The Standard Model 3, Weak Interactions and Weinberg Salam Model

(1) In the μ decay process,

$$\mu^-(p) \to e^-(k) + \overline{\nu}_e(q) + \nu_\mu(q') \ ,$$

suppose the initial μ is polarised. The spin polarisation may be represented by a 4-vector s^μ, with $s.p = 0$, where in the muon rest frame $s^\mu = (0, \mathbf{s})$, and the corresponding Dirac spinor then satisfies $u_\mu(p)\overline{u}_\mu(p) = (\gamma.p + m_\mu)\frac{1}{2}(1 + \gamma_5 \gamma.s)$. Show that if $\mathcal{M}$ is the matrix element of $\mathcal{L}_W(0)$ for this process then

$$\sum_{\text{spins } e,\overline{\nu}_e,\nu_\mu} |\mathcal{M}|^2 = 64 G_F{}^2\, q'.k\ q.(p - m_\mu s) \ .$$

Hence obtain, neglecting the electron mass, for the differential decay rate in the μ rest frame for the final electron of energy E_e emitted into a solid angle $\mathrm{d}\Omega(\hat{\mathbf{k}})$ about the direction $\hat{\mathbf{k}}$

$$\mathrm{d}\Gamma = \frac{G_F{}^2 m_\mu^5}{24(2\pi)^4}\, x^2 \big(3 - 2x - (2x-1)\,\hat{\mathbf{k}}.\mathbf{s}\big)\, \mathrm{d}x\, \mathrm{d}\Omega(\hat{\mathbf{k}}) \ .$$

where $x = 2E_e/m_\mu$, $0 \le x \le 1$. Explain why this decay distribution is a direct indication of the breakdown of parity invariance.

(2) If $A_{\alpha 4+i5}(x)$ is $\Delta S = \Delta Q = 1$ weak hadronic current then F_K is defined by

$$\langle 0| A_{\alpha 4+i5}(0) | K^-(p)\rangle = i\sqrt{2} F_K p_\alpha \ .$$

Calculate the decay rate

$$\Gamma_{K^- \to \mu^- + \overline{\nu}_\mu} = \frac{G_F{}^2 F_K^2 \sin^2\theta_C}{4\pi} m_\mu^2 m_K \bigg(1 - \frac{m_\mu^2}{m_K^2}\bigg)^2 \ .$$

(3)* The differential cross section for scattering of two particles, with momenta p_1, p_2, in an initial state $|i\rangle$ producing a final state $|f\rangle$ is

$$\mathrm{d}\sigma = \frac{1}{F}\,\mathrm{d}\rho_f(P_i)\,|\mathcal{M}_{fi}|^2 \ , \quad \langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4 \delta^4(P_f - P_i)\mathcal{M}_{fi} \ , \quad P_i = p_1 + p_2 \ ,$$

where $\mathrm{d}\rho_f(P)$ is the differential phase space for states f with total momentum P, if $\sum_f |f\rangle\langle f| = 1$ then $\sum_f (2\pi)^4\delta^4(P_f - P)|f\rangle\langle f| = \int \mathrm{d}\rho_f(P)\,|f\rangle\langle f|\big|_{P_f = P}$, and with standard normalisations the flux $F = 4p_1{}^0 p_2{}^0 v$ for v the relative speed of the particles initially.
Using the current-current form of the weak interaction show that the cross-section for the process, $\nu_\mu(q) + e^-(p) \to \mu^-(k) + \nu_e(q')$, neglecting all masses, is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(k.p)} = \frac{2G_F{}^2}{\pi} \ .$$

Show that the total cross section is $\sigma_{\text{tot}} = 2G_F{}^2\, q.p/\pi$.

