2} \right) + 3 g$ , $s = (p_1 + p_2)^2$ , $t = (p_3 - p_1)^2$ , $u = (p_4 - p_1)^2$. Verify that $s + t + u = 0$ and hence show that for $\alpha$ particles with low energies $E$ we have $\mathcal{M} = O(E^4)$. 2