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Superconformal Ward Identities for Correlation Functions in Three Dimensions

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1. Conformal and Superconformal Transformations in Three Dimensions

As usual with supersymmetry it is convenient to adopt a spinorial notation. In three dimensions the gamma matrices are expressible in terms of symmetric real 2×2 matrices

$$(\sigma_a)_{\alpha\beta} = (\sigma_a)_{\beta\alpha} \, , \qquad (\tilde{\sigma}_a)^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(\sigma_a)_{\gamma\delta} \, , \tag{1.1}$$

with $\alpha, \beta = 1, 2$ and

$$\sigma_a \, \tilde{\sigma}_b + \sigma_b \, \tilde{\sigma}_a = -2\eta_{ab} \, \mathrm{I} \, , \tag{1.2}$$

with η_{ab} the 3-dimensional Minkowski metric with signature $(-1,1,1)$ and I the identity matrix. Any 3-vector x^a is then equivalent to a symmetric $2{\times}2$ matrix $\begin{pmatrix} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{pmatrix}$ or, using the σ-matrices in (1.1),

$$x^a \to \mathrm{x}_{\alpha\beta} = (x^a \sigma_a)_{\alpha\beta} \, , \qquad \tilde{\mathrm{x}}^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\mathrm{x}_{\gamma\delta} \, , \tag{1.3}$$

so that

$$\mathrm{x}\tilde{\mathrm{x}} = -x^2 \, \mathrm{I} \, , \qquad x^2 = -\tfrac{1}{2}\mathrm{x}_{\alpha\beta}\tilde{\mathrm{x}}^{\alpha\beta} = -\det \mathrm{x} \, . \tag{1.4}$$

For x^a real $\mathrm{x} = \mathrm{x}^*$, on analytic continuation to a Euclidean metric $\tilde{\mathrm{x}} = -\mathrm{x}^*$. We also define

$$\partial_{\alpha\beta} = (\sigma^a \partial_a)_{\alpha\beta} \, , \qquad \tilde{\partial}^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\partial_{\gamma\delta} \, , \tag{1.5}$$

so that

$$\partial_{\alpha\beta} \, \tilde{\mathrm{x}}^{\gamma\delta} = -\delta_\alpha{}^\gamma \delta_\beta{}^\delta - \delta_\alpha{}^\delta \delta_\beta{}^\gamma \, . \tag{1.6}$$

Conformal transformations are determined in terms of conformal Killing vectors $v^a(x)$ which, with the above notation, can be written as

$$\tilde{\mathrm{v}}^{\alpha\beta} = \tilde{\mathrm{a}}^{\alpha\beta} - \tilde{\mathrm{x}}^{\gamma\beta}\omega_\gamma{}^\alpha - \tilde{\mathrm{x}}^{\alpha\gamma}\omega_\gamma{}^\beta + \lambda \, \tilde{\mathrm{x}}^{\alpha\beta} + \tilde{\mathrm{x}}^{\alpha\beta} \, \mathrm{b}_{\gamma\delta} \, \tilde{\mathrm{x}}^{\delta\beta} \, , \quad \omega_\gamma{}^\gamma = 0 \, . \tag{1.7}$$

For any $\tilde{\mathrm{v}}^{\alpha\beta}$ of the form (1.7)

$$\big[\partial_{\alpha\beta}, \tfrac{1}{2}\tilde{\mathrm{v}}^{\gamma\delta}\partial_{\gamma\delta}\big] = \hat{\omega}_\alpha{}^\gamma \partial_{\gamma\beta} + \hat{\omega}_\beta{}^\gamma \partial_{\alpha\gamma} - \hat{\lambda} \, \partial_{\alpha\beta} \, , \qquad \hat{\omega}_\alpha{}^\alpha = 0 \, , \tag{1.8}$$

which defines $\hat{\omega}_\beta{}^\alpha = \omega_\beta{}^\alpha - \mathrm{b}_{\beta\gamma}\tilde{\mathrm{x}}^{\gamma\alpha} + \frac{1}{2} \, \delta_\beta{}^\alpha \, \mathrm{b}_{\gamma\delta}\tilde{\mathrm{x}}^{\gamma\delta}$, $\hat{\lambda} = \lambda + \mathrm{b}_{\gamma\delta}\tilde{\mathrm{x}}^{\gamma\delta}$.

For a conformal primary field of spin s, $\Phi_{\alpha_1 \dots \alpha_{2s}}(x) = \Phi_{(\alpha_1 \dots \alpha_{2s})}(x)$, then conformal transformations have the form

$$\delta_v \Phi_{\alpha_1 \dots \alpha_{2s}} = \big(\tfrac{1}{2} \, \tilde{\mathrm{v}}^{\alpha\beta}\partial_{\alpha\beta} - \Delta \, \hat{\lambda}\,\big)\Phi_{\alpha_1 \dots \alpha_{2s}} + 2s \, \hat{\omega}_{(\alpha_1}{}^\alpha \, \Phi_{\alpha_2 \dots \alpha_{2s})\alpha} \, , \tag{1.9}$$

with Δ the scale dimension. Writing the transformation (1.9) as

$$i\,\delta_v \Phi_{\alpha_1\dots\alpha_{2s}} = \big[\tfrac12\,\tilde{\rm a}^{\alpha\beta} P_{\alpha\beta} - i\lambda\, H + i\,\omega_\beta{}^\alpha M_\alpha{}^\beta + \tfrac12\,{\rm b}_{\alpha\beta} K^{\alpha\beta}, \Phi_{\alpha_1\dots\alpha_{2s}}\big]\,, \tag{1.10}$$

this gives

$$\begin{aligned}
\big[P_{\alpha\beta},\, \Phi_{\alpha_1\dots\alpha_{2s}}(x)\big] &= i\partial_{\alpha\beta}\,\Phi_{\alpha_1\dots\alpha_{2s}}(x)\ \ , \\
\big[M_\alpha{}^\beta,\, \Phi_{\alpha_1\dots\alpha_{2s}}(x)\big] &= \ - \big(\tilde{\rm x}^{\beta\gamma}\partial_{\gamma\alpha} - \tfrac12\,\delta_\alpha{}^\beta\,\tilde{\rm x}^{\gamma\delta}\partial_{\gamma\delta}\big)\,\Phi_{\alpha_1\dots\alpha_{2s}}(x) \\
&\quad\ + 2s\big(\delta_{(\alpha_1}{}^\beta \Phi_{\alpha_2\dots\alpha_{2s})\alpha}(x) - \tfrac12\,\delta_\alpha{}^\beta\,\Phi_{\alpha_1\dots\alpha_{2s}}(x)\big)\,, \\
\big[H, \Phi_{\alpha_1\dots\alpha_{2s}}(x)\big] &= \big(-\tfrac12 \tilde{\rm x}^{\alpha\beta}\partial_{\alpha\beta} + \Delta\big)\,\Phi_{\alpha_1\dots\alpha_{2s}}(x)\,, \\
\big[K^{\alpha\beta},\, \Phi_{\alpha_1\dots\alpha_{2s}}(x)\big] &= i\big(\tilde{\rm x}^{\alpha\gamma}\tilde{\rm x}^{\beta\delta}\,\partial_{\gamma\delta} - 2(\Delta - s)\,\tilde{\rm x}^{\alpha\beta}\big)\,\Phi_{\alpha_1\dots\alpha_{2s}}(x)\,, \\
&\quad\ - i\,2s\,\delta_{(\alpha_1}{}^\alpha \tilde{\rm x}^{\beta\gamma}\,\Phi_{\alpha_2\dots\alpha_{2s})\gamma}(x) - i\,2s\,\delta_{(\alpha_1}{}^\beta \tilde{\rm x}^{\alpha\gamma}\,\Phi_{\alpha_2\dots\alpha_{2s})\gamma}(x)\,.
\end{aligned} \tag{1.11}$$

Since

$$\big[\delta_{v_2}, \delta_{v_1}\big] = \delta_{v'}\,, \qquad \tilde{\rm v}'^{\,\alpha\beta} = \tfrac12\big(\tilde{\rm v}_1{}^{\gamma\delta}\partial_{\gamma\delta}\,\tilde{\rm v}_2{}^{\alpha\beta} - \tilde{\rm v}_2{}^{\gamma\delta}\partial_{\gamma\delta}\,\tilde{\rm v}_1{}^{\alpha\beta}\big)\,, \tag{1.12}$$

then (1.10) implies the commutation relations for the Lie algebra of the three dimensional conformal group $SO(3,2)$,[1]

$$\begin{aligned}
\big[M_\alpha{}^\beta, M_\gamma{}^\delta\big] &= \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta\,, \quad \big[M_\alpha{}^\beta, P_{\gamma\delta}\big] = \delta_\gamma{}^\beta P_{\alpha\delta} + \delta_\delta{}^\beta P_{\gamma\alpha} - \delta_\alpha{}^\beta P_{\gamma\delta}\,, \\
\big[H, P_{\alpha\beta}\big] &= P_{\alpha\beta}\,, \qquad \big[H, K^{\alpha\beta}\big] = -K^{\alpha\beta}\,, \qquad \big[H, M_\alpha{}^\beta\big] = 0\,, \\
\big[M_\alpha{}^\beta, K^{\gamma\delta}\big] &= \ - \delta_\alpha{}^\gamma K^{\beta\delta} - \delta_\alpha{}^\delta K^{\gamma\beta} + \delta_\alpha{}^\beta K^{\gamma\delta}\,, \\
\big[P_{\alpha\beta}, K^{\gamma\delta}\big] &= 4\,\delta_{(\alpha}{}^{(\gamma} M_{\beta)}{}^{\delta)} + 4\,\delta_{(\alpha}{}^\gamma \delta_{\beta)}{}^\delta H\,, \qquad \big[P_{\alpha\beta}, P_{\gamma\delta}\big] = \big[K^{\alpha\beta}, K^{\gamma\delta}\big] = 0\,.
\end{aligned} \tag{1.13}$$

$P_{\alpha\beta} = P_{\beta\alpha}$ is the momentum operator, $M_\alpha{}^\beta$ the generator of three dimensional Lorentz transformations while H is the generator for scale transformations and $K^{\alpha\beta} = K^{\beta\alpha}$ for special conformal transformations.

Standard quantum hermeticity conditions require

$$(P_{\alpha\beta})^\dagger = P_{\alpha\beta}\,, \quad (K^{\alpha\beta})^\dagger = K^{\alpha\beta}\,, \quad H^\dagger = -H\,, \quad (M_\alpha{}^\beta)^\dagger = -M_\alpha{}^\beta\,. \tag{1.14}$$

However for states $|\Phi_{\alpha_1\dots\alpha_{2s}}\rangle = \Phi_{\alpha_1\dots\alpha_{2s}}(0)|0\rangle$, which are annihilated by $K^{\alpha\beta}$ and $H|\Phi_{\alpha_1\dots\alpha_{2s}}\rangle = \Delta|\Phi_{\alpha_1\dots\alpha_{2s}}\rangle$, then there is an associated scalar product defining a conjugation such that

$$(P_{\alpha\beta})^+ = -K^{\alpha\beta}\,, \quad (K^{\alpha\beta})^+ = -P_{\alpha\beta}\,, \qquad H^+ = H\,, \qquad (M_\alpha{}^\beta)^+ = M_\beta{}^\alpha\,. \tag{1.15}$$

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[1] For $P_{\alpha\beta} = (\sigma^a P_a)_{\alpha\beta}$, $K^{\alpha\beta} = (\tilde\sigma^a K_a)^{\alpha\beta}$ and $M_\alpha{}^\beta = -\frac12 i(\sigma^a\tilde\sigma^b)_\alpha{}^\beta M_{ab}$, $M_{ab} = -M_{ba}$, then P_a, K_a, M_{ab}, H satisfy the commutation relations for the standard $SO(d,2)$ conformal Lie algebra in any dimension d, with in particular $[P_a, K_b] = -2i\,M_{ab} - 2\eta_{ab}\,H$, $[M_{ab}, P_c] = i(\eta_{ac}P_b - \eta_{bc}P_c)$.

$M_\alpha{}^\beta, H$ are then the generators of the maximal compact subgroup $SO(3) \times SO(2)$.

Superconformal symmetry transformations are expressed in terms of superconformal anti-commuting Killing spinors, linear in x,

$$\hat\epsilon^\alpha(x) = \epsilon^\alpha + i\,\bar\eta_\beta\,\tilde{\mathrm{x}}^{\beta\alpha}\,. \tag{1.16}$$

Here $\hat\epsilon^\alpha(x)$ is regarded as a $\mathcal{N}$-component row vector, although additional indices are suppressed. The associated column vector formed the transpose is written as

$$\hat{\bar\epsilon}^\alpha(x) = \hat\epsilon^\alpha(x)^T = \bar\epsilon^\alpha + i\,\tilde{\mathrm{x}}^{\alpha\beta}\eta_\beta \in \mathbb{R}^{\mathcal{N}}\,. \tag{1.17}$$

If $\Phi_{\alpha_1\dots\alpha_{2s}}(x)$ is a superconformal primary field belonging to a vector space $\mathcal{V}_\Phi$ then a superconformal transformation gives

$$\delta_{\hat\epsilon}\Phi_{\alpha_1\dots\alpha_{2s}} = \hat\epsilon^\alpha\,\Psi_{\alpha,\alpha_1\dots\alpha_{2s}} = -\bar\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}\,\hat{\bar\epsilon}^\alpha\,, \tag{1.18}$$

with $\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}(x) \in \mathbb{R}^{\mathcal{N}} \times \mathcal{V}_\Phi$. Closure of superconformal transformations requires

$$\delta_{\hat\epsilon_2}\delta_{\hat\epsilon_1} - \delta_{\hat\epsilon_1}\delta_{\hat\epsilon_2} = 2\,\delta_v + 2\,\delta_r\,, \tag{1.19}$$

for $\delta_v\Phi_{\alpha_1\dots\alpha_{2s}}$ defined as in (1.9) with

$$\tilde{\mathrm{v}}^{\alpha\beta} = \tilde{\mathrm{v}}^{\beta\alpha} = i\big(\hat\epsilon_1{}^\alpha\,\hat{\bar\epsilon}_2{}^\beta - \hat\epsilon_2{}^\alpha\,\hat{\bar\epsilon}_1{}^\beta\big)\,, \tag{1.20}$$

and δ_r an infinitesimal R-symmetry transformation so that

$$\delta_r\Phi_{\alpha_1\dots\alpha_{2s}} = r_I\,L_I\Phi_{\alpha_1\dots\alpha_{2s}}\,. \tag{1.21}$$

The R-symmetry transformations define a group G_R and then in (1.21) L_I belong to a representation of the generators of G_R acting on $\mathcal{V}$. For the commutator of R-symmetry transformations

$$\delta_{r_2}\delta_{r_1} - \delta_{r_1}\delta_{r_2} = \delta_{r'}\,, \qquad r'_I = r_{1J}r_{2K}\,c_{JKI} \quad \Leftrightarrow \quad [L_I, L_J] = c_{IJK}\,L_K\,. \tag{1.22}$$