(4)* For the decay of a neutron, $n(p_1) \to p(p_2) + e^-(k) + \overline{\nu}_e(q)$, the matrix elements of the vector and axial currents appearing in the weak interaction $\mathcal{L}_W$ can be approximated if both the neutron and proton are slowly moving (neglecting v/c corrections) by

$$\langle p(p_2 s_2)|V^0_{1+i2}(0)|n(p_1 s_1)\rangle = g_V\, 2M u(s_2)^\dagger u(s_1) \;,$$
$$\langle p(p_2 s_2)|\mathbf{A}_{1+i2}(0)|n(p_1 s_1)\rangle = g_A\, 2M u(s_2)^\dagger \boldsymbol{\sigma} u(s_1) \;,$$

where $u(s)$ are non relativistic two-component spinors, $\boldsymbol{\sigma}$ are the Pauli spin matrices and M is the nucleon mass ($M_n \approx M_p$). The isospin raising operator $I_+ = \int \mathrm{d}^3x\, V_{1+i2}(x)$ and isospin symmetry requires $I_+|n(ps)\rangle = |p(ps)\rangle$. Assuming isospin invariance show that $g_V = 1$. Show that the differential decay rate for a neutron at rest, neglecting $1/M$ corrections, becomes

$$\mathrm{d}\Gamma = (1+3g_A^2)\frac{2G_F^{\,2}\cos^2\theta_C}{(2\pi)^4}\big(M_n - M_p - E_e\big)^2 \mathrm{d}^3k \;,$$

for E_e the energy of the emitted electron. Hence obtain

$$\Gamma_{n\to p+e^-+\overline{\nu}_e} = (1+3g_A^2)\frac{4G_F^{\,2}\cos^2\theta_C\, m_e^5}{(2\pi)^3} f(W_0) \;, \quad W_0 = \frac{M_n - M_p}{m_e} \;,$$
$$f(W_0) = \int_1^{W_0} (W_0 - x)^2 (x^2-1)^{\frac{1}{2}} x\, \mathrm{d}x \;.$$

Assuming isospin invariance, so that $I_+|\pi^-(p)\rangle = \sqrt{2}|\pi^0(p)\rangle$, show that the matrix element of the vector current $\langle \pi^0(p)|V^\mu_{1+i2}(0)|\pi^-(p)\rangle = 2\sqrt{2}p^\mu$ and also, using parity and Lorentz invariance $\langle \pi^0(p_2)|A^\mu_{1+i2}(0)|\pi^-(p_1)\rangle = 0$. Find the analogous formula for the decay rate for $m_{\pi^-} - m_{\pi^0} \ll m_\pi$

$$\Gamma_{\pi^-\to\pi^0+e^-+\overline{\nu}_e} = \frac{G_F^{\,2}\cos^2\theta_C\, m_e^5}{\pi^3} f(W_0) \;, \quad W_0 = \frac{m_{\pi^-} - m_{\pi^0}}{m_e} \;.$$

(5)* Use the interaction $\mathcal{L}_W = -G_F/\sqrt{2}\, J_\alpha^{\text{hadrons}\dagger}\, \overline{\nu}_\tau \gamma^\alpha(1-\gamma_5)\tau$ to show that the total decay rate for $\tau^- \to \nu_\tau + \text{hadrons}$ is

$$\Gamma_{\tau^-\to\nu_\tau+\text{hadrons}} = \frac{G_F^{\,2} m_\tau^3}{32\pi^2}\int_0^{m_\tau^2} \mathrm{d}\sigma \left(1 - \frac{\sigma}{m_\tau^2}\right)^2 \left(\rho_0(\sigma) + \left(1 + \frac{2\sigma}{m_\tau^2}\right)\rho_1(\sigma)\right) ,$$

where

$$\sum_X (2\pi)^4 \delta^4(P_X - k)\langle 0|J_\alpha^{\text{hadrons}}|X\rangle\langle X|J_\beta^{\text{hadrons}\dagger}|0\rangle = k_\alpha k_\beta \rho_0(k^2) + (-g_{\alpha\beta}k^2 + k_\alpha k_\beta)\rho_1(k^2) \,.$$

If X is restricted to the π^- show that $\rho_0(\sigma) = 4\pi F_\pi^2 \cos^2\theta_C \delta(\sigma - m_\pi^2)$ and hence find $\Gamma_{\tau^-\to\nu_\tau+\pi^-}$.

