The Standard Model 4, QCD

(1) In QCD let $a = g^2/(4\pi)^2$. The running coupling $a(\mu^2)$ is defined by

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} a = \beta(a) \, , \quad \beta(a) = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 + \mathrm{O}(a^5) \, .$$

Show that for a suitable choice of Λ

$$\frac{1}{a(\mu^2)} = \beta_0 \log \frac{\mu^2}{\Lambda^2} + \frac{\beta_1}{\beta_0} \log\log \frac{\mu^2}{\Lambda^2} + \mathrm{O}\left(1 \Big/ \log \frac{\mu^2}{\Lambda^2}\right) .$$

If the coupling is redefined so that

$$\bar{a} = f(a) = a + v_1 a^2 + v_2 a^3 + \mathrm{O}(a^4) \, ,$$

show that $\bar{\beta}(\bar{a}) = \beta(a) f'(a)$ and hence $\bar{\beta}_0 = \beta_0, \ \bar{\beta}_1 = \beta_1, \ \bar{\beta}_2 = \beta_2 + \beta_0(v_2 - v_1^2) - \beta_1 v_1$. If $\bar{a}(\mu^2) = f(a(\mu^2))$ is written in terms of $\bar{\Lambda}$ in the same form as a in terms of Λ above show that

$$\log \frac{\bar{\Lambda}^2}{\Lambda^2} = \frac{v_1}{\beta_0} \, .$$

Suppose a physical observable R has a perturbative expansion in terms of the coupling $a(\mu^2)$ of the form $R = a^N \big[r_0 + r_1 a + r_2 a^2 + \dots \big]$. Under the above redefinition $r_1 \to \bar{r}_1 = r_1 + N v_1 r_0, \ r_2 \to \bar{r}_2 = r_2 + N v_2 r_0 + \frac{1}{2} N(N-1) v_1^2 r_0 + (N+1) v_1 r_1$. Show that

$$\hat{r}_1 = r_1 - \beta_0 N r_0 \log \frac{\mu^2}{\Lambda^2} \, , \qquad \hat{r}_2 = r_2 + N \frac{\beta_2}{\beta_0} \, r_0 - \frac{N+1}{2N} \, \frac{r_1^2}{r_0} - \frac{\beta_1}{\beta_0} \, r_1 \, ,$$

are invariant under redefinitions of the couplings.

(2) For a fundamental complex scalar field φ the electromagnetic current has the form $J^\mu = i(\varphi^* \partial^\mu \varphi - \partial^\mu \varphi^* \, \varphi)$. Assuming that this field corresponds to the charged constituents of a hadron H obtain, treating the constituents as if they were free,

$$W_H{}^{\nu\mu}(q, P) \sim \int \mathrm{d}^4 k \; W_\varphi{}^{\nu\mu}(q, k) \big(\Gamma_H(P, k) + \overline{\Gamma}_H(P, k) \big) \, ,$$

where

$$W_\varphi{}^{\nu\mu}(q, k) = \tfrac{1}{2} (2k + q)^\nu (2k + q)^\mu \, \delta((k + q)^2) \, .$$

Hence obtain for $P \cdot q \; W_2(P \cdot q, -q^2) = F_2(x, -q^2)$, $W_1(P \cdot q, -q^2) = F_1(x, -q^2)$, $x = -q^2/2P \cdot q$, the asymptotic forms in the deep inelastic limit $-q^2 \to \infty$

$$F_2(x, -q^2) \sim x \big(f(x) + \overline{f}(x) \big) \, , \qquad F_1(x, -q^2) \sim 0 \, ,$$

where, for $0 < x < 1$, and taking $k = \xi P + k'$ with $k' \cdot q$ bounded

$$f(x) = x \int \mathrm{d}^4 k \, \delta(\xi - x) \, \Gamma_H(P, k) \, , \quad \overline{f}(x) = x \int \mathrm{d}^4 k \, \delta(\xi - x) \, \overline{\Gamma}_H(P, k) \, .$$

(3)* Show that the tensor which appears in the discussion of deep inelastic electron scattering on a hadron H can be written as

$$W^{\nu\mu}(q,P) = \frac{1}{4\pi}\int \mathrm{d}^4x\, e^{iq\cdot x}\langle H,P|J^\nu(x)J^\mu(0)|H,P\rangle\,.$$

Assume that the current is expressible in terms of free quark fields by $J^\mu = \,:\!\bar q\gamma^\mu Q q\!:$, where Q is a matrix of quark charges in the space of quark flavours and $:\ :$ denotes normal ordering. For free quarks show that

$$\langle 0|q(x)\bar q(0)|0\rangle = \frac{i}{2\pi^2}\,\frac{\gamma\cdot x}{(x^2 - i\epsilon x^0)^2}\,.$$

