The Standard Model 2, Spontaneous Symmetry Breakdown

(1) A field theory is described in terms of the elements of a complex $N \times N$ matrix M by a Lagrangian

$$\mathcal{L} = \mathrm{tr}(\partial^\mu M^\dagger \partial_\mu M) - \tfrac{1}{2}\lambda\, \mathrm{tr}(M^\dagger M M^\dagger M) - k\, \mathrm{tr}(M^\dagger M)\,,$$

where tr denotes the matrix trace and $\lambda > 0$. Show that this theory is invariant under the symmetry group $U(N) \times U(N)/U(1)$ for transformations given by $M \to AMB^{-1}$ for $A, B \in U(N)$ and where the $U(1)$ corresponds to $A = B = e^{i\theta} I$ (note that if H is a subgroup of G then G/H is a group if H belongs to the centre of G, i.e. $hg = gh$ for all $h \in H$, $g \in G$). Show that if $k < 0$ spontaneous symmetry breakdown occurs and that in the ground state ${M_0}^\dagger M_0 = v^2 I$ for some v. What is the unbroken symmetry group and how many Goldstone modes are there?
If $\mathcal{L} \to \mathcal{L} + \mathcal{L}'$ where

$$\mathcal{L}' = h\big(\det(M) + \det(M^\dagger)\big)\,,$$

what is the symmetry group and how many Goldstone modes are there now after spontaneous symmetry breakdown? (assume the ground state still satisfies ${M_0}^\dagger M_0 = v^2 I$)
[Note $U(N) = SU(N) \times U(1)/Z_N$ where Z_N is the finite group corresponding to the complex numbers $e^{2\pi i k/N}$, $k = 0, \dots N-1$, under multiplication.]

(2)* A field theory has 5 real scalar fields ϕ_a which are expressed in terms of a symmetric traceless 3×3 matrix $\Phi = \sum_1^5 \phi_a t_a$ where t_a are a basis of symmetric traceless matrices with $\mathrm{tr}(t_a t_b) = \delta_{ab}$, where tr denotes the trace. The Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}\mathrm{tr}(\partial^\mu \Phi \partial_\mu \Phi) - V(\Phi)\,, \quad V(\Phi) = g\big(\tfrac{1}{4}\mathrm{tr}(\Phi^4) + \tfrac{1}{3} b\, \mathrm{tr}(\Phi^3) + \tfrac{1}{2} c\, \mathrm{tr}(\Phi^2)\big)\,,$$

where $g > 0$. Show that this theory has an $SO(3)$ symmetry. Let $\mathcal{M}_0 = \{\Phi_0 : V(\Phi_0) = V_{\mathrm{min}}\}$. Assume $SO(3)$ acts transitively on $\mathcal{M}_0$, i.e. all points in $\mathcal{M}_0$ can be linked by an $SO(3)$ transformation. Show that then all $\Phi_0 \in \mathcal{M}_0$ have the same eigenvalues, which add up to zero, and that we may choose Φ_0 so that it is diagonal. Describe how the eigenvalues of Φ_0 determine the unbroken subgroup of $SO(3)$.
For this theory show that $\mathcal{M}_0$ is determined by the equation

$$\Phi_0^3 + b\, \Phi_0^2 + c\, \Phi_0 = \mu\, I\,, \quad 3\mu = \mathrm{tr}(\Phi_0^3) + b\, \mathrm{tr}(\Phi_0^2)\,.$$

(μ may be regarded as a Lagrange multiplier for the condition $\mathrm{tr}(\Phi) = 0$ when varying $V(\Phi)$). Verify that there is a potential solution in which the unbroken subgroup is $SO(2)$ if $b^2 > 12c$ (note that in this case Φ_0 may be given in terms of a single eigenvalue).
For 3×3 traceless matrices $\mathrm{tr}(M^4) = \frac{1}{2}\big(\mathrm{tr}(M^2)\big)^2$. Show that if $b = 0$ the initial symmetry is in fact $SO(5)$ and that $V_{\mathrm{min}} = -\frac{1}{2}\, gc^2$ with an unbroken group $SO(4)$.
How do the results on possible unbroken symmetry groups generalise to the analogous theory with $SO(N)$ symmetry defined in terms of $N \times N$ symmetric traceless matrices?

(3) Consider a $SU(2)$ gauge theory coupled to a two component complex scalar field ϕ acting on which the $SU(2)$ generators are represented by $\frac{1}{2}\boldsymbol{\tau}$, for $\boldsymbol{\tau}$ the usual Pauli matrices,

$$\mathcal{L} = -\tfrac{1}{4}\, \mathbf{F}^{\mu\nu}\!\cdot\! \mathbf{F}_{\mu\nu} + (D^\mu \phi)^\dagger D_\mu \phi - \tfrac{1}{2}\lambda\big(\phi^\dagger \phi - \tfrac{1}{2}v^2\big)^2\,,$$

where

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g\,\mathbf{A}_\mu \times \mathbf{A}_\nu\,, \quad D_\mu\phi = \partial_\mu\phi - ig\,\mathbf{A}_\mu{\cdot}\tfrac{1}{2}\boldsymbol{\tau}\phi\,.$$

Explain why we may choose $\phi = \frac{1}{\sqrt{2}}(v+f)\begin{pmatrix}0\\1\end{pmatrix}$ and that the $SU(2)$ gauge symmetry is completely broken. What are the masses of the elementary particle states neglecting any quantum corrections?