(6) Show that the interaction of the Z boson with a lepton field ℓ can be written as

$$\mathcal{L}_I = \frac{g}{2\cos\theta_W}\,\overline{\ell}\gamma^\mu(v - a\gamma_5)\ell\, Z_\mu \;,$$

where for the electron $v = 2\sin^2\theta_W - \frac{1}{2}, a = -\frac{1}{2}$ while for ν_e then $v = a = \frac{1}{2}$. Calculate the decay rate $\Gamma_{Z\to\ell\overline{\ell}}$ neglecting the lepton mass.

(7) The Lagrangian density for the Weinberg-Salam theory with gauge fields $\mathbf{A}_\mu, B_\mu$, a complex scalar field ϕ and fermion fields ψ may be written as

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - \tfrac{1}{4}\lambda(\phi^\dagger\phi - \tfrac{1}{2}v^2)^2 + \bar\psi i\gamma^\mu D_\mu\psi\,,\\ &-\big(\bar\psi\Gamma_2\phi\,\tfrac{1}{2}(1+\gamma_5)\psi_2 + \bar\psi\Gamma_1\phi^c\,\tfrac{1}{2}(1+\gamma_5)\psi_1 + \text{hermitian conjugate}\big)\,,\end{aligned}$$

where, with $\boldsymbol{\tau}$ the usual Pauli matrices,

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g\mathbf{A}_\mu\times\mathbf{A}_\nu\,,\quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,,$$

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\,,\quad \phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\,,\quad \phi^c = i\tau_2\phi^*\,,\quad D_\mu\phi = \partial_\mu\phi - i(g\mathbf{A}_\mu\cdot\tfrac{1}{2}\boldsymbol{\tau} + g'B_\mu Y)\phi\,,$$

$$D_\mu\psi = \partial_\mu\psi - i(g\mathbf{A}_\mu\cdot\tfrac{1}{2}\boldsymbol{\tau} + g'B_\mu y)\tfrac{1}{2}(1-\gamma_5)\psi - ig'B_\mu(y+Y\tau_3)\tfrac{1}{2}(1+\gamma_5)\psi\,.$$

Identify the gauge group of $\mathcal{L}$ and show how each term in $\mathcal{L}$ is gauge invariant (note that Y only enters in the product $g'Y$ so that its value is essentially arbitrary). Show that a mass term for the gauge fields of the form

$$m_W^2 W^{\mu\dagger}W_\mu + \tfrac{1}{2}\,m_Z^2 Z^\mu Z_\mu$$

is produced where $W_\mu = \frac{1}{\sqrt{2}}(A_{1\mu} - iA_{2\mu})$ and $Z_\mu = \cos\theta_W A_{3\mu} - \sin\theta_W B_\mu$ for a suitable choice of the angle θ_W. How is m_Z/m_W related to θ_W? Explain why it is impossible to introduce mass terms for the fermion fields into $\mathcal{L}$ compatible with gauge invariance. What are the fermion charges which give the coupling to the photon?

(8) Calculate the decay rate of a Higgs boson to a $\ell\bar\ell$ pair in the form, if $m_H > 2m_\ell$,

$$\Gamma_{H\to\ell\bar\ell} = \frac{G_F}{\sqrt{2}\,4\pi}\frac{m_\ell^2}{m_H^2}(m_H^2 - 4m_\ell^2)^{\frac{3}{2}}\,.$$