Hence show, using Wick's theorem, that as $x^2 \to 0$ the currents are expected to satisfy an operator product expansion of the form

$$\begin{aligned} J^\nu(x)J^\mu(0) \sim{}& \frac{1}{\pi^4}\,\frac{1}{(x^2-i\epsilon x^0)^4}\big(g^{\nu\mu}x^2 - 2x^\nu x^\mu\big)\mathrm{tr}(Q^2)\\ &+\frac{i}{2\pi^2}\,\frac{1}{(x^2-i\epsilon x^0)^2}\big(x^\nu\mathcal{V}^\mu(x|0) + x^\mu\mathcal{V}^\nu(x|0) - g^{\nu\mu}x\cdot\mathcal{V}(x|0) + i\epsilon^{\nu\mu\alpha\beta}\mathcal{A}_\beta(x|0)\big)\,, \end{aligned}$$

where $\mathcal{V}^\mu(x|0)$, $\mathcal{A}^\mu(x|0)$ are bilocal operators,

$$\begin{aligned} \mathcal{V}^\mu(x|0) &= \,:\!\bar q(x)\gamma^\mu Q^2 q(0)\!: - :\!\bar q(0)\gamma^\mu Q^2 q(x)\!:\,,\\ \mathcal{A}^\mu(x|0) &= \,:\!\bar q(x)\gamma^\mu\gamma_5 Q^2 q(0)\!: + :\!\bar q(0)\gamma^\mu\gamma_5 Q^2 q(x)\!:\ . \end{aligned}$$

Let

$$\langle H,P|\mathcal{V}^\mu(x|0)|H,P\rangle\big|_{x^2=0} = 2\big(P^\mu f_1(x\cdot P) + x^\mu f_2(x\cdot P)\big)\,,$$

and using, in the deep inelastic limit $-q^2, \nu = q\cdot P \to \infty$ with $x_B = -q^2/2\nu$ fixed,

$$\frac{1}{4\pi}\int \mathrm{d}^4x\, e^{iq\cdot x}\frac{x^\mu}{(x^2-i\epsilon x^0)^2}\,f(x\cdot P) \sim -\frac{\pi^2 i}{2\nu}\,q^\mu\tilde f(x_B)\ \ \text{for}\ \ x>0\,,\quad \tilde f(y) = \frac{1}{2\pi}\int \mathrm{d}u\, e^{-iyu}f(u)\,,$$

obtain in this limit $W_1(\nu,-q^2) \sim \frac12\tilde f_1(x_B)$, $\nu W_2(\nu,-q^2) \sim x_B\tilde f_1(x_B)$. From their original definition show that $f_1(u) = -f_1(u)^* = -f_1(-u)$ and hence that $\tilde f_1(x_B)$ is real.

(4)* For two couplings g, y show that a solution of the coupled renormalisation flow equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}g_\tau^2 = b\,g_\tau^4\,,\qquad \frac{\mathrm{d}}{\mathrm{d}\tau}y_\tau^2 = y_\tau^2\big(r\,g_\tau^2 - s\,y_\tau^2\big)\,,$$

with $b, r, s > 0$, is given by

$$\frac{g_\tau^2}{y_\tau^2} = \frac{1}{X} + \left(\frac{g_0^2}{y_0^2} - \frac{1}{X}\right)\left(\frac{g_0^2}{g_\tau^2}\right)^{\frac{r}{b}-1}\,,\qquad \frac{g_0^2}{g_\tau^2} = 1 - b g_0^2\tau\,,\qquad X = \frac{r-b}{s}\,.$$

Hence if $r > b$ explain why for any initial y_0, g_0 we may expect $y_\tau^2 \to X g_\tau^2$ as τ increases.
The coupling of the top quark to the neutral Higgs field ϕ_0 in the standard model is given by $\mathcal{L}_{\phi,t} = -\sqrt{2}y\big(\overline{t_L}t_R\phi_0{}^\dagger + \overline{t_R}t_L\phi_0\big)$ where SSB gives $\langle\phi_0\rangle = v/\sqrt{2}$ so that the top mass $m_t = yv$ with $v = (\sqrt{2}G_F)^{-\frac12} \approx 250\mathrm{GeV}$. The β-functions for the Yukawa coupling y and the QCD coupling g are

$$\beta^y = \frac{1}{16\pi^2}(9y^3 - 8yg^2)\,,\qquad \beta^g = -\frac{1}{16\pi^2}\,7g^3\,.$$

Determine the running couplings $y(\mu)$, $g(\mu)$ from the above solution taking $\tau = -\ln(\mu^2/\mu_0{}^2)$. Suppose the flow equations are applied with μ_0 very large so that for $\mu \approx v$ we may assume that $y(\mu)$ is given in terms of $g(\mu)$ by the above fixed point. Determine m_t choosing a suitable value of μ if $g(m_Z)^2/4\pi = 0.117$, $m_Z = 91$ GeV. A better approximation for determing y_τ consists of assuming that $\mu_0 \approx 10^{16}$ GeV and that in the above we can then neglect g_0^2/y_0^2. What is m_t then?