(4) A triplet gauge field $\mathbf{A}_\mu$ is coupled to a real triplet field $\boldsymbol{\phi}$ with the Lagrangian,

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}{\cdot}\mathbf{F}_{\mu\nu} + \tfrac{1}{2}(D^\mu\boldsymbol{\phi}){\cdot}D_\mu\boldsymbol{\phi} - \tfrac{1}{8}\lambda\big(\boldsymbol{\phi}^2 - v^2\big)^2\,,\\ \mathbf{F}_{\mu\nu} &= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + e\,\mathbf{A}_\mu \times \mathbf{A}_\nu\,, \quad D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + e\,\mathbf{A}_\mu \times \boldsymbol{\phi}\,.\end{aligned}$$

Show that this theory is invariant under $SU(2)$ gauge transformations but that this is broken by the ground state to $U(1)$. Rewrite the theory in terms of physical fields and determine their masses and couplings.
For a complex triplet field $\boldsymbol{\phi}$ suppose the Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}{\cdot}\mathbf{F}_{\mu\nu} + (D^\mu\boldsymbol{\phi})^*{\cdot}D_\mu\boldsymbol{\phi} + \tfrac{1}{2}g^2(\boldsymbol{\phi}^* \times \boldsymbol{\phi})^2\,.$$

Show that in the classical ground state the potential may be minimised, up to a freedom of gauge transformations, by choosing $\boldsymbol{\phi}_0 = \frac{1}{\sqrt{2}}v\,\mathbf{e}_3$ for any complex v where $\mathbf{e}_3$ is the unit vector in the 3-direction. Explain why $v \sim -v$ under residual gauge transformations. Why is it possible to impose the conditions $Re(v^*\boldsymbol{\phi}{\cdot}\mathbf{e}_1) = Re(v^*\boldsymbol{\phi}{\cdot}\mathbf{e}_2) = 0$? Determine the masses of the physical fields. Why are theories with different values of v^2 inequivalent?

(5) A gauge theory for the group G is described by the Lagrangian,

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}\,F^{\mu\nu}{}_a F_{\mu\nu a} + \tfrac{1}{2}\,(D^\mu\phi){\cdot}D_\mu\phi - V(\phi)\,,\\ F_{\mu\nu a} &= \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g\,c_{abc}A_{\mu b}A_{\nu c}\,, \quad D_\mu\phi = \partial_\mu\phi + g\,A_{\mu a}\theta_a\phi\,,\end{aligned}$$

with $a = 1,\dots \dim G$ and θ_a matrices representing the Lie algebra of G, $[\theta_a, \theta_b] = c_{abc}\theta_c$ and c_{abc} is completely antisymmetric. Assuming $V'(\phi){\cdot}\theta_a\phi = 0$ and $\phi'{\cdot}(\theta_a\phi) = -(\theta_a\phi'){\cdot}\phi$ show that $\mathcal{L}$ is invariant under G gauge transformations.
Suppose $V(\phi)$ is minimised at $\phi = \phi_0$ and that we add a gauge fixing term of the form

$$\mathcal{L}_{\text{g.f.}} = -\tfrac{1}{2}\big(\partial^\mu A_{\mu a} - g(\theta_a\phi_0){\cdot}\phi\big)\big(\partial^\nu A_{\nu a} - g(\theta_a\phi_0){\cdot}\phi\big)\,.$$

If $\phi = \phi_0 + f$ derive the decoupled linearised equations of motion for the vector, scalar fields,

$$\partial^2 A_{\mu a} + g^2(\theta_a\phi_0){\cdot}(\theta_b\phi_0)\,A_{\mu b} = 0\,, \quad \partial^2 f + \mathcal{M}{\cdot}f + g^2(\theta_a\phi_0)\,(\theta_a\phi_0){\cdot}f = 0\,,$$

where $\mathcal{M}$ is a matrix determined by the second derivatives of $V(\phi)$ at $\phi = \phi_0$. Show that the mass eigenstates form multiplets of the unbroken gauge group H, for which the corresponding gauge fields are massless (it is sufficient to show that the mass matrices appearing in the linear field equations commute with the generators of H in the appropriate representation).

(6)* Let $\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - \frac{1}{2}g(\phi^*\phi - \frac{1}{2}v^2)^2$ be the Lagrangian for a complex scalar field ϕ. Writing $\phi = \frac{1}{\sqrt{2}}(v+f+i\alpha)$ show that the α field is massless whereas the f field has a mass $\sqrt{gv^2}$. Consider the scattering amplitude $\mathcal{M}$ for α particle scattering which is defined by $\langle\alpha(p_3)\alpha(p_4)|T|\alpha(p_1)\alpha(p_2)\rangle = (2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\mathcal{M}$ where $S = 1 - iT$. Neglecting any Feynman diagrams with loops, show that

$$\mathcal{M} = g^2v^2\bigg(\frac{1}{s-gv^2} + \frac{1}{t-gv^2} + \frac{1}{u-gv^2}\bigg) + 3g\,, \quad s = (p_1+p_2)^2\,,\ t = (p_3-p_1)^2\,,\ u = (p_4-p_1)^2\,.$$

Verify that $s+t+u = 0$ and hence show that for α particles with low energies E we have $\mathcal{M} = \mathrm{O}(E^4)$.