(9)* Show that the coupling of the W to the electromagnetic field A is described by

$$\begin{aligned}\mathcal{L}_{W,A} &= -\tfrac{1}{2}F^{W\mu\nu\dagger}F^W{}_{\mu\nu} + ie\,W^\mu W^{\nu\dagger}F_{\mu\nu}\,,\\ F^W{}_{\mu\nu} &= d_\mu W_\nu - d_\nu W_\mu\,,\quad d_\mu = \partial_\mu - ieA_\mu\,,\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.\end{aligned}$$

Determine the contribution of the W field to the electromagnetic current j^μ. Show that if $|p\epsilon\rangle$ denotes a W particle, with $p^2 = m_W^2$ and spin described by the polarisation vector ϵ, show that

$$\langle p_2\epsilon_2|j^\mu|p_1\epsilon_1\rangle = -\epsilon_2{}^*.\epsilon_1(p_2{}^\mu + p_1{}^\mu) + 2(\epsilon_2{}^{\mu*}\epsilon_1{}^\nu - \epsilon_1{}^\mu\epsilon_2{}^{\nu*})q_\nu\,,$$

where $q = p_2 - p_1$. Verify that $q_\mu\langle p_2\epsilon_2|j^\mu|p_1\epsilon_1\rangle = 0$.

(10) For two generations of quarks show that if the quark mass matrices for the charge $\frac{2}{3}, -\frac{1}{3}$ quarks are respectively, taking $\mathcal{L}_m = -\big(\,\overline{q}_+m_+\tfrac{1}{2}(1+\gamma_5)q_+ + \overline{q}_-m_-\tfrac{1}{2}(1+\gamma_5)q_- + \text{hermitian conjugate}\big)$,

$$m_+ = \begin{pmatrix}0 & a\\ a^* & b\end{pmatrix}\,,\quad m_- = \begin{pmatrix}0 & c\\ c^* & d\end{pmatrix}\,,\quad b,d\ \text{real}\,.$$

If $R(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ define θ_+ by $R(\theta_+)\begin{pmatrix}0 & |a|\\ |a| & |b|\end{pmatrix}R(\theta_+)^{-1} = \begin{pmatrix}m_u & 0\\ 0 & -m_c\end{pmatrix}$. If θ_- is similarly defined verify that the Cabbibo angle is given by

$$\theta_C = \theta_- - \theta_+ = \tan^{-1}\sqrt{\frac{m_d}{m_s}} - \tan^{-1}\sqrt{\frac{m_u}{m_c}} \approx \sqrt{\frac{m_d}{m_s}}\,,$$

where m_d, m_s, m_u, m_c are the d, s, u, c quark masses.

(11)* Show that including the Z as well as the photon the differential cross section for $e^- e^+ \to \ell\bar{\ell}$ where $\ell = e, \mu, \tau$ has the form, neglecting lepton masses,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4q^2}\Big\{(1+\cos^2\theta)\big(1+2v^2 D + (v^2+a^2)^2 D^2\big) + 4\cos\theta\big(a^2 D + 2v^2 a^2 D^2\big)\Big\} \; ,$$

where $v = 2\sin^2\theta_W - \frac{1}{2}$, $a = -\frac{1}{2}$ and $D = \dfrac{G_F}{\sqrt{2}}\,\dfrac{m_Z^2}{q^2 - m_Z^2}\Big/\dfrac{2\pi\alpha}{q^2}$. For $q^2 \approx m_Z^2$ the total cross section behaves like

$$\sigma_{e^-e^+\to\ell\bar{\ell}} \sim 12\pi\,\frac{\Gamma_{Z\to e^-e^+}\Gamma_{Z\to\ell\bar{\ell}}}{(q^2-m_Z^2)^2 + m_Z^2\Gamma^2} \; ,$$

where Γ is the total decay width. Show that the formula for the differential cross section is compatible with the result for $\Gamma_{Z\to\ell\bar{\ell}}$.