Along with $[\delta_v, \delta_r] = 0$ we also require

$$\big[\delta_{\hat\epsilon}, \delta_r\big] = \delta_{\hat\epsilon_r}\,, \quad \hat\epsilon_r{}^\alpha = -\hat\epsilon^\alpha\,r_I\rho_I \quad \Rightarrow \quad [\rho_I, \rho_J] = c_{IJK}\,\rho_K\,, \tag{1.23}$$

and

$$\big[\delta_{\hat\epsilon}, \delta_v\big] = \delta_{\hat\epsilon'}\,, \tag{1.24}$$

for

$$\hat{\epsilon}'^{\beta} = \tfrac{1}{2}\big(\tilde{\mathrm{v}}^{\gamma\delta}\partial_{\gamma\delta} + \hat{\lambda}\big)\hat{\epsilon}^{\beta} - \hat{\epsilon}^{\gamma}\,\hat{\omega}_{\gamma}{}^{\beta}\,, \tag{1.25}$$

so that

$$\epsilon'^{\beta} = -\epsilon^{\gamma}\omega_{\gamma}{}^{\beta} + \tfrac{1}{2}\,\lambda\,\epsilon^{\beta} - i\,\bar{\eta}_{\alpha}\,\tilde{\mathrm{a}}^{\alpha\beta}\,, \qquad \bar{\eta}'_{\alpha} = \omega_{\alpha}{}^{\gamma}\bar{\eta}_{\gamma} - \tfrac{1}{2}\,\lambda\,\bar{\eta}_{\alpha} - i\,\epsilon^{\gamma}\,\mathrm{b}_{\gamma\alpha}\,. \tag{1.26}$$

For $\tilde{\mathrm{v}}^{\alpha\beta}$ in (1.20) to be a G_R singlet we must have

$$(\rho_I)^T = -\rho_I\,, \tag{1.27}$$

so that from (1.23)

$$\hat{\bar{\epsilon}}_r{}^{\alpha} = r_I\rho_I\,\hat{\bar{\epsilon}}^{\alpha}\,. \tag{1.28}$$

As a consequence of (1.27) the R-symmetry group $G_R \subset SO(\mathcal{N})$. We also assume

$$c_{IJK} = c_{[IJK]}\,. \tag{1.29}$$

Conformal and R-symmetry transformations are easily extended to $\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}$, as defined in (1.18), by taking

$$\begin{aligned}\delta_v\Psi_{\alpha,\alpha_1\dots\alpha_{2s}} = {}& \big(\tfrac{1}{2}\,\tilde{\mathrm{v}}^{\alpha\beta}\partial_{\alpha\beta} - (\Delta+\tfrac{1}{2})\,\hat{\lambda}\,\big)\Psi_{\alpha,\alpha_1\dots\alpha_{2s}} \\ & + 2s\,\hat{\omega}_{(\alpha_1}{}^{\alpha}\,\Phi_{\alpha_2\dots\alpha_{2s})\alpha} + \hat{\omega}_{\alpha}{}^{\beta}\,\Psi_{\beta,\alpha_1\dots\alpha_{2s}}\,,\end{aligned} \tag{1.30}$$

and

$$\delta_r\Psi_{\alpha,\alpha_1\dots\alpha_{2s}} = r_I(\rho_I + L_I)\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}\,. \tag{1.31}$$

The required superconformal transformations have the form

$$\begin{aligned}\delta_{\hat{\epsilon}}\Psi_{\alpha,\alpha_1\dots\alpha_{2s}} = {}& \hat{\bar{\epsilon}}^{\beta}\,i\partial_{\beta\alpha}\Phi_{\alpha_1\dots\alpha_{2s}} + 2(\Delta-s)\,\eta_\alpha\,\Phi_{\alpha_1\dots\alpha_{2s}} + 4s\,\eta_{(\alpha_1}\Phi_{\alpha_2\dots\alpha_{2s})\alpha} \\ & + 2\,\rho_I\eta_\alpha\,L_I\Phi_{\alpha_1\dots\alpha_{2s}} \\ & + a\,\hat{\bar{\epsilon}}^{\beta}\,i\partial_{[\beta(\alpha_1}\Phi_{\alpha_2\dots\alpha_{2s})\alpha]} \\ & + \rho_I\hat{\bar{\epsilon}}^{\beta}\,L_I\,i\big(b\,\partial_{\beta\alpha}\Phi_{\alpha_1\dots\alpha_{2s}} + 2c\,\partial_{(\beta(\alpha_1}\Phi_{\alpha_2\dots\alpha_{2s})\alpha)} + d\,\partial_{(\alpha_1\alpha_2}\Phi_{\alpha_3\dots\alpha_{2s})\beta\alpha}\big) \\ & + J_{\alpha\beta,\alpha_1\dots\alpha_{2s}}\,\hat{\bar{\epsilon}}^{\beta} + F_{\alpha_1\dots\alpha_{2s}}\,\varepsilon_{\alpha\beta}\,\hat{\bar{\epsilon}}^{\beta}\,.\end{aligned} \tag{1.32}$$

This is in accord with (1.19) for $[\delta_{\hat{\epsilon}_2},\delta_{\hat{\epsilon}_1}]\Phi_{\alpha_1\dots\alpha_{2s}}$, where the last three lines in (1.32) do not contribute so long as (1.27) holds and also

$$J_{\alpha\beta,\alpha_1\dots\alpha_{2s}} = J_{\beta\alpha,\alpha_1\dots\alpha_{2s}} = -(J_{\alpha\beta,\alpha_1\dots\alpha_{2s}})^T\,, \qquad F_{\alpha_1\dots\alpha_{2s}} = (F_{\alpha_1\dots\alpha_{2s}})^T\,, \tag{1.33}$$

as $\mathcal{N}\times\mathcal{N}$ matrices, so that $J_{\alpha\beta,\alpha_1\dots\alpha_{2s}} \in (\mathbb{R}^{\mathcal{N}}\times\mathbb{R}^{\mathcal{N}})_A\times\mathcal{V}_\Phi$, $F_{\alpha_1\dots\alpha_{2s}} \in (\mathbb{R}^{\mathcal{N}}\times\mathbb{R}^{\mathcal{N}})_S\times\mathcal{V}_\Phi$. Assuming (1.21) then (1.19) requires

$$r_I = \hat{\epsilon}_1{}^{\alpha}\rho_I\,\eta_{2\alpha} - \hat{\epsilon}_2{}^{\alpha}\rho_I\,\eta_{1\alpha} = \epsilon_1{}^{\alpha}\rho_I\,\eta_{2\alpha} - \epsilon_2{}^{\alpha}\rho_I\,\eta_{1\alpha}\,, \tag{1.34}$$

where the terms linear in x vanish owing to (1.27).

The coefficients a, b, c, d in (1.32) are determined, from (1.24), by imposing, using (1.9) and (1.30), $[\delta_{\hat\epsilon}, \delta_v]\Psi_{\alpha,\alpha_1\dots\alpha_{2s}} - \delta_{\hat\epsilon'}\Psi_{\alpha,\alpha_1\dots\alpha_{2s}} = 0$, with $\hat\epsilon'$ given by (1.25). All terms cancel except those those arising from $\partial_{\alpha\beta}\hat\lambda = -2\mathrm{b}_{\alpha\beta}$, $\partial_{\alpha\beta}\hat\omega_\gamma{}^\delta = \mathrm{b}_{\gamma\alpha}\,\delta_\beta{}^\delta + \mathrm{b}_{\gamma\beta}\,\delta_\alpha{}^\delta - \mathrm{b}_{\alpha\beta}\,\delta_\gamma{}^\delta$. For these contributions to be compatible with (1.24) we require

$$a = \frac{2s}{\Delta - 1}\,, \tag{1.35}$$

and b, c, d must satisfy

$$(\Delta - s)b + c = 1\,, \qquad s\,b + \Delta\,c + d = 0\,, \qquad (2s-1)c + (\Delta + s - 2)d = 0\,. \tag{1.36}$$

This gives

$$b + c = \frac{\Delta + s - 1}{(\Delta - 1)(\Delta + s)}\,, \qquad c + d = -\frac{s}{(\Delta - 1)(\Delta + s)}\,. \tag{1.37}$$

With these results then from (1.32) we may obtain

$$\begin{aligned}\delta_{\hat\epsilon}\Psi_{1,1\dots1} = {}& \frac{1}{\Delta + s}\big((\Delta + s)\hat{\bar\epsilon}{}^1 + \rho_I\hat{\bar\epsilon}{}^1 L_I\big)\, i\partial_{11}\Phi_{1\dots1} \\ & + \frac{1}{(\Delta - 1)(\Delta + s)}\big((\Delta + s)\hat{\bar\epsilon}{}^2 + \rho_I\hat{\bar\epsilon}{}^2 L_I\big)\big((\Delta + s - 1)\, i\partial_{21}\Phi_{1\dots1} - s\, i\partial_{11}\Phi_{1\dots12}\big) \\ & + 2\big((\Delta + s)\eta_1 + \rho_I\eta_1 L_I\big)\,\Phi_{1\dots1}\,, \end{aligned} \tag{1.38}$$

and

$$\begin{aligned} & \delta_{\hat\epsilon}\big(\Psi_{2,1\dots1} - \Psi_{1,1\dots12}\big) \\ & = \frac{1}{\Delta - 1}\big((\Delta - 1 - s)\hat{\bar\epsilon}{}^1 + \rho_I\hat{\bar\epsilon}{}^1 L_I\big)\big(i\partial_{12}\Phi_{1\dots1} - i\partial_{11}\Phi_{1\dots12}\big) \\ & \quad + \frac{1}{2(\Delta - 1)(\Delta - 1 - s)}\big((\Delta - 1 - s)\hat{\bar\epsilon}{}^2 + \rho_I\hat{\bar\epsilon}{}^2 L_I\big) \\ & \qquad \times\big((2\Delta - 3)\, i\partial_{22}\Phi_{1\dots1} - 2(\Delta + s - 2)\, i\partial_{21}\Phi_{1\dots12} + (2s-1)\, i\partial_{11}\Phi_{1\dots122}\big) \\ & \quad + 2\big((\Delta - 1 - s)\eta_2 + \rho_I\eta_2 L_I\big)\,\Phi_{1\dots1} - 2\big((\Delta - 1 - s)\eta_1 + \rho_I\eta_1 L_I\big)\,\Phi_{1\dots12}\,. \end{aligned} \tag{1.39}$$

For $s = 0$ then (1.32) simplifies to

$$\begin{aligned}\delta_{\hat\epsilon}\Psi_\alpha = {}& \frac{1}{\Delta}\big(\Delta\,\hat{\bar\epsilon}{}^\beta + \rho_I\hat{\bar\epsilon}{}^\beta L_I\big)\, i\partial_{\beta\alpha}\Phi + 2\big(\Delta\,\eta_\alpha + \rho_I\eta_\alpha L_I\big)\Phi \\ & + J_{\alpha\beta}\,\hat{\bar\epsilon}{}^\beta + F\,\varepsilon_{\alpha\beta}\,\hat{\bar\epsilon}{}^\beta\,. \end{aligned} \tag{1.40}$$

Superconformal transformations as in (1.18) or (1.32) are generated by the supersymmetry charges Q_α and their superpartners S^α,

$$\delta_{\hat\epsilon}\Phi_{\alpha_1\dots\alpha_{2s}} = \big[\epsilon^\alpha Q_\alpha + \bar\eta_\alpha\bar S^\alpha,\, \Phi_{\alpha_1\dots\alpha_{2s}}\big]\,, \qquad \bar Q_\alpha = (Q_\alpha)^T\,,\ \ S^\alpha = (\bar S^\alpha)^T\,. \tag{1.41}$$

By virtue of (1.19) and (1.10) we have

$$\big\{Q_\alpha, \bar Q_\beta\big\} = 2P_{\alpha\beta}\,\mathbb{I}\,, \qquad \big\{\bar S^\alpha, S^\beta\big\} = -2K^{\alpha\beta}\,\mathbb{I}\,, \tag{1.42}$$

where $\mathbb{I}$ is the identity on $\mathbb{R}^{\mathcal{N}}$, and

$$\big\{Q_\alpha, S^\beta\big\} = 2\big(M_\alpha{}^\beta + \delta_\alpha{}^\beta H\big)\,\mathbb{I} - 2\,\rho_I R_I\,\delta_\alpha{}^\beta\,. \tag{1.43}$$

In (1.43) R_I is the R-symmetry generator so that in (1.21)

$$\delta_r \Phi_{\alpha_1\dots\alpha_{2s}} = -r_I\big[R_I, \Phi_{\alpha_1\dots\alpha_{2s}}\big]\,, \qquad \big[R_I, \Phi_{\alpha_1\dots\alpha_{2s}}\big] = -L_I\Phi_{\alpha_1\dots\alpha_{2s}}\,, \tag{1.44}$$

so that R_I obeys the same Lie algebra as L_I in (1.22).

Using (1.24) with (1.26) we get

$$\begin{aligned}
&\big[M_\gamma{}^\delta, Q_\alpha\big] = \delta_\alpha{}^\delta Q_\gamma - \tfrac{1}{2}\delta_\gamma{}^\delta\, Q_\alpha\,, \qquad &&\big[H, Q_\alpha\big] = \tfrac{1}{2}\, Q_\alpha\,,\\
&\big[P_{\gamma\delta}, Q_\alpha\big] = 0\,, &&\big[K^{\gamma\delta}, Q_\alpha\big] = -\delta_\alpha{}^\gamma \bar S^\delta - \delta_\alpha{}^\delta \bar S^\gamma\,,\\
&\big[M_\gamma{}^\delta, \bar S^\alpha\big] = -\,\delta_\gamma{}^\alpha \bar S^\delta + \tfrac{1}{2}\delta_\gamma{}^\delta\, \bar S^\alpha\,, &&\big[H, \bar S^\alpha\big] = -\tfrac{1}{2}\,\bar S^\alpha\,,\\
&\big[P_{\gamma\delta}, \bar S^\alpha\big] = -\,\delta_\gamma{}^\alpha Q_\delta - \delta_\delta{}^\alpha Q_\gamma\,, &&\big[P_{\gamma\delta}, \bar S^\alpha\big] = 0\,.
\end{aligned} \tag{1.45}$$

Furthermore for the R-symmetry charges defined by (1.44), (1.22) leads to

$$\big[R_I, Q_\alpha\big] = -\rho_I Q_\alpha\,, \qquad \big[R_I, \bar S^\alpha\big] = -\rho_I \bar S^\alpha\,. \tag{1.46}$$

Imposing the Jacobi identities requires that the antisymmetric matrices ρ_I satisfy the completeness condition

$$(\rho_I)_{rs}\,(\rho_I)_{uv} = \delta_{ru}\,\delta_{sv} - \delta_{rv}\,\delta_{su}\,. \tag{1.47}$$

This ensures that we must identify $G_R \simeq SO(\mathcal{N})$.