(12)* If V denotes the CKM matrix show that the low energy lagrangian for $\Delta S = 1$ processes can be written as $\mathcal{L}_W{}^{\Delta S=1} = -2\sqrt{2}G_F \sum_{q',q=u,c,t} V_{q'd}^{\,*} V_{qs}\, \overline{d_L}\gamma^\mu q'\, \bar{q}\gamma_\mu s_L$. Show that there is a second order contribution to the S operator which describes $\Delta S = 2$ processes of the form

$$S^{(2)} = \tfrac{1}{2}(2\sqrt{2}G_F)^2 \int \mathrm{d}^4x\, \mathrm{d}^4x'\, \overline{d_L}(x)\gamma^\mu S(x-x')\gamma^\nu s_L(x')\, \overline{d_L}(x')\gamma_\nu S(x'-x)\gamma_\mu s_L(x) \; ,$$

where $S(x) = \sum_{q=u,c,t} \xi_q S_F(x, m_q)$, $\xi_q = V_{qd}^{\,*} V_{qs}$, with S_F the fermion propagator,

$$S_F(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, e^{-ip\cdot x}\,\frac{\gamma{\cdot}p + m_q}{p^2 - m_q^2 + i\epsilon} \; .$$

At low energies the fields may be regarded as slowly varying so that $s_L(x')$ and $\overline{d_L}(x')$ may be expanded about x. Neglecting derivatives the essential integral in the second order expression for S then becomes

$$\int \mathrm{d}^4x\, S(-x)\times S(x) = i\big(\tfrac{1}{4}K_1\,\gamma^\lambda \times \gamma_\lambda + K_2\, 1\times 1\big) \; .$$

Show that after a rotation to Euclidean space, $p_0 \to ip_4$ and $p^2 \to -p_E^2$ then

$$K_1 = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4p_E\, p_E^2 \sum_{q',q=u,c,t} \frac{\xi_q \xi_{q'}}{(p_E^2 + m_q{}^2)(p_E^2 + m_{q'}{}^2)} \; .$$

Using $\mathrm{d}^4p_E \to \pi^2 \mathrm{d}p_E^2\, p_E^2$ and $\sum_q \xi_q = 0$, from the unitarity of the matrix V, evaluate the integral to obtain

$$K_1 = -\frac{1}{16\pi^2}\bigg(\sum_q \xi_q{}^2 m_q{}^2 + \sum_{q\neq q'} \xi_q\xi_{q'}\,\frac{m_q{}^2 m_{q'}{}^2}{m_{q'}{}^2 - m_q{}^2}\ln\frac{m_{q'}{}^2}{m_q{}^2}\bigg) \; .$$

Hence verify that

$$S^{(2)} \approx i\, G_F{}^2 K_1 \int \mathrm{d}^4x\, \overline{d_L}\gamma^\mu\gamma^\lambda\gamma^\nu s_L\, \overline{d_L}\gamma_\nu\gamma_\lambda\gamma_\mu s_L \; .$$

Using $\gamma^\mu\gamma^\lambda\gamma^\nu = g^{\mu\lambda}\gamma^\nu + g^{\lambda\nu}\gamma^\mu - g^{\mu\nu}\gamma^\lambda - i\epsilon^{\mu\nu\lambda\rho}\gamma_\rho\gamma_5$ and $\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda\sigma} = -6\delta^\rho{}_\sigma$ show that, neglecting the u quark contribution since m_u is very small, there is an effective $\Delta S = 2$ interaction at low energies given by

$$\mathcal{L}_{\mathrm{eff}}{}^{\Delta S=2} = -\frac{1}{4\pi^2}G_F{}^2\Big(\xi_c{}^2 m_c^2 + \xi_t^2 m_t^2 + 2\xi_c\xi_t\,\frac{m_c{}^2 m_t^2}{m_t^2 - m_c{}^2}\ln\frac{m_t{}^2}{m_c{}^2}\Big)\, \overline{d_L}\gamma^\lambda s_L\, \overline{d_L}\gamma_\lambda s_L \; .$$