As operators the hermitian conjugation in (1.14) extends to

$$(Q_\alpha)^\dagger = \bar Q_\alpha\,, \qquad (\bar S^\alpha)^\dagger = -S^\alpha\,, \qquad R_I{}^\dagger = -R_I\,, \tag{1.48}$$

where the conjugation includes a $\mathbb{R}^{\mathcal{N}}$ matrix transpose and we assume with (1.27) that ρ_I is real. Corresponding to (1.15) we have

$$(Q_\alpha)^+ = S^\alpha\,, \qquad (S^\alpha)^+ = Q_\alpha\,. \tag{1.49}$$

2. Shortened Superconformal Representations

For a superconformal primary $\Phi_{\alpha_1\dots\alpha_{2s}} \in \mathcal{V}_\Phi$, which transforms as in (1.18) and corresponding to a representation $\mathcal{R}_\Phi$ for the R-symmetry group $SO(\mathcal{N})$, then, for general

Δ, $\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}$ can be decomposed into fields which transform according to all the irreducible representations $\mathcal{R}_i$ which appear in the tensor product decomposition

$$\mathcal{R}_\mathcal{N} \otimes \mathcal{R}_\Phi \simeq \oplus_j \mathcal{R}_j \, , \tag{2.1}$$

where $\mathcal{R}_\mathcal{N}$ denotes the basic $\mathcal{N}$ dimensional representation, with generators ρ_I, defined on $\mathbb{R}^\mathcal{N}$. In addition $\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}$ can also be decomposed into fields with spin $s \pm \frac{1}{2}$. As is well known for particular Δ there are truncated representations in which some representations are absent. The critical condition is

$$\big(\Delta\, \epsilon + \rho_I \epsilon\, L_I\big)\, \Phi \notin \mathcal{U}_\Psi \subset \mathbb{R}^\mathcal{N} \otimes \mathcal{V}_\Phi \, , \qquad \epsilon \in \mathbb{R}^\mathcal{N} \, , \ \Phi \in \mathcal{V}_\Phi \, , \tag{2.2}$$

where $\mathcal{U}_\Psi \simeq \oplus_{i\in U} \mathcal{V}_i$ for a particular set of the representation spaces $\mathcal{V}_i$ associated with the representations $\mathcal{R}_i$ labelled by $i \in U$. If $C_{\mathcal{R}_i}$ is the quadratic Casimir for the representation $\mathcal{R}_i$, defined so that for $\Phi \in \mathcal{V}_\Phi$, $L_I L_I \Phi = -C_{\mathcal{R}_\Phi} \Phi$, this then requires

$$\Delta = \lambda = \tfrac{1}{2}\big(C_{\mathcal{R}_i} - C_{\mathcal{R}_\Phi} - (\mathcal{N}-1)\big) \quad \text{for all} \quad i \in U \, , \tag{2.3}$$

since, from (1.47), $\rho_I \rho_I = -(\mathcal{N}-1)\, \mathbb{I}$. As a consequence of (1.40) for $\delta_{\bar\epsilon} \Psi_\alpha$ this is directly applicable to the case of a spinless superconformal primary Φ. Subject to the absence of contributions to Ψ_α belonging to $\mathcal{U}_\psi$ there is a short superconformal representation in this case. In (1.40) $J_{\alpha\beta}$ and F are also constrained by the requirement $\Psi_\alpha \notin \mathcal{U}_\Psi$, so that we may decompose

$$(\mathbb{R}^\mathcal{N} \times \mathbb{R}^\mathcal{N})_A \times \mathcal{V}_\Phi \simeq \mathcal{V}_J \oplus \mathcal{U}_J \, , \qquad J_{\alpha\beta} \in \mathcal{V}_J \, . \tag{2.4}$$

Applying the same analysis to (1.39) then if

$$\Delta = 1 + s + \lambda \, , \tag{2.5}$$

there are corresponding semi-short representations where $\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}$ lacks contributions belonging to $\mathcal{U}_\Psi$ with spin $s - \frac{1}{2}$,

$$\varepsilon^{\alpha\alpha_1} \Psi_{\alpha,\alpha_1\dots\alpha_{2s}} \notin \mathcal{U}_\Psi \, . \tag{2.6}$$

In general in both (2.3) and (2.5) unitarity requires $\lambda \geq 0$.

As an illustration for $\mathcal{N} = 3$ and $G_R = SO(3)$ then as usual representations are labelled by $r = 0, \frac{1}{2}, 1, \dots$. The basic three dimensional representation $\mathcal{R}_3$ corresponds to $r = 1$ and, with $\mathcal{R}_\Phi$ the representation labelled by r, $C_{\mathcal{R}_\Phi} = r(r+1)$. In this case in the tensor product (2.1) $r_i = r \pm 1, r$. Requiring the $r+1$ representation is absent then it is easy to see that in (2.3) and (2.5) $\lambda = r$. For $\mathcal{N} = 6$, $G_R = SO(6)$ then the irreducible representations may be labelled by Dynkin indices $[r_1, r_2, r_3]$ and the Casimir operator is then

$$C_{[r_1,r_2,r_3]} = r_1(r_1+3) + r_3(r_3+3) - \tfrac{1}{4}(r_1-r_3)^2 + r_2(r_1+r_2+r_3+4)\,. \qquad (2.7)$$

The basic six dimensional representation $\mathcal{R}_6$ has Dynkin indices $[0,1,0]$, of course from (2.7) $C_{[0,1,0]} = 5$. Assumed $\mathcal{R}_\Phi$ has Dynkin labels $[r_1,r_2,r_3]$ the decomposition (2.1) becomes now

$$\begin{aligned}[0,1,0]\otimes[r_1,r_2,r_3] \simeq{}& [r_1,r_2{+}1,r_3]\oplus[r_1{+}1,r_2{-}1,r_3{+}1]\oplus[r_1{+}1,r_2,r_3{-}1]\\ &\oplus[r_1{-}1,r_2,r_3{+}1]\oplus[r_1{-}1,r_2{+}1,r_3{-}1]\oplus[r_1,r_2{-}1,r_3]\,.\end{aligned} \qquad (2.8)$$

Requiring that the omitted representation $\mathcal{R}_i$ in the superconformal transformation $\delta_\epsilon\Phi$ is that labelled by $[r_1,r_2{+}1,r_3]$ then, according to (2.3), $\lambda = \frac{1}{2}(r_1+2r_2+r_3)$.

For a short representation with a superconformal scalar primary $\varphi \in \mathcal{V}_\varphi$, with scale dimension Δ_φ constrained by (2.3), superconformal transformations, as in (1.18) with $\Phi \to \varphi$, are expressible in terms of $\psi_\alpha \in \mathcal{V}_\psi \simeq \oplus_{j\notin U}\mathcal{V}_j$, and $\bar\psi_\alpha = {\psi_\alpha}^T$, where

$$\Psi_\alpha = \mathcal{D}_\psi\,\psi_\alpha\,, \qquad \bar\Psi_\alpha = \bar\psi_\alpha \overleftarrow{\mathcal{D}}_\psi\,, \qquad (2.9)$$

for $\mathcal{D}_\psi : \mathcal{V}_\psi \to \mathcal{V}_\varphi \times \mathbb{R}^{\mathcal{N}}$ and $\bar\psi\,\overleftarrow{\mathcal{D}}_\psi = (\mathcal{D}_\psi\psi)^T$. Hence (1.18) becomes

$$\delta_{\hat\epsilon}\,\varphi = \hat\epsilon^\alpha\,\mathcal{D}_\psi\psi_\alpha = -\bar\psi_\alpha\overleftarrow{\mathcal{D}}_\psi\,\hat{\bar\epsilon}^\alpha\,. \qquad (2.10)$$

Then (1.40) shows that the associated superconformal transformations for ψ_α can be written as

$$\delta_{\hat\epsilon}\,\psi_\alpha = \mathcal{D}_\varphi\, i\partial_{\alpha\beta}\varphi\,\hat{\bar\epsilon}^\beta + 2\Delta_\varphi\,\mathcal{D}_\varphi\,\varphi\,\eta_\alpha + \mathcal{D}_J J_{\alpha\beta}\,\hat{\bar\epsilon}^\beta + \mathcal{D}_F F\,\varepsilon_{\alpha\beta}\,\hat{\bar\epsilon}^\beta\,, \qquad (2.11)$$

for $\mathcal{D}_\varphi$ such that $\mathcal{D}_\varphi : \mathcal{V}_\varphi \times \mathbb{R}^{\mathcal{N}} \to \mathcal{V}_\psi$, where

$$\mathcal{D}_\psi\mathcal{D}_\varphi\,\varphi = \Big(\mathbb{I} + \frac{1}{\Delta_\varphi}\,\rho_I\,L_I\Big)\varphi\,, \qquad (2.12)$$

and also $\mathcal{D}_J : \mathcal{V}_J \times \mathbb{R}^{\mathcal{N}} \to \mathcal{V}_\psi$, $\mathcal{D}_F : \mathcal{V}_F \times \mathbb{R}^{\mathcal{N}} \to \mathcal{V}_\psi$, where the supersymmetry algebra requires

$$\mathcal{D}_\psi\mathcal{D}_J = -\big(\mathcal{D}_\psi\mathcal{D}_J\big)^T = \rho_I\,M_I\,, \qquad \mathcal{D}_\psi\mathcal{D}_F = \big(\mathcal{D}_\psi\mathcal{D}_F\big)^T\,. \qquad (2.13)$$

Since $\operatorname{coker}\mathcal{D}_\psi \simeq \mathcal{U}_\psi$, $\operatorname{coker}\mathcal{D}_J \simeq \mathcal{U}_J$, are non empty there are $\mathcal{Q}_\psi$, $\mathcal{Q}_J$ such that

$$\mathcal{Q}_\psi\mathcal{D}_\psi = 0\,, \qquad \mathcal{Q}_J\mathcal{D}_J = 0\,. \qquad (2.14)$$

The existence of $\mathcal{Q}_\psi, \mathcal{Q}_J$ lead to non trivial constraints in superconformal Ward identities.

3. Ward Identities for Three Point Functions

For simplicity we first analyse the superconformal constraints for the three point function involving three scalar superconformal primary fields $\varphi_1, \varphi_2, \varphi_3$, with scale dimensions $\Delta_1, \Delta_2, \Delta_3$ belonging to short supermultiplets. Conformal invariance determines a unique form for the three point function in this case

$$\big\langle \varphi_1(x_1)\, \varphi_2(x_2)\, \varphi_3(x_3) \big\rangle = \mathcal{C}_{\Delta_1\Delta_2\Delta_3}(x_1,x_2,x_3)\ C\,, \tag{3.1}$$

where $C \in (\mathcal{V}_{\varphi_1} \times \mathcal{V}_{\varphi_2} \times \mathcal{V}_{\varphi_3})_{\text{sym.}}$ and we define

$$\mathcal{C}_{\Delta_1\Delta_2\Delta_3}(x_1,x_2,x_3) = \frac{1}{(x_{12}{}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)}\,(x_{23}{}^2)^{\frac{1}{2}(\Delta_3+\Delta_2-\Delta_1)}\,(x_{31}{}^2)^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2)}}\,. \tag{3.2}$$

Superconformal Ward identities are obtained by requiring

$$\delta_{\hat\epsilon}\big\langle \psi_{r\alpha}(x_r)\, \varphi_s(x_s)\, \varphi_t(x_t) \big\rangle = 0\,. \tag{3.3}$$

To determine $\langle \delta_{\hat\epsilon}\psi_{i\alpha}(x_r)\, \varphi_s(x_s)\, \varphi_t(x_t)\rangle$ in (3.3) as determined by (2.11) we may use

$$\partial_{r\alpha\beta}\, \frac{1}{(x_{rs}{}^2)^\lambda}\, \hat{\bar\epsilon}{}^\beta(x_r) - 2i\,\lambda\, \frac{1}{(x_{rs}{}^2)^\lambda}\, \eta_\alpha = -2\lambda\, \frac{1}{(x_{rs}{}^2)^{\lambda+1}}\, \mathrm{x}_{rs\,\alpha\beta}\, \hat{\bar\epsilon}{}^\beta(x_s)\,, \tag{3.4}$$

to obtain

$$\begin{aligned} &\partial_{r\,\alpha\beta}\, \mathcal{C}_{\Delta_1\Delta_2\Delta_3}(x_1,x_2,x_3)\, \hat{\bar\epsilon}{}^\beta(x_1) - 2i\,\Delta_r\, \eta_\alpha\, \mathcal{C}_{\Delta_1\Delta_2\Delta_3}(x_1,x_2,x_3) \\ &\quad = -\Big((\Delta_r + \Delta_s - \Delta_t)\frac{1}{x_{rs}{}^2}\, \mathrm{x}_{rs\,\alpha\beta}\, \hat{\bar\epsilon}{}^\beta(x_j) + (\Delta_r + \Delta_t - \Delta_s)\frac{1}{x_{rt}{}^2}\, \mathrm{x}_{rt\,\alpha\beta}\, \hat{\bar\epsilon}{}^\beta(x_t)\Big) \\ &\qquad \times \mathcal{C}_{\Delta_1\Delta_2\Delta_3}(x_1,x_2,x_3)\,. \end{aligned} \tag{3.5}$$

By virtue of (2.11) and (2.10) other contributions to the superconformal identities are determined in terms of

$$\big\langle J_{r\alpha\beta}(x_r)\, \varphi_s(x_s)\, \varphi_t(x_t) \big\rangle = i\, \mathrm{X}_{r[st]\alpha\beta}\, \mathcal{C}_{\Delta_1\Delta_2\Delta_3}(x_1,x_2,x_3)\, K_{r,st}\,, \tag{3.6}$$

and

$$\big\langle \psi_{r\alpha}(x_r)\, \bar\psi_{s\beta}(x_s)\, \varphi_t(x_t) \big\rangle = i\, \frac{1}{x_{rs}{}^2}\, \mathrm{x}_{rs\,\alpha\beta}\, \mathcal{C}_{\Delta_1\Delta_2\Delta_3}(x_1,x_2,x_3)\, P_{rs}\,. \tag{3.7}$$

In (3.6)

$$\mathrm{X}_{r[st]\alpha\beta} = \mathrm{X}_{r[st]\beta\alpha} = \frac{1}{x_{rs}{}^2\, x_{rt}{}^2}\, (\mathrm{x}_{rs}\tilde{\mathrm{x}}_{st}\mathrm{x}_{tr})_{\alpha\beta} = \frac{1}{x_{rs}{}^2}\, \mathrm{x}_{rs\,\alpha\beta} - \frac{1}{x_{rt}{}^2}\, \mathrm{x}_{rt\,\alpha\beta}\,, \tag{3.8}$$

where

$$\mathrm{X}_{r[st]}\,\tilde{\mathrm{X}}_{r[ts]} = \frac{x_{st}{}^2}{x_{rs}{}^2\, x_{rt}{}^2}\,\mathrm{I}\,. \tag{3.9}$$

From the definitions (3.6) and (3.7) we must require

$$K_{r,st} = -K_{r,ts}\,, \qquad P_{rs} = P_{sr}{}^T\,. \tag{3.10}$$

The superconformal identities arising from (3.3) are then

$$\begin{aligned}
&\Delta_r\,\mathcal{D}_{\varphi_r} C\Big(\frac{1}{x_{rs}{}^2}\,\mathrm{x}_{rs\,\alpha\beta}\,\hat{\bar\epsilon}^{\,\beta}(x_s) + \frac{1}{x_{rt}{}^2}\,\mathrm{x}_{rt\,\alpha\beta}\,\hat{\bar\epsilon}^{\,\beta}(x_t)\Big)\\
&+ \big((\Delta_s-\Delta_t)\,\mathcal{D}_{\varphi_r}C - \mathcal{D}_{J_r}K_{r,st}\big)\,\mathrm{X}_{r[st]\alpha\beta}\,\hat{\bar\epsilon}^{\,\beta}(x_r)\\
&+ P_{rs}\overleftarrow{\mathcal{D}}_{\psi_s}\,\frac{1}{x_{rs}{}^2}\,\mathrm{x}_{rs\alpha\beta}\,\hat{\bar\epsilon}^{\,\beta}(x_s) + P_{rt}\overleftarrow{\mathcal{D}}_{\psi_t}\,\frac{1}{x_{rt}{}^2}\,\mathrm{x}_{rt\alpha\beta}\,\hat{\bar\epsilon}^{\,\beta}(x_t) = 0\,.
\end{aligned} \tag{3.11}$$

From this we may extract

$$2\Delta_r\,\mathcal{D}_{\varphi_r}C = -\,P_{rs}\overleftarrow{\mathcal{D}}_{\psi_s} - P_{rt}\overleftarrow{\mathcal{D}}_{\psi_t}\,, \tag{3.12a}$$

$$(\Delta_s-\Delta_t)\,\mathcal{D}_{\varphi_r}C = \mathcal{D}_{J_r}K_{r,st} - \tfrac{1}{2}\big(P_{rs}\overleftarrow{\mathcal{D}}_{\psi_s} - P_{rt}\overleftarrow{\mathcal{D}}_{\psi_t}\big)\,. \tag{3.12b}$$

Using (3.12a) with (3.10) we may obtain

$$\textstyle\sum_r \Delta_r\,\mathcal{D}_{\psi_r}\mathcal{D}_{\varphi_r}\,C = \sum_r\Delta_r\,\big(\mathcal{D}_{\psi_r}\,\mathcal{D}_{\varphi_r}\,C\big)^T\,, \tag{3.13}$$

or using (2.12),

$$\textstyle\sum_r L_{r,I}\,C = 0\,, \tag{3.14}$$

which just implies that C is a $SO(\mathcal{N})$ invariant. Combining (3.12a) and (3.12b) to give

$$(\Delta_s-\Delta_t-\Delta_r)\,\mathcal{D}_{\psi_r}\mathcal{D}_{\varphi_r}C = \mathcal{D}_{\psi_r}\mathcal{D}_{J_r}K_{r,st} + \mathcal{D}_{\psi_r}P_{rt}\overleftarrow{\mathcal{D}}_{\psi_t}\,, \tag{3.15}$$

from we may also eliminate P_{rt}

$$(\Delta_s-\Delta_t-\Delta_r)\Big(\frac{1}{\Delta_r}\,L_{r,I} + \frac{1}{\Delta_t}\,L_{t,I}\Big)\,C = M_{r,I}K_{r,st} + M_{t,I}K_{t,sr}\,. \tag{3.16}$$

This gives three independent equations but one linear combination just gives (3.14) again.

If $\Delta_r = 1$ the current $J_{r\alpha\beta}$ is may be conserved,

$$\tilde\partial_r{}^{\alpha\beta}J_{r\alpha\beta} = 0\,. \tag{3.17}$$

For $r=1$ then

$$\tilde\partial_1{}^{\alpha\beta}\bigg(\frac{1}{(x_{12}{}^2\,x_{13}{}^2)^{\frac12}}\,X_{1[23]\alpha\beta}\bigg) = 0\,, \tag{3.18}$$

and hence from (3.6) and (3.2) $\tilde\partial_1{}^{\alpha\beta}\big\langle J_{1\alpha\beta}(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\big\rangle = 0$ for $\Delta_1 = 1$ and also so long as $\Delta_2 = \Delta_3$.

4. Further Superconformal Identities for Three Point Functions

We here discuss the form for the superconformal Ward identity for a three point function involving two scalar superconformal primary fields φ_1, φ_2, with scale dimensions Δ_1, Δ_2 belonging to short supermultiplets, and a superconformal primary with spin s, $\Phi_{\alpha_1\ldots\alpha_{2s}}$. For notational convenience here we write, in terms of (3.8), $X_{\alpha\beta} = \mathrm{X}_{1[23]\alpha\beta}$, $Z_{\alpha\beta} = \mathrm{X}_{3[12]\alpha\beta}$

Instead of (3.1) conformal invariance now requires

$$\big\langle \varphi_1(x_1)\, \varphi_2(x_2)\, \Phi_{\alpha_1\ldots\alpha_{2s}}(x_3) \big\rangle = \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3)\ C_{\varphi_1\varphi_2\Phi}\,, \tag{4.1}$$

where we define, for $s = 0,1,2,\ldots$,

$$\begin{aligned}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3) &= \frac{1}{(x_{13}{}^2)^{\Delta_1}(x_{23}{}^2)^{\Delta_2}}\, (Z^2)^{\frac12(\Delta-\Delta_1-\Delta_2-s)}\, Z_{(\alpha_1\alpha_2}\ldots Z_{\alpha_{2s-1}\alpha_{2s})} \\ &= \frac{Z_{(\alpha_1\alpha_2}\ldots Z_{\alpha_{2s-1}\alpha_{2s})}}{(x_{12}{}^2)^{\frac12(\Delta_1+\Delta_2-\Delta+s)}\,(x_{23}{}^2)^{\frac12(\Delta+\Delta_2-\Delta_1-s)}\,(x_{31}{}^2)^{\frac12(\Delta+\Delta_1-\Delta_2-s)}}\,.\end{aligned} \tag{4.2}$$

The conformal functions of x_1,x_2,x_3 given by (4.2) satisfy the identities

$$\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3) = (-1)^s\,\mathcal{C}^{(s)}_{\Delta_2\Delta_1\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_2,x_1,x_3)\,, \tag{4.3}$$

and

$$\begin{aligned}&\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha\alpha_1\ldots\alpha_{2s-1}}(x_1,x_2,x_3) = x_{13}{}^2\, Z_{\alpha(\alpha_1}\mathcal{C}^{(s-1)}_{\Delta_1+1\,\Delta_2\,\Delta,\alpha_2\ldots\alpha_{2s-1})}(x_1,x_2,x_3)\,,\\ &(2s+1)\,\mathcal{C}^{(s+1)}_{\Delta_1\Delta_2\,\Delta+1,\alpha\beta\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3)\\ &\qquad = Z_{\alpha\beta}\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3) + 2s\, Z_{\alpha(\alpha_1}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_2\ldots\alpha_{2s})\beta}(x_1,x_2,x_3)\,,\end{aligned} \tag{4.4}$$

with also

$$\begin{aligned}Z_{\alpha\beta}\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3) &= \mathcal{C}^{(s+1)}_{\Delta_1\Delta_2\,\Delta+1,\alpha\beta\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3)\\ &\quad + \frac{2s}{2s+1}\,\varepsilon_{\alpha(\alpha_1}\varepsilon_{\beta|\alpha_2}\,\mathcal{C}^{(s-1)}_{\Delta_1\Delta_2\,\Delta+1,\alpha_3\ldots\alpha_{2s})}(x_1,x_2,x_3)\,.\end{aligned} \tag{4.5}$$

In particular cases the conformal functions (4.2) satisfy identities. For instance using

$$\partial_{1\alpha\beta}\, Z_{\gamma\delta} = 2\,\frac{1}{(x_{13}{}^2)^2}\,\mathrm{x}_{13\,\alpha(\gamma}\mathrm{x}_{13\,\beta|\delta)}\,, \tag{4.6}$$

we may find

$$\begin{aligned}&\partial_1{}^2\mathcal{C}^{(s)}_{\frac12\,\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3)\\ &\qquad = (\Delta-\Delta_2-\tfrac12-s)(\Delta-\Delta_2+\tfrac12+s)\,\mathcal{C}^{(s)}_{\frac52\,\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3)\,.\end{aligned} \tag{4.7}$$

The relevant superconformal Ward identities are derived by requiring

$$\delta_{\hat\epsilon}\big\langle \psi_{1\alpha}(x_1)\, \varphi_2(x_2)\, \Phi_{\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle = 0\,. \tag{4.8}$$

To determine $\langle \delta_{\hat\epsilon}\psi_{1\alpha}(x_1)\, \varphi_2(x_2)\, \Phi_{\alpha_1\dots\alpha_{2s}}(x_3)\rangle$, with $\delta_{\hat\epsilon}\psi_{1\alpha}$ given by (2.11), we may use (3.4) and (4.6) to obtain

$$\begin{aligned}
&\partial_{1\,\alpha\beta}\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\, \hat{\bar\epsilon}^{\beta}(x_1) - 2i\,\Delta_1\, \eta_\alpha\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\\
&\quad = -\Big((\Delta_1+\Delta_2-\Delta+s)\frac{1}{x_{12}{}^2}\, \mathrm{x}_{12\,\alpha\beta}\, \hat{\bar\epsilon}^{\beta}(x_2) + (\Delta_1+\Delta_2-\Delta+s)\frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\,\alpha\beta}\, \hat{\bar\epsilon}^{\beta}(x_3)\Big)\\
&\qquad\quad \times \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\\
&\qquad + 2s\,\frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\;\alpha(\alpha_1}\mathrm{x}_{13\beta|\alpha_2}\, \hat{\bar\epsilon}^{\beta}(x_1)\, \mathcal{C}^{(s-1)}_{\Delta_1+1\,\Delta_2\,\Delta,\alpha_3\dots\alpha_{2s})}(x_1,x_2,x_3)\,.
\end{aligned} \tag{4.9}$$

The other contributions to this superconformal identity are determined in terms of

$$\begin{aligned}
&\big\langle J_{1\alpha\beta}(x_1)\, \varphi_2(x_2)\, \Phi_{\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle\\
&\qquad = iX_{\alpha\beta}\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\, K_{J_1\varphi_2\Phi}\\
&\qquad\quad + i\,\frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\;\alpha(\alpha_1}\mathrm{x}_{13\beta|\alpha_2}\, \mathcal{C}^{(s-1)}_{\Delta_1+1\,\Delta_2\,\Delta,\alpha_3\dots\alpha_{2s})}(x_1,x_2,x_3)\, L_{J_1\varphi_2\Phi}\,,
\end{aligned} \tag{4.10}$$

and, using (2.10) for $\delta_{\hat\epsilon}\varphi_2$,

$$\begin{aligned}
&\big\langle \psi_{1\alpha}(x_1)\, \bar\psi_{2\beta}(x_2)\, \Phi_{\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle\\
&\qquad = i\,\frac{1}{x_{12}{}^2}\, \mathrm{x}_{12\alpha\beta}\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\, P_{\psi_1\bar\psi_2\Phi}\\
&\qquad\quad + i\,\frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\alpha(\alpha_1}\mathrm{x}_{23\beta|\alpha_2}\, \mathcal{C}^{(s-1)}_{\Delta_1+1\,\Delta_2\,\Delta,\alpha_3\dots\alpha_{2s})}(x_1,x_2,x_3)\, Q_{\psi_1\bar\psi_2\Phi}\,.
\end{aligned} \tag{4.11}$$

We also require, using (1.18),

$$\begin{aligned}
&\big\langle \psi_{1\alpha}(x_1)\, \varphi_2(x_2)\, \bar\Psi_{\beta,\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle\\
&\qquad\qquad = i\,\frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\alpha\beta}\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\, P_{\psi_1\varphi_2\bar\Psi}\\
&\qquad\qquad\quad + i\,\frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\alpha(\alpha_1}\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_2\dots\alpha_{2s})\,\beta}(x_1,x_2,x_3)\, Q_{\psi_1\varphi_2\bar\Psi}\,.
\end{aligned} \tag{4.12}$$

If we define

$$P_{\psi_1\varphi_2\bar\Psi+} = P_{\psi_1\varphi_2\bar\Psi} + Q_{\psi_1\varphi_2\bar\Psi}\,, \qquad P_{\psi_1\varphi_2\bar\Psi-} = 2s\, P_{\psi_1\varphi_2\bar\Psi} - Q_{\psi_1\varphi_2\bar\Psi}\,, \tag{4.13}$$

then $P_{\psi_1\varphi_2\bar\Psi\pm}$ correspond to $\bar\Psi$ having spin $s\pm\frac{1}{2}$.

Combining the different contributions the superconformal identity (4.8) then becomes

$$
\begin{aligned}
&\Big\{\Delta_1\, \mathcal{D}_{\varphi_1} C_{\varphi_1\varphi_2\Phi}\Big(\frac{1}{x_{12}{}^2}\, \mathrm{x}_{12\,\alpha\beta}\, \hat{\bar\epsilon}^{\,\beta}(x_2) + \frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\,\alpha\beta}\, \hat{\bar\epsilon}^{\,\beta}(x_3)\Big) \\
&+ \big((\Delta_2-\Delta+s)\, \mathcal{D}_{\varphi_1} C_{\varphi_1\varphi_2\Phi} - \mathcal{D}_{J_1} K_{J_1\varphi_2\Phi}\big)\, X_{\alpha\beta}\, \hat{\bar\epsilon}^{\,\beta}(x_1)\Big\}\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3) \\
&- \big(2s\, \mathcal{D}_{\varphi_1} C_{\varphi_1\varphi_2\Phi} + \mathcal{D}_{J_1} L_{J_1\varphi_2\Phi}\big)\, \frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\,\alpha(\alpha_1}\mathrm{x}_{13\,\beta|\alpha_2}\, \hat{\bar\epsilon}^{\,\beta}(x_1)\, \mathcal{C}^{(s-1)}_{\Delta_1+1\,\Delta_2\,\Delta,\alpha_3\ldots\alpha_{2s})}(x_1,x_2,x_3) \\
&+ P_{\psi_1\bar\psi_2\Phi}\overleftarrow{\overline{\mathcal{D}}}_{\psi_2}\, \frac{1}{x_{12}{}^2}\, \mathrm{x}_{12\,\alpha\beta}\, \hat{\bar\epsilon}^{\,\beta}(x_2)\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3) \\
&+ Q_{\psi_1\bar\psi_2\Phi}\overleftarrow{\overline{\mathcal{D}}}_{\psi_2}\, \frac{1}{x_{23}{}^2}\, \mathrm{x}_{13\,\alpha(\alpha_1}\mathrm{x}_{23\,\beta|\alpha_2}\, \hat{\bar\epsilon}^{\,\beta}(x_2)\, \mathcal{C}^{(s-1)}_{\Delta_1+1\,\Delta_2\,\Delta,\alpha_3\ldots\alpha_{2s})}(x_1,x_2,x_3) \\
&+ P_{\psi_1\varphi_2\bar\Psi}\, \frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\,\alpha\beta}\, \hat{\bar\epsilon}^{\,\beta}(x_3)\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3) \\
&+ Q_{\psi_1\varphi_2\bar\Psi}\, \frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\,(\alpha_1|\beta}\, \hat{\bar\epsilon}^{\,\beta}(x_3)\, \mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_2\ldots\alpha_{2s})\,\alpha}(x_1,x_2,x_3) \\
&= 0\,.
\end{aligned}
\tag{4.14}
$$

From this we may extract

$$
\begin{aligned}
2\Delta_1\, \mathcal{D}_{\varphi_1} C_{\varphi_1\varphi_2\Phi} &= -P_{\psi_1\bar\psi_2\Phi}\overleftarrow{\overline{\mathcal{D}}}_{\psi_2} - P_{\psi_1\varphi_2\bar\Psi}\,, \\
0 &= Q_{\psi_1\bar\psi_2\Phi}\overleftarrow{\overline{\mathcal{D}}}_{\psi_2} + Q_{\psi_1\varphi_2\bar\Psi}\,,
\end{aligned}
\tag{4.15}
$$

and also

$$
\begin{aligned}
(\Delta-\Delta_1-\Delta_2-s)\, \mathcal{D}_{\varphi_1} C_{\varphi_1\varphi_2\Phi} &= -\mathcal{D}_{J_1} K_{J_1\varphi_2\Phi} + P_{\psi_1\bar\psi_2\Phi}\overleftarrow{\overline{\mathcal{D}}}_{\psi_2}\,, \\
2s\, \mathcal{D}_{\varphi_1} C_{\varphi_1\varphi_2\Phi} &= -\mathcal{D}_{J_1} L_{J_1\varphi_2\Phi} + Q_{\psi_1\bar\psi_2\Phi}\overleftarrow{\overline{\mathcal{D}}}_{\psi_2}\,.
\end{aligned}
\tag{4.16}
$$

The result (4.16) provides a constraint on three point functions in that if the representation $\mathcal{R}_\Phi \in \mathcal{R}_{\varphi_1}\otimes\mathcal{R}_{\varphi_2}$ but $\mathcal{R}_\Phi \notin \mathcal{R}_{J_1}\otimes\mathcal{R}_{\varphi_2}$, $\mathcal{R}_{\psi_1}\otimes\mathcal{R}_{\psi_2}$ then $C_{\varphi_1\varphi_2\Phi}\neq 0$ is possible only if

$$
\Delta = \Delta_1+\Delta_2\,, \qquad s=0\,. \tag{4.17}
$$

When $\Delta_1 = 1$ we may impose the additional condition (3.17) for $r=1$. Using

$$
\begin{aligned}
&\tilde\partial_1{}^{\alpha\beta}\big(X_{\alpha\beta}\, \mathcal{C}^{(s)}_{1\,\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3)\big) = 2(\Delta_2-\Delta)\, \mathcal{C}^{(s)}_{3\,\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3)\,, \\
&\tilde\partial_1{}^{\alpha\beta}\Big(\frac{1}{x_{13}{}^2}\, \mathrm{x}_{13\,\alpha(\alpha_1}\mathrm{x}_{13\,\beta|\alpha_2}\, \mathcal{C}^{(s-1)}_{2\,\Delta_2\,\Delta,\alpha_3\ldots\alpha_{2s})}(x_1,x_2,x_3)\Big) \\
&\qquad\qquad = (\Delta_2-\Delta+s+1)\, \mathcal{C}^{(s)}_{3\,\Delta_2\,\Delta,\alpha_1\ldots\alpha_{2s}}(x_1,x_2,x_3)\,,
\end{aligned}
\tag{4.18}
$$

in (4.10), (3.17) requires

$$
2(\Delta-\Delta_2)\, K_{J_1\varphi_2\Phi} + (\Delta-\Delta_2-s-1)\, L_{J_1\varphi_2\Phi} = 0\,. \tag{4.19}
$$

Combining (4.19) with (4.15), (4.16) gives

$$\begin{aligned}2(\Delta-\Delta_2+s+1)(\Delta-\Delta_2-s)\,\mathcal{D}_{\varphi_1}C_{\varphi_1\varphi_2\Phi}&\\ =\,-2(\Delta-\Delta_2)\,P_{\psi_1\varphi_2\bar\Psi}-(\Delta-\Delta_2-s-1)\,Q_{\psi_1\varphi_2\bar\Psi}\,.&\end{aligned}\tag{4.20}$$

Further relations for the three point function in (4.1) obtained from considering the superconformal transformations of $\Phi_{\alpha_1\dots\alpha_{2s}}$, starting from the identity

$$\delta_{\hat\epsilon}\big\langle\varphi_1(x_1)\,\varphi_2(x_2)\,\bar\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle=0\,.\tag{4.21}$$

To evaluate $\langle\varphi_1(x_1)\,\varphi_2(x_2)\,\delta_{\hat\epsilon}\bar\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}(x_3)\rangle$ we use, from (1.32) and (1.27),

$$\begin{aligned}\delta_{\hat\epsilon}\bar\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}={}&\hat\epsilon^{\beta}\,i\partial_{\beta\alpha}\Phi_{\alpha_1\dots\alpha_{2s}}+2(\Delta-s)\,\bar\eta_\alpha\,\Phi_{\alpha_1\dots\alpha_{2s}}+4s\,\bar\eta_{(\alpha_1}\Phi_{\alpha_2\dots\alpha_{2s})\alpha}\\&-2\,\bar\eta_\alpha\,\rho_I L_I\Phi_{\alpha_1\dots\alpha_{2s}}\\&+a\,\hat\epsilon^{\beta}\,i\partial_{[\beta(\alpha_1}\Phi_{\alpha_2\dots\alpha_{2s})\alpha]}\\&-\hat\epsilon^{\beta}\,\rho_I L_I\,i\big(b\,\partial_{\beta\alpha}\Phi_{\alpha_1\dots\alpha_{2s}}+2c\,\partial_{(\beta(\alpha_1}\Phi_{\alpha_2\dots\alpha_{2s})\alpha)}+d\,\partial_{(\alpha_1\alpha_2}\Phi_{\alpha_3\dots\alpha_{2s})\beta\alpha}\big)\\&+\hat\epsilon^{\beta}\,J_{\alpha\beta,\alpha_1\dots\alpha_{2s}}-\hat\epsilon^{\beta}\varepsilon_{\beta\alpha}\,F_{\alpha_1\dots\alpha_{2s}}\,.\end{aligned}\tag{4.22}$$

With the aid of

$$\partial_{3\,\alpha\beta}\,Z_{\gamma\delta}=2\,R_{(\alpha(\gamma}Z_{\delta)\beta)}\,,\qquad R_{\alpha\beta}=\frac{1}{x_{23}{}^2}\,\mathrm{x}_{23\alpha\beta}+\frac{1}{x_{13}{}^2}\,\mathrm{x}_{13\alpha\beta}\,,\tag{4.23}$$

the action of the derivatives on the three point function (4.1) is given by, for a given by (1.35),

$$\begin{aligned}&\partial_{3\,\beta\alpha}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)+a\,\partial_{3\,[\beta(\alpha_1}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_2\dots\alpha_{2s})\alpha]}(x_1,x_2,x_3)\\&=-(\Delta_1-\Delta_2)\,Z_{\alpha\beta}\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\\&\quad+(\Delta-s)\,R_{\alpha\beta}\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)+2s\,R_{\beta(\alpha_1}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_2\dots\alpha_{2s})\alpha}(x_1,x_2,x_3)\,,\end{aligned}\tag{4.24}$$

and also, with b,c,d satisfying (1.37),

$$\begin{aligned}&b\,\partial_{3\,\beta\alpha}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)+2c\,\partial_{3\,(\beta(\alpha_1}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_2\dots\alpha_{2s})\alpha)}(x_1,x_2,x_3)\\&+d\,\partial_{3\,(\alpha_1\alpha_2}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s})\beta\alpha}(x_1,x_2,x_3)\\&=\,-(\Delta_1-\Delta_2)\,\frac{1}{\Delta-s-1}\,Z_{\beta\alpha}\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\\&\quad+(\Delta_1-\Delta_2)\,\frac{2s+1}{(\Delta-s-1)(\Delta+s)}\,\mathcal{C}^{(s+1)}_{\Delta_1\Delta_2\,\Delta,\beta\alpha\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\\&\quad+R_{\beta\alpha}\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\,.\end{aligned}\tag{4.25}$$

Defining

$$\hat\epsilon^{\beta}(x_3)R_{\beta\alpha} - 2i\,\bar\eta_\alpha = \frac{1}{{x_{23}}^2}\,\hat\epsilon^{\beta}(x_2)\,\mathrm{x}_{2\beta\alpha} + \frac{1}{{x_{13}}^2}\,\hat\epsilon^{\beta}(x_1)\,\mathrm{x}_{13\beta\alpha} = e_\alpha\,, \tag{4.26}$$

we then find

$$\begin{aligned}
&\big\langle \varphi_1(x_1)\,\varphi_2(x_2)\,\delta_{\hat\epsilon}\bar\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle \\
&= -i(\Delta_1-\Delta_2)\,\hat\epsilon^{\beta}(x_3)\Big(\mathbb{I} - \frac{1}{\Delta-s-1}\,\rho_I L_{\Phi,I}\Big)C_{\varphi_1\varphi_2\Phi}\;Z_{\beta\alpha}\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\\
&\quad - i(\Delta_1-\Delta_2)\,\frac{2s+1}{(\Delta-s-1)(\Delta+s)}\;\hat\epsilon^{\beta}(x_3)\,\rho_I L_{\Phi,I}\,C_{\varphi_1\varphi_2\Phi}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times\mathcal{C}^{(s+1)}_{\Delta_1\Delta_2\,\Delta+1,\beta\alpha\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\\
&\quad + i\,e_\alpha\big((\Delta-s)\,\mathbb{I} - \rho_I L_{\Phi,I}\big)C_{\varphi_1\varphi_2\Phi}\;\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\\
&\quad + i\,2s\;C_{\varphi_1\varphi_2\Phi}\;e_{(\alpha_1}\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_2\dots\alpha_{2s})\,\alpha}(x_1,x_2,x_3)\\
&\quad + \hat\epsilon^{\beta}(x_3)\big\langle\varphi_1(x_1)\,\varphi_2(x_2)\,J_{\alpha\beta,\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle - \hat\epsilon^{\beta}(x_3)\varepsilon_{\beta\alpha}\,\big\langle\varphi_1(x_1)\,\varphi_2(x_2)\,F_{\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle\,.
\end{aligned} \tag{4.27}$$

In (4.27) conformal invariance requires

$$\begin{aligned}
\big\langle\varphi_1(x_1)\,\varphi_2(x_2)\,J_{\alpha\beta,\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle &= i\,\mathcal{C}^{(s+1)}_{\Delta_1\Delta_2\,\Delta+1,\alpha\beta\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\;C_{\varphi_1\varphi_2 J_+}\\
&\quad + i\,\varepsilon_{\alpha(\alpha_1}\varepsilon_{\beta|\alpha_2}\,\mathcal{C}^{(s-1)}_{\Delta_1\Delta_2\,\Delta+1,\alpha_3\dots\alpha_{2s})}(x_1,x_2,x_3)\;C_{\varphi_1\varphi_2 J_-}\\
\big\langle\varphi_1(x_1)\,\varphi_2(x_2)\,F_{\alpha_1\dots\alpha_{2s}}(x_3)\big\rangle &= i\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta+1,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\;C_{\varphi_1\varphi_2 F}\,.
\end{aligned} \tag{4.28}$$

The superconformal identity arising from (4.21) also requires, assuming (2.10), contributions from $\langle\psi_{1\beta}(x_1)\,\varphi_2(x_2)\,\bar\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}(x_3)\rangle$, which is given by (4.12), and also $\langle\varphi_1(x_1)\,\psi_{2\beta}(x_2)\,\bar\Psi_{\alpha,\alpha_1\dots\alpha_{2s}}(x_3)\rangle$, which has a corresponding definition in terms of $P_{\varphi_1\psi_2\bar\Psi}$ and $Q_{\varphi_1\psi_2\bar\Psi}$. Since

$$\hat\epsilon^{\beta}(x_3)\,Z_{\beta\alpha} = \frac{1}{{x_{23}}^2}\,\hat\epsilon^{\beta}(x_2)\,\mathrm{x}_{23\beta\alpha} - \frac{1}{{x_{13}}^2}\,\hat\epsilon^{\beta}(x_1)\,\mathrm{x}_{13\beta\alpha}\,, \tag{4.29}$$

and with the definition (4.26) the superconformal identity implies, with notation as in (4.13),

$$2\big((\Delta+s)\,\mathbb{I} - \rho_I L_{\Phi,I}\big)C_{\varphi_1\varphi_2\Phi} = -\mathcal{D}_{\psi_1}P_{\psi_1\varphi_2\bar\Psi+} - \mathcal{D}_{\psi_2}P_{\varphi_1\psi_2\bar\Psi+}\,, \tag{4.30}$$

and, for $s>0$,

$$4s\,\big((\Delta-s-1)\,\mathbb{I} - \rho_I L_{\Phi,I}\big)C_{\varphi_1\varphi_2\Phi} = -\mathcal{D}_{\psi_1}P_{\psi_1\varphi_2\bar\Psi-} - \mathcal{D}_{\psi_2}P_{\varphi_1\psi_2\bar\Psi-}\,, \tag{4.31}$$

together with

$$2C_{\varphi_1\varphi_2 J_+} = 2(\Delta_1-\Delta_2)\Big(\mathbb{I} - \frac{1}{\Delta+s}\,\rho_I L_{\Phi,I}\Big)C_{\varphi_1\varphi_2\Phi} + \mathcal{D}_{\psi_1}P_{\psi_1\varphi_2\bar\Psi+} - \mathcal{D}_{\psi_2}P_{\varphi_1\psi_2\bar\Psi+} \tag{4.32}$$

and

$$
\begin{aligned}
2(2s+1)\, C_{\varphi_1\varphi_2 J_-} = {}& 4s\,(\Delta_1-\Delta_2)\Big(\mathbb{I}-\frac{1}{\Delta-s-1}\,\rho_I L_{\Phi,I}\Big)C_{\varphi_1\varphi_2\Phi} \\
& + \mathcal{D}_{\psi_1}P_{\psi_1\varphi_2\bar\Psi-} - \mathcal{D}_{\psi_2}P_{\varphi_1\psi_2\bar\Psi-}\,.
\end{aligned}
\tag{4.33}
$$

Together (4.30) and (4.31) give

$$
\begin{aligned}
& -2(\Delta-1)\big(\mathcal{D}_{\psi_1}P_{\psi_1\varphi_2\bar\Psi}+\mathcal{D}_{\psi_2}P_{\varphi_1\psi_2\bar\Psi}\big)-(\Delta-s-2)\big(\mathcal{D}_{\psi_1}Q_{\psi_1\varphi_2\bar\Psi}+\mathcal{D}_{\psi_2}Q_{\varphi_1\psi_2\bar\Psi}\big) \\
& = -\frac{1}{2s+1}\Big(2(s+1)(\Delta-s-1)\big(\mathcal{D}_{\psi_1}P_{\psi_1\varphi_2\bar\Psi+}+\mathcal{D}_{\psi_2}P_{\varphi_1\psi_2\bar\Psi+}\big) \\
& \qquad\qquad + (\Delta+s)\big(\mathcal{D}_{\psi_1}P_{\psi_1\varphi_2\bar\Psi-}+\mathcal{D}_{\psi_2}P_{\varphi_1\psi_2\bar\Psi-}\big)\Big) \\
& = 4\big((\Delta+s)(\Delta-s-1)\,\mathbb{I}-(\Delta-1)\,\rho_I L_{\Phi,I}\big)C_{\varphi_1\varphi_2\Phi}\,.
\end{aligned}
\tag{4.34}
$$

Assuming both $\Delta_1=\Delta_2=1$ and using (4.20) together with the corresponding equation for $\mathcal{D}_{\varphi_2}C_{\varphi_1\varphi_2\Phi}$ then leads to

$$
\begin{aligned}
& (\Delta+s)(\Delta-s-1)\big(\mathcal{D}_{\psi_1}\mathcal{D}_{\varphi_1}+\mathcal{D}_{\psi_2}\mathcal{D}_{\varphi_2}\big)C_{\varphi_1\varphi_2\Phi} \\
& \qquad = 2\big((\Delta+s)(\Delta-s-1)\,\mathbb{I}-(\Delta-1)\,\rho_I L_{\Phi,I}\big)C_{\varphi_1\varphi_2\Phi}\,.
\end{aligned}
\tag{4.35}
$$

(2.12) then implies

$$
(\Delta+s)(\Delta-s-1)\,\big(L_{1,I}+L_{2,I}\big)C_{\varphi_1\varphi_2\Phi} = -2(\Delta-1)\,L_{\Phi,I}\,C_{\varphi_1\varphi_2\Phi}\,, \tag{4.36}
$$

or, since $(L_{1,I}+L_{2,I}+L_{\Phi,I})C_{\varphi_1\varphi_2\Phi}=0$,

$$
(\Delta+s-1)(\Delta-s-2)\,L_{\Phi,I}\,C_{\varphi_1\varphi_2\Phi}=0\,. \tag{4.37}
$$

Unless $L_{\Phi,I}\,C_{\varphi_1\varphi_2\Phi}=0$ and Φ is a singlet this implies that $\Delta=s+2$ or $\Delta=1$, $s=0$, assuming unitarity constraints on Δ.

5. Ward Identities for Four Point Functions

There are also corresponding superconformal Ward identities for higher point functions. Here we consider the application of such identities to a four point function $\langle\varphi_1(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\rangle$ for superconformal primary fields φ_r belonging to short supermultiplets such that (2.10) and (2.11) give the associated superconformal transformations.

The superconformal transformations $\delta_{\hat{\epsilon}}\big\langle\psi_{1\alpha}(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\big\rangle = 0$ lead to a basic Ward identity for the four point function of the form

$$\begin{aligned} i\partial_{1\alpha\beta}\,\mathcal{D}_{\varphi_1}\,&\big\langle\varphi_1(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\big\rangle\,\hat{\bar{\epsilon}}{}^{\beta}(x_1)\\ &+ 2\Delta_1\,\mathcal{D}_{\varphi_1}\,\big\langle\varphi_1(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\big\rangle\,\eta_\alpha\\ =&\; -\mathcal{D}_{J_1}\big\langle J_{1\alpha\beta}(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\big\rangle\,\hat{\bar{\epsilon}}{}^{\beta}(x_1)\\ &+ \big\langle\psi_{1\alpha}(x_1)\,\bar{\psi}_{2\beta}(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\big\rangle\,\overleftarrow{\mathcal{D}}_{\psi_2}\,\hat{\bar{\epsilon}}{}^{\beta}(x_2)\\ &+ \big\langle\psi_{1\alpha}(x_1)\,\varphi_2(x_2)\,\bar{\psi}_{3\beta}(x_3)\,\varphi_4(x_4)\big\rangle\,\overleftarrow{\mathcal{D}}_{\psi_3}\,\hat{\bar{\epsilon}}{}^{\beta}(x_3)\\ &+ \big\langle\psi_{1\alpha}(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\bar{\psi}_{4\beta}(x_4)\,\big\rangle\,\overleftarrow{\mathcal{D}}_{\psi_4}\,\hat{\bar{\epsilon}}{}^{\beta}(x_4)\,. \end{aligned} \tag{5.1}$$

By virtue of conformal invariance

$$\big\langle\varphi_1(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\big\rangle = \mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4)\,F(u,v)\,, \tag{5.2}$$

where u,v are conformal invariants

$$u = \frac{{x_{12}}^2\,{x_{34}}^2}{{x_{13}}^2\,{x_{24}}^2}\,, \qquad v = \frac{{x_{23}}^2\,{x_{14}}^2}{{x_{13}}^2\,{x_{24}}^2}\,, \qquad x_{rs} = x_r - x_s\,. \tag{5.3}$$

The choice for $\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}$, depending on ${x_{rs}}^2$, in (5.2) is arbitrary so long as it has the required conformal weight at each x_r, for convenience we assume here

$$\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4) = \frac{({x_{23}}^2)^{E-\Delta_3}\,({x_{24}}^2)^{E-\Delta_4}}{({x_{12}}^2)^{\Delta_1}\,({x_{34}}^2)^{E}}\,, \tag{5.4}$$

with

$$E = \tfrac{1}{2}(\Delta_1+\Delta_3+\Delta_4-\Delta_2)\,, \tag{5.5}$$

Using (3.4) and noting that

$$\hat{\bar{\epsilon}}{}^{\alpha}(x_3) = \frac{1}{{x_{12}}^2}\,(\tilde{\mathrm{x}}_{32}\mathrm{x}_{21})^{\alpha}{}_{\beta}\,\hat{\bar{\epsilon}}{}^{\beta}(x_1) + \frac{1}{{x_{12}}^2}\,(\tilde{\mathrm{x}}_{31}\mathrm{x}_{12})^{\alpha}{}_{\beta}\,\hat{\bar{\epsilon}}{}^{\beta}(x_2)\,, \tag{5.6}$$

with a similar result for $x_3 \to x_4$, then, taking $\hat{\bar{\epsilon}}(x_1)$ and $\hat{\bar{\epsilon}}(x_2)$ as linearly independent, (5.1) reduces to

$$\begin{aligned} \mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}&(x_1,x_2,x_3,x_4)\,i\partial_{1\alpha\beta}\,\mathcal{D}_{\varphi_1}F(u,v)\\ &= -\,{x_{12}}^2\,\mathcal{D}_{J_1}\big\langle J_{1\alpha\beta}(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\big\rangle\\ &\quad+ \big\langle\psi_{1\alpha}(x_1)\,\varphi_2(x_2)\,\bar{\psi}_{3\gamma}(x_3)\,\varphi_4(x_4)\big\rangle\,\overleftarrow{\mathcal{D}}_{\psi_3}\,(\tilde{\mathrm{x}}_{32}\mathrm{x}_{21})^{\gamma}{}_{\beta}\\ &\quad+ \big\langle\psi_{1\alpha}(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\bar{\psi}_{4\gamma}(x_4)\,\big\rangle\,\overleftarrow{\mathcal{D}}_{\psi_4}\,(\tilde{\mathrm{x}}_{42}\mathrm{x}_{21})^{\gamma}{}_{\beta}\,, \end{aligned} \tag{5.7}$$

and

$$
\begin{aligned}
-\,\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4)\;2\Delta_1\, i\mathrm{x}_{12\alpha\beta}\,\mathcal{D}_{\varphi_1}F(u,v)&\\
= {x_{12}}^2\,\big\langle\psi_{1\alpha}(x_1)\,\bar\psi_{2\beta}(x_2)\,\varphi_3(x_3)\,\varphi_4(x_4)\big\rangle\,\overleftarrow{\mathcal{D}}_{\psi_2}&\\
+\big\langle\psi_{1\alpha}(x_1)\,\varphi_2(x_2)\,\bar\psi_{\gamma}(x_3)\,\varphi_4(x_4)\big\rangle\,\overleftarrow{\mathcal{D}}_{\psi_3}\,(\tilde{\mathrm{x}}_{31}\mathrm{x}_{12})^{\gamma}{}_{\beta}&\\
+\big\langle\psi_{1,\alpha}(x_1)\,\varphi_2(x_2)\,\varphi_3(x_3)\,\bar\psi_{4\gamma}(x_4)\,\big\rangle\,\overleftarrow{\mathcal{D}}_{\psi_4}\,(\tilde{\mathrm{x}}_{41}\mathrm{x}_{12})^{\gamma}{}_{\beta}\,.&
\end{aligned}
\tag{5.8}
$$

To analyse (5.7) and (5.8) we use conformal invariance to write

$$
\begin{aligned}
&\big\langle\psi_{r\alpha}(x_r)\,\bar\psi_{s\beta}(x_s)\,\varphi_t(x_t)\,\varphi_u(x_u)\big\rangle\\
&\quad=\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4)\bigg(\frac{1}{{x_{rs}}^2}\,i\mathrm{x}_{rs\alpha\beta}\,R_{rs}(u,v)+\frac{1}{{x_{rt}}^2\,{x_{su}}^2}\,i(\mathrm{x}_{rt}\tilde{\mathrm{x}}_{tu}\mathrm{x}_{us})_{\alpha\beta}\,S_{rs}(u,v)\bigg)\,,\\
&r<s\,,\ \ (r,s,t,u)=(1,2,3,4),(1,3,4,2),(1,4,3,2),(2,3,4,1),(2,4,1,3),(3,4,1,2)\,,\\
&r>s\,,\ \ (r,s,t,u)=(2,1,4,3),(3,1,2,4),(4,1,2,3),(3,2,1,4),(4,2,3,1),(4,3,2,1)\,,
\end{aligned}
\tag{5.9}
$$

and

$$
\begin{aligned}
&\big\langle J_{r\alpha\beta}(x_r)\,\varphi_s(x_s)\,\varphi_t(x_t)\,\varphi_u(x_u)\big\rangle\\
&\quad=\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4)\big(i\mathrm{X}_{r[st]\alpha\beta}\,I_r(u,v)+i\mathrm{X}_{r[tu]\alpha\beta}\,J_r(u,v)\big)\,,\\
&(r,s,t,u)=(1,2,3,4),(2,3,4,1),(3,4,1,2),(4,1,2,3)\,.
\end{aligned}
\tag{5.10}
$$

With the prescription in (5.9)

$$
R_{rs}={R_{sr}}^T\,,\qquad S_{rs}={S_{sr}}^T\,.
\tag{5.11}
$$

In (5.9) the choices given for r,s,t,u are sufficient by virtue of the identity

$$
\mathrm{x}_{rt}\,\tilde{\mathrm{x}}_{tu}\,\mathrm{x}_{us}+\mathrm{x}_{ru}\,\tilde{\mathrm{x}}_{ut}\,\mathrm{x}_{ts}={x_{tu}}^2\mathrm{x}_{rs}\,.
\tag{5.12}
$$

Similarly (5.10) is sufficient given

$$
\mathrm{X}_{r[st]}=-\mathrm{X}_{r[ts]}\,,\qquad \mathrm{X}_{r[st]}+\mathrm{X}_{r[tu]}+\mathrm{X}_{r[us]}=0\,.
\tag{5.13}
$$

The superconformal identity (5.1) reduces to two 2×2 equations. First from (5.8)

$$
\begin{aligned}
-2\Delta_1\,\mathrm{I}\,\mathcal{D}_{\varphi_1}F(u,v)=\big(\mathrm{I}\,R_{12}(u,v)+\mathrm{X}\,S_{12}(u,v)\big)\,\overleftarrow{\mathcal{D}}_{\psi_2}&\\
+\big(\mathrm{I}\,R_{13}(u,v)+(\,\mathrm{I}-\mathrm{X})^{-1}\,S_{13}(u,v)\big)\,\overleftarrow{\mathcal{D}}_{\psi_3}&\\
+\big(\mathrm{I}\,R_{14}(u,v)+(\,\mathrm{I}-\mathrm{X})\,S_{14}(u,v)\big)\,\overleftarrow{\mathcal{D}}_{\psi_4}\,,&
\end{aligned}
\tag{5.14}
$$

for

$$\mathrm{X} = \frac{1}{x_{13}{}^2\, x_{24}{}^2}\, \mathrm{x}_{13}\tilde{\mathrm{x}}_{34}\mathrm{x}_{42}\tilde{\mathrm{x}}_{21} = \mathrm{I} - \frac{1}{x_{13}{}^2\, x_{24}{}^2}\, \mathrm{x}_{13}\tilde{\mathrm{x}}_{32}\mathrm{x}_{24}\tilde{\mathrm{x}}_{41}\,. \tag{5.15}$$

Secondly from (5.7)

$$\begin{aligned} 2\bigg(\mathrm{X}_{1[23]}\, u\frac{\partial}{\partial u} + \mathrm{X}_{1[43]}\, v\frac{\partial}{\partial v}\bigg)\mathcal{D}_{\varphi_1}F(u,v) = {} & -\mathcal{D}_{J_1}\big(\mathrm{X}_{1[23]}\, I_1(u,v) + \mathrm{X}_{1[34]}\, J_1(u,v)\big) \\ & + \big(\,\mathrm{X}_{1[32]}\, R_{13}(u,v) + \mathrm{X}_{1[42]}\, S_{13}(u,v)\big)\overleftarrow{\mathcal{D}}_{\psi_3} \\ & + \big(\,\mathrm{X}_{1[42]}\, R_{14}(u,v) + \mathrm{X}_{1[32]}\, S_{14}(u,v)\big)\overleftarrow{\mathcal{D}}_{\psi_4}\,. \end{aligned} \tag{5.16}$$

or using $\mathrm{X}_{1[34]}\mathrm{X}_{1[23]}{}^{-1} = \mathrm{X}\,(\mathrm{I}-\mathrm{X})^{-1},\ \mathrm{X}_{1[42]}\mathrm{X}_{1[23]}{}^{-1} = -(\mathrm{I}-\mathrm{X})^{-1}$

$$\begin{aligned} 2\bigg(u\frac{\partial}{\partial u} - \mathrm{X}\,(\mathrm{I}-\mathrm{X})^{-1}\, v\frac{\partial}{\partial v}\bigg)\mathcal{D}_{\varphi_1}F(u,v) = {} & -\mathcal{D}_{J_1}\big(I_1(u,v) + \mathrm{X}(\mathrm{I}-\mathrm{X})^{-1}\, J_1(u,v)\big) \\ & - \big(R_{13}(u,v) + (\mathrm{I}-\mathrm{X})^{-1}\, S_{13}(u,v)\big)\overleftarrow{\mathcal{D}}_{\psi_3} \\ & - \big((\mathrm{I}-\mathrm{X})^{-1}R_{14}(u,v) + S_{14}(u,v)\big)\overleftarrow{\mathcal{D}}_{\psi_4}\,. \end{aligned} \tag{5.17}$$

If X is diagonalised, with eigenvalues $x, \bar{x}$, so that from (5.3) and (5.15)[2]

$$u = \det \mathrm{X} = x\bar{x}\,, \qquad v = \det(\,\mathrm{I}-\mathrm{X}) = (1-x)(1-\bar{x})\,, \tag{5.18}$$

then

$$\frac{1}{x}\, u\frac{\partial}{\partial u} - \frac{1}{1-x}\, v\frac{\partial}{\partial v} = \frac{\partial}{\partial x}\,. \tag{5.19}$$

Defining

$$\begin{aligned} T_{12}(x,\bar{x}) &= R_{12}(u,v) + x\, S_{12}(u,v)\,, \\ T_{13}(x,\bar{x}) &= R_{13}(u,v) + (1-x)^{-1}S_{13}(u,v)\,, \\ T_{14}(x,\bar{x}) &= R_{14}(u,v) + (1-x)\, S_{14}(u,v)\,. \end{aligned} \tag{5.20}$$

(5.14) becomes

$$-2\Delta_1\, \mathcal{D}_{\varphi_1}F = T_{12}\,\overleftarrow{\mathcal{D}}_{\psi_2} + T_{13}\,\overleftarrow{\mathcal{D}}_{\psi_3} + T_{14}\,\overleftarrow{\mathcal{D}}_{\psi_4}\,. \tag{5.21}$$

Using (5.19) with

$$K_1(x,\bar{x}) = \frac{1}{x}\, I_1(u,v) + \frac{1}{1-x}\, J_1(u,v)\,, \tag{5.22}$$

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[2] By using a conformal transformation we may take

$$\mathrm{x}_1 \to \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}, \quad \mathrm{x}_2 \to 0\,, \quad \mathrm{x}_3 \to \infty\,, \quad \mathrm{x}_4 \to \mathrm{I}\,.$$

In this case $\mathrm{X} \to \begin{pmatrix} \bar{x} & 0 \\ 0 & x \end{pmatrix}$ whereas $\mathrm{X}_{1[32]} \to \begin{pmatrix} \frac{1}{\bar{x}} & 0 \\ 0 & \frac{1}{x} \end{pmatrix}$ and $\mathrm{X}_{1[43]} \to \begin{pmatrix} \frac{1}{1-\bar{x}} & 0 \\ 0 & \frac{1}{1-x} \end{pmatrix}$.

(5.16) becomes

$$-2\,\frac{\partial}{\partial x}\,\mathcal{D}_{\varphi_1}F = \mathcal{D}_{J_1}K_1 + \frac{1}{x}\,T_{13}\,\overleftarrow{\mathcal{D}}_{\psi_3} + \frac{1}{x(1-x)}\,T_{14}\,\overleftarrow{\mathcal{D}}_{\psi_4}\,, \tag{5.23}$$

For both (5.21) and (5.23) there are associated conjugate equations obtained by $x \leftrightarrow \bar{x}$.

The identities (5.21) and (5.23) may be extended by using, from (5.4),

$$\begin{aligned}\mathcal{C}_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4) &= u^{-\Delta_1}v^{\Sigma-\Delta_3}\,\mathcal{C}_{\Delta_2\Delta_3\Delta_4\Delta_1}(x_2,x_3,x_4,x_1)\\ &= (v/u)^{\Sigma-\Delta_2-\Delta_3}\,\mathcal{C}_{\Delta_3\Delta_4\Delta_1\Delta_3}(x_3,x_4,x_1,x_2)\\ &= u^{\Delta_2-\Sigma}v^{\Delta_4}\,\mathcal{C}_{\Delta_4\Delta_1\Delta_2\Delta_3}(x_4,x_1,x_2,x_3)\,.\end{aligned} \tag{5.24}$$

Following the same derivation as before (5.21) generalises to

$$-2\Delta_r\,\mathcal{D}_{\varphi_r}F = \textstyle\sum_{s\neq r} T_{rs}\,\overleftarrow{\mathcal{D}}_{\psi_s} \tag{5.25}$$

where in addition to (5.20) for $r < s$

$$\begin{aligned}T_{23}(x,\bar{x}) &= R_{23}(u,v) + (1-x)\,S_{23}(u,v)\,,\\ T_{24}(x,\bar{x}) &= R_{24}(u,v) + x^{-1}S_{24}(u,v)\,,\\ T_{34}(x,\bar{x}) &= R_{34}(u,v) + x\,S_{34}(u,v)\,,\end{aligned} \tag{5.26}$$

and, using (5.11),

$$T_{s\,r} = T_{rs}{}^{T}\,. \tag{5.27}$$

Furthermore in addition to (5.23) we have

$$\begin{aligned}2\bigg(\frac{\partial}{\partial x} + \frac{\Delta_2-\Delta_1}{x} + \frac{\Delta_2+\Delta_3-\Sigma}{1-x}\bigg)\mathcal{D}_{\varphi_2}F &= \mathcal{D}_{J_2}K_2 - \frac{1}{x(1-x)}\,T_{23}\,\overleftarrow{\mathcal{D}}_{\psi_3} - \frac{1}{x}\,T_{24}\,\overleftarrow{\mathcal{D}}_{\psi_4}\,,\\ -2\bigg(\frac{\partial}{\partial x} + \frac{\Delta_2+\Delta_3-\Sigma}{x(1-x)}\bigg)\mathcal{D}_{\varphi_3}F &= \mathcal{D}_{J_3}K_3 + \frac{1}{x}\,T_{31}\,\overleftarrow{\mathcal{D}}_{\psi_1} + \frac{1}{x(1-x)}\,T_{32}\,\overleftarrow{\mathcal{D}}_{\psi_2}\,,\\ 2\bigg(\frac{\partial}{\partial x} + \frac{\Delta_2+\Delta_4-\Sigma}{x}\bigg)\mathcal{D}_{\varphi_4}F &= \mathcal{D}_{J_4}K_4 - \frac{1}{x(1-x)}\,T_{41}\,\overleftarrow{\mathcal{D}}_{\psi_1} - \frac{1}{x}\,T_{42}\,\overleftarrow{\mathcal{D}}_{\psi_2}\,,\end{aligned} \tag{5.28}$$

for

$$\begin{aligned}K_3(x,\bar{x}) &= \frac{1}{x}\,I_3(u,v) + \frac{1}{1-x}\,J_3(u,v)\,,\\ K_r(x,\bar{x}) &= \frac{1}{1-x}\,I_r(u,v) + \frac{1}{x}\,J_r(u,v)\,, \quad r=2,4\,.\end{aligned} \tag{5.29}$$

By considering $\sum_r \mathcal{D}_{\psi_r}\mathcal{D}_{\phi_r}F$ and using (5.25) with (2.12) then it is easy to see that we must have

$$\textstyle\sum_r L_{r,I}\,F = 0\,, \tag{5.30}$$

as a consistency condition, in a similar fashion to the three point case.

Applying the conservation condition (3.17) for $r=1$ and using

$$\tilde{\partial}_1{}^{\alpha\beta}\Big(\frac{1}{x_{12}{}^2}\,\mathrm{X}_{1[23]\alpha\beta}\Big) = \frac{2x_{23}{}^2}{(x_{12}{}^2)^2\,x_{13}{}^2}\,, \quad \tilde{\partial}_1{}^{\alpha\beta}\Big(\frac{1}{x_{12}{}^2}\,\mathrm{X}_{1[43]\alpha\beta}\Big) = \frac{2x_{23}{}^2}{(x_{12}{}^2)^2\,x_{13}{}^2}\,\Big(1-\frac{1}{v}\Big)\,, \tag{5.31}$$

gives in this case

$$2\frac{\partial}{\partial u}I_1 - \frac{1}{u}(1-u-v)\frac{\partial}{\partial v}I_1 - 2\frac{\partial}{\partial v}J_1 + \frac{1}{v}(1-u-v)\frac{\partial}{\partial u}J_1 - \frac{1}{u}\,I_1 + \frac{1}{u}\Big(1-\frac{1}{v}\Big)J_1 = 0\,. \tag{5.32}$$

Equivalently from (5.22), and with $\bar{K}_1(x,\bar{x}) = K_1(\bar{x},x)$,

$$\frac{\partial}{\partial x}\Big(\frac{\bar{K}_1}{u}\Big) + \frac{\partial}{\partial \bar{x}}\Big(\frac{K_1}{u}\Big) = \frac{1}{x-\bar{x}}\,\Big(\frac{K_1}{u} - \frac{\bar{K}_1}{u}\Big)\,. \tag{5.33}$$

6. $\mathcal{N}=3$ Superconformal Algebra

For $\mathcal{N}=3$ we may use the isomorphism $SO(3)\simeq SU(2)/\mathbb{Z}_2$ to write the six supercharges as $Q_{\alpha ij} = Q_{\alpha ji}$, $i,j=1,2$ and for their superconformal partners $S^{\alpha ij} = S^{\alpha ji}$. In this case we define

$$\bar{Q}_\alpha{}^{ij} = \varepsilon^{ik}\varepsilon^{jl}\,Q_{\alpha kl}\,, \qquad \bar{S}^\alpha{}_{ij} = \varepsilon_{ik}\varepsilon_{jl}\,S^{\alpha kl}\,. \tag{6.1}$$

The commutation relations (1.42) are equivalent to the algebra

$$\begin{aligned} \big\{Q_{\alpha ij}, Q_{\beta kl}\big\} &= (\varepsilon_{ik}\,\varepsilon_{jl} + \varepsilon_{il}\,\varepsilon_{jk})P_{\alpha\beta}\,, \\ \big\{S^{\alpha ij}, S^{\beta kl}\big\} &= -\,\big(\varepsilon^{ik}\,\varepsilon^{jl} + \varepsilon^{il}\,\varepsilon^{jk}\big)K^{\alpha\beta}\,. \end{aligned} \tag{6.2}$$

and (1.43) becomes

$$\big\{Q_{\alpha ij}, S^{\beta kl}\big\} = 2\,\delta_i{}^{(k}\delta_j{}^{l)}\big(M_\alpha{}^\beta + \delta_\alpha{}^\beta H\big) - 2\,\delta_\alpha{}^\beta\,R_{(i}{}^{(k}\delta_{j)}{}^{l)}\,, \tag{6.3}$$

with $R_i{}^j$, $R_i{}^i=0$, the $SU(2)$ R-symmetry generators

$$\big[R_i{}^j, R_k{}^l\big] = \delta_k{}^j\,R_i{}^l - \delta_i{}^l\,R_k{}^j\,. \tag{6.4}$$

The supercharges form the three dimensional vector representation so that (1.46) in this case becomes

$$\big[R_i{}^j, Q_{\alpha kl}\big] = \delta_k{}^j Q_{\alpha il} + \delta_l{}^j Q_{\alpha ki} - \delta_i{}^j Q_{\alpha kl}\,, \quad \big[R_i{}^j, S^{\alpha kl}\big] = -\delta_i{}^k S^{\alpha jl} - \delta_i{}^l S^{\alpha kj} + \delta_i{}^j S^{\alpha kl}\,. \tag{6.5}$$

In terms of the usual $SU(2)$ or $SO(3)$ generators

$$[R_i{}^j] = \begin{pmatrix} R_3 & R_+ \\ R_- & -R_3 \end{pmatrix} , \tag{6.6}$$

and $(R_i{}^j)^\dagger = R_j{}^i = -\varepsilon^{ik}\varepsilon_{jl}R_k{}^l$.

In a quantum field theory context we require unitary representations in which H has a positive real spectrum but the operators are unitary with respect to a modified scalar product than that implicit in (1.14). Such representations can be constructed by using as a representation space the Verma module for which a basis is given by the action of the generators $(P_{\alpha\beta})^{n_{\alpha\beta}}\,(M_2{}^1)^{n_M}\,(R_-)^{n_R}\,(Q_{\alpha ij})^{m_{\alpha ij}}$ with a definite ordering and for $n_{11}, n_{12}, n_{22}, n_M, n_R = 0, 1, 2, \ldots$, $m_{\alpha ij} = 0, 1$, $i \leq j$, on a highest weight state $|\Delta, s, r\rangle_{\rm hw}$ satisfying

$$\begin{aligned} H|\Delta, s, r\rangle_{\rm hw} &= \Delta|\Delta, s, r\rangle_{\rm hw} \,, \\ R_3|\Delta, s, r\rangle_{\rm hw} &= r|\Delta, s, r\rangle_{\rm hw} \,, \qquad M_1{}^1|\Delta, s, r\rangle_{\rm hw} = s|\Delta, s, r\rangle_{\rm hw} \,, \\ S^{\alpha ij}|\Delta, s, r\rangle_{\rm hw} &= R_+|\Delta, s, r\rangle_{\rm hw} = M_1{}^2|\Delta, s, r\rangle_{\rm hw} = 0 \,, \end{aligned} \tag{6.7}$$

where unitarity requires $r, s = 0, \frac{1}{2}, 1, \ldots$, s labels the finite dimensional representations of $Sl(2,\mathbb{R})_{\rm spin}$ with generators $M_\alpha{}^\beta$. The condition involving $S^{\alpha ij}$ implies also that $|\Delta, s, r\rangle_{\rm hw}$ is annihilated by $K^{\alpha\beta}$. In general states satisfying $K^{\alpha\beta}|\psi\rangle = 0$ are conformal primary. The states formed by the action of $R_-, M_2{}^1$ on $|\Delta, s, r\rangle_{\rm hw}$ form the superconformal primary states and have dimension $(2s+1)(2r+1)$, the supercharges $Q_{\alpha ij}$ acting on this generate $2^6(2s+1)(2r+1)$ conformal primary states.

Except in special cases when $|\Delta, s, r\rangle_{\rm hw}$ satisfies additional constraints, unitarity requires also $\Delta \geq 1+s+r$. Truncated representations are formed when some $Q_{\alpha ij}$ annihilate the highest weight state. For the case of interest here we require

$$Q_{\alpha 11}|\Delta, s, r\rangle_{\rm hw} = 0 \,, \tag{6.8}$$

where the anti-commutator (6.3) then requires

$$\Delta = r \,, \qquad s = 0 \,. \tag{6.9}$$

In this case there remain four supercharges $Q_{\alpha 12}, Q_{\alpha 22}$ generating the Verma module. Denoting representations of $SU(2)_R \times Sl(2,\mathbb{R})_{\rm spin}$ by r_s the action of the supercharges on the highest weight state generate representations formed from conformal primary states according to,

$$r_0 \xrightarrow{Q} \begin{matrix} r_{\frac{1}{2}} \\ (r-1)_{\frac{1}{2}} \end{matrix} \xrightarrow{Q^2} \begin{matrix} r_0, (r-1)_0, (r-2)_0 \\ (r-1)_1 \end{matrix} \xrightarrow{Q^3} \begin{matrix} (r-1)_{\frac{1}{2}} \\ (r-2)_{\frac{1}{2}} \end{matrix} \xrightarrow{Q^4} (r-2)_0 \ , \tag{6.10}$$

where there are both $16r-8$ boson and fermion states. When $r=\frac{1}{2}$ if we use the Racah-Speiser algorithm so that in a decomposition in terms of $SU(2)_R$ representations $(-r)\simeq -(r-1)$ then (6.10) becomes

$$\tfrac{1}{2}{}_0 \xrightarrow{Q} \tfrac{1}{2}{}_{\frac{1}{2}} \xrightarrow{Q^2} \phi \xrightarrow{Q^3} -\tfrac{1}{2}{}_{\frac{1}{2}} \xrightarrow{Q^4} -\tfrac{1}{2}{}_0 \, . \tag{6.11}$$

For $r=1$ with a similar use of the Racah-Speiser algorithm

$$1_0 \xrightarrow{Q} 1_{\frac{1}{2}},\, 0_{\frac{1}{2}} \xrightarrow{Q^2} 1_0,\, 0_1 \xrightarrow{Q^3} \phi \xrightarrow{Q^4} -0_0 \, . \tag{6.12}$$

In (6.11) and (6.12) ϕ denotes that there are no contributing representations, the negative contributions reflect that is necessary to impose derivative constraints on the antecedent representations. Hence (6.11) describes the fundamental hypermultiplet which is formed by conformal primary states $|\varphi_i\rangle$, $|\psi_{\alpha i}\rangle$, which have $\Delta=\frac{1}{2},1$, such that

$$Q_{\alpha ij}|\varphi_k\rangle = \tfrac{1}{2}\big(\varepsilon_{ik}\,|\psi_{\alpha j}\rangle + \varepsilon_{jk}\,|\psi_{\alpha i}\rangle\big)\,, \qquad Q_{\alpha ij}|\psi_{\beta k}\rangle = -\varepsilon_{ik}P_{\alpha\beta}|\varphi_j\rangle - \varepsilon_{jk}P_{\alpha\beta}|\varphi_i\rangle\,, \tag{6.13}$$

where we require also

$$\tilde{P}^{\alpha\beta}P_{\alpha\beta}|\varphi_i\rangle = 0\,, \qquad \tilde{P}^{\alpha\beta}|\psi_{\beta i}\rangle = 0\,, \tag{6.14}$$

which ensure the absence of the negatively contributing representations in (6.11). For the supermultiplet with $r=1$ described by (6.12) there is a $SU(2)_R$ singlet vector state $|J_{\alpha\beta}\rangle = |J_{\beta\alpha}\rangle$, with $\Delta=2$, satisfying the condition

$$\tilde{P}^{\alpha\beta}|J_{\alpha\beta}\rangle = 0\,. \tag{6.15}$$

(6.14) are just the dynamical equations for free hypermultiplet fields while (6.15) corresponds to a conserved current.

7. Superconformal field transformations for $\mathcal{N}=3$

We here consider the Ward identities for $\mathcal{N}=3$ superconformal symmetry involving superconformal primary fields belonging to the truncated supermultiplet described by (6.10). The superconformal primary fields for $r=\frac{1}{2}n$ are symmetric rank n $SU(2)$ tensors which are expressible in terms of homogeneous functions of $SU(2)$ spinors t^i, $i=1,2$, of degree n according to

$$\varphi_{i_1\dots i_n}(x) \to \varphi^{(n)}(x,t) = \varphi_{i_1\dots i_n}(x)\, t^{i_1}\dots t^{i_n}\,, \qquad \varphi^{(n)}(x,\lambda t) = \lambda^n \varphi^{(n)}(x,t)\,. \tag{7.1}$$

For superconformal transformations then, following (6.10) and ensuring homogeneity in t is preserved,

$$\delta_{\hat\epsilon}\varphi^{(n)}(t) = \hat\epsilon^{\alpha\, ij}\,\tilde t_i \partial_j \psi_\alpha^{(n)}(t) + \hat\epsilon^{\alpha\, ij}\,\tilde t_i \tilde t_j\, \chi_\alpha^{(n-2)}(t)\,, \tag{7.2}$$

where we define

$$\tilde t_i = \varepsilon_{ij}\, t^j\,, \qquad \tilde\partial^i = \varepsilon^{ij}\partial_j\,, \tag{7.3}$$

and the Killing spinor in (1.16) is now

$$\hat\epsilon^{\alpha\, ij}(x) = \epsilon^{\alpha\, ij} + i\,\bar\eta_\beta{}^{ij}\, \tilde{\rm x}^{\beta\alpha}\,. \tag{7.4}$$

The corresponding transformations for the spinor fields ψ, χ, generated by the action of the supercharges, are then from (6.10)

$$\begin{aligned}
\delta_{\hat\epsilon}\psi_\alpha^{(n)}(t) ={}& \hat\epsilon^{\beta\, kl}\big(a\,\tilde t_k\partial_l\, i\partial_{\alpha\beta}\varphi^{(n)}(t) + \tilde t_k\tilde t_l\, J^{(n-2)}_{\alpha\beta}(t)\big) + na\,\bar\eta_\alpha{}^{kl}\tilde t_k\partial_l\,\varphi^{(n)}(t)\\
&+ \varepsilon_{\alpha\beta}\,\hat\epsilon^{\beta\, kl}\big((n-1)\,\tilde t_k\partial_l f^{(n)}(t) + (n-2)\,\tilde t_k\tilde t_l f^{(n-2)}(t)\big)\,,\\
\delta_{\hat\epsilon}\chi_\alpha^{(n-2)}(t) ={}& -\hat\epsilon^{\beta\, kl}\big(a\,\partial_k\partial_l\, i\partial_{\alpha\beta}\varphi^{(n)}(t) + \tilde t_k\partial_l\, J^{(n-2)}_{\alpha\beta}(t)\big) - na\,\bar\eta_\alpha{}^{kl}\partial_k\partial_l\,\varphi^{(n)}(t)\\
&+ \varepsilon_{\alpha\beta}\,\hat\epsilon^{\beta\, kl}\big(\partial_k\partial_l f^{(n)}(t) + (n+2)\,\tilde t_k\partial_l f^{(n-2)}(t) + \tilde t_k\tilde t_l f^{(n-4)}(t)\big)\,.
\end{aligned} \tag{7.5}$$

The coefficients in (7.5) are determined by requiring closure of the algebra which determines $\Delta = \frac{1}{2}n$. In general we have

$$(\delta_{\hat\epsilon_2}\delta_{\hat\epsilon_1} - \delta_{\hat\epsilon_1}\delta_{\hat\epsilon_2})\varphi^{(n)}(t) = U^{\alpha\beta,ij,kl}\, f_{\alpha\beta,ij,kl}(t) + V^{ij,kl}\, g_{ij,kl}(t) + W^{ij,kl}\, h_{ij,kl}(t)\,, \tag{7.6}$$

where

$$\begin{aligned}
U^{\alpha\beta,ij,kl} &= \hat\epsilon_1^{(\alpha\, ij}\,\hat\epsilon_2^{\beta)\, kl} - \hat\epsilon_2^{(\alpha\, ij}\,\hat\epsilon_1^{\beta)\, kl} = U^{\alpha\beta,kl,ij}\,,\\
V^{ij,kl} &= \varepsilon_{\alpha\beta}\big(\hat\epsilon_1^{\alpha\, ij}\,\hat\epsilon_2^{\beta\, kl} - \hat\epsilon_2^{\alpha\, ij}\,\hat\epsilon_1^{\beta\, kl}\big) = -V^{kl,ij}\,,\\
W^{ij,kl} &= \hat\epsilon_1^{\alpha\, ij}\,\bar\eta_{2\,\alpha}^{\,kl} - \hat\epsilon_2^{\alpha\, ij}\,\bar\eta_{1\,\alpha}^{\,kl}\,.
\end{aligned} \tag{7.7}$$

With the identity

$$2\,\tilde t_{[j}\,\partial_{k]} = \varepsilon_{jk}\, t^m\partial_m\,, \tag{7.8}$$

we have, using the symmetry conditions in (7.7), for the individual contributions to (7.6)

$$\begin{aligned}
& V^{ij,kl}\big((n-1)\,\tilde t_i\partial_j\,\tilde t_k\partial_l + \tilde t_i\tilde t_j\,\partial_k\partial_l\big) f^{(n)}(t)\\
&\qquad = V^{ij,kl}\big(-(n-1)\,\varepsilon_{jk}\,\tilde t_i\partial_l + 2\,\tilde t_i\,\tilde t_{[j}\partial_{k]}\partial_l\big) f^{(n)}(t) = 0\,,\\
& V^{ij,kl}\big((n-2)\,\tilde t_i\partial_j\,\tilde t_k\tilde t_l + (n+2)\,\tilde t_i\tilde t_j\,\tilde t_k\partial_l\big) f^{(n-2)}(t)\\
&\qquad = V^{ij,kl}\big(-2(n-2)\,\varepsilon_{jl}\,\tilde t_i\partial_k + 4\,\tilde t_i\tilde t_k\,\tilde t_{[j}\partial_{l]}\big) f^{(n-2)}(t) = 0\,,\\
& U^{\alpha\beta,ij,kl}\big(\tilde t_i\partial_j\,\tilde t_k\partial_l - \tilde t_i\tilde t_j\,\partial_k\partial_l\big)\,\partial_{\alpha\beta}\varphi^{(n)}(t)\\
&\quad = -U^{\alpha\beta,ij,kl}\big(\varepsilon_{jk}\,\tilde t_i\partial_l + \varepsilon_{jk}\,\tilde t_i\, t^m\partial_m\,\partial_l\big)\,\partial_{\alpha\beta}\varphi^{(n)}(t) = -\tfrac{1}{2}n^2\, U^{\alpha\beta,ij,kl}\varepsilon_{jk}\varepsilon_{il}\,\partial_{\alpha\beta}\varphi^{(n)}(t)\,,\\
& U^{\alpha\beta,ij,kl}\big(\tilde t_i\partial_j\,\tilde t_k\tilde t_l - \tilde t_i\tilde t_j\,\tilde t_k\partial_l\big) J^{(n-2)}_{\alpha\beta}(t) = 0\,,\\
& W^{ij,kl}\big(\tilde t_i\partial_j\,\tilde t_k\partial_l - \tilde t_i\tilde t_j\,\partial_k\partial_l\big)\varphi^{(n)}(t) = -n\, W^{ij,kl}\varepsilon_{jk}\,\tilde t_i\partial_l\,\varphi^{(n)}(t)\,.
\end{aligned} \tag{7.9}$$

Choosing $a = -2/n^2$ we have

$$\begin{aligned}
&(\delta_{\hat\epsilon_2}\delta_{\hat\epsilon_1} - \delta_{\hat\epsilon_1}\delta_{\hat\epsilon_2})\varphi^{(n)}(x,t) = v^{\alpha\beta}(x)\, i\partial_{\alpha\beta}\varphi^{(n)}(x,t) - 2w_j{}^i(x)\big(t^j\partial_i - n\,\delta_i^j\big)\,\varphi^{(n)}(x,t)\,,\\
&v^{\alpha\beta} = \varepsilon_{ik}\varepsilon_{jl}\big(\hat\epsilon_1^{\alpha\, ij}\,\hat\epsilon_2^{\beta\, kl} - \hat\epsilon_2^{\alpha\, ij}\,\hat\epsilon_1^{\beta\, kl}\big)\,, \qquad w_j{}^i = \hat\epsilon_1^{\alpha\, ik}\,\eta_{2\,\alpha\, jk} - \hat\epsilon_2^{\alpha\, ik}\,\eta_{1\,\alpha\, jk}\,.
\end{aligned} \tag{7.10}$$

This result is in accord with (6.2) and (6.3), where the action of $P_{\alpha\beta}, M_\alpha{}^\beta, H, K^{\alpha\beta}$ is just as in (1.11) with $\Delta = \frac{1}{2}n$, and

$$\big[R_i{}^j, \varphi^{(n)}(x,t)\big] = -L_i{}^j\varphi^{(n)}(x,t)\,, \tag{7.11}$$

for

$$L_i{}^j\varphi^{(n)}(t) = (-t^j\partial_i + \tfrac{1}{2}n\,\delta_i^j)\varphi^{(n)}(t)\,, \qquad L_i{}^jL_j{}^i\varphi^{(n)}(t) = \tfrac{1}{2}n(n+2)\,\varphi^{(n)}(t)\,. \tag{7.12}$$

These results for $\mathcal{N} = 3$ superconformal transformations can be expressed in the general form described in section 2 by taking $\varphi \to \varphi^{(n)}(x,t)$ and the fermion fields ψ_α are then given by a column vector formed by the two $SU(2)$ representation appearing in (7.2),

$$\psi_\alpha(x,t) = \begin{pmatrix} \psi_\alpha^{(n)}(x,t) \\ \chi_\alpha^{(n-2)}(x,t) \end{pmatrix}\,, \qquad \bar\psi_\alpha(x,t) = \psi_\alpha(x,t)^T\,, \tag{7.13}$$

with also corresponding results the vector and scalar fields $J_{\alpha\beta}$ and F,

$$J_{\alpha\beta}^{(n-2)}(x,t)\,, \qquad F(x,t) = \begin{pmatrix} f^{(n)}(x,t) \\ f^{(n-2)}(x,t) \\ f^{(n-4)}(x,t) \end{pmatrix}\,. \tag{7.14}$$

The superconformal transformation (7.2) is then in the form (2.10) if

$$\mathcal{D}_{\psi,ij}\psi_\alpha(x,t) = \big(\tilde t_{(i}\partial_{j)} \quad \tilde t_i\,\tilde t_j\big)\,\psi_\alpha(x,t)\,, \tag{7.15}$$

and (7.5) may also be expressed as in (2.11) with the definitions

$$\begin{aligned}
\mathcal{D}_{\varphi,ij}\varphi^{(n)}(x,t) &= \frac{2}{n^2}\begin{pmatrix} \tilde t_{(i}\partial_{j)} \\ -\partial_i\,\partial_j \end{pmatrix}\varphi^{(n)}(x,t)\,,\\
\mathcal{D}_{J,ij}J_{\alpha\beta}^{(n-2)}(x,t) &= \begin{pmatrix} \tilde t_i\,\tilde t_j \\ -\tilde t_{(i}\partial_{j)} \end{pmatrix} J_{\alpha\beta}^{(n-2)}(x,t)\,,
\end{aligned} \tag{7.16}$$

and

$$\mathcal{D}_{F,ij}F(x,t) = \begin{pmatrix} (n-1)\,\tilde t_{(i}\partial_{j)} & (n-2)\,\tilde t_i\,\tilde t_j & 0 \\ \partial_i\,\partial_j & (n+2)\,\tilde t_{(i}\partial_{j)} & \tilde t_i\,\tilde t_j \end{pmatrix} F(x,t)\,. \tag{7.17}$$

Using (7.16) and (7.15) we get

$$\mathcal{D}_{\psi,ij}\bar{\mathcal{D}}_{\varphi}{}^{kl}\varphi^{(n)}(t) = \Big(\delta_{(i}{}^{k}\delta_{j)}{}^{l} + \frac{2}{n}\,\delta_{(i}{}^{(k}L_{j)}{}^{l)}\Big)\varphi^{(n)}(t)\,, \qquad \bar{\mathcal{D}}_{\varphi}{}^{kl} = \varepsilon^{km}\varepsilon^{ln}\mathcal{D}_{\varphi,mn}\,, \qquad (7.18)$$

as required by (2.12). Also, with $\bar{\mathcal{D}}_{J}{}^{kl} = \varepsilon^{km}\varepsilon^{ln}\mathcal{D}_{J,mn}$,

$$\mathcal{D}_{\psi,ij}\bar{\mathcal{D}}_{\mathcal{J}}{}^{kl}J^{(n-2)}(t) = \delta_{(i}{}^{(k}M_{j)}{}^{l)}J^{(n-2)}(t)\,, \quad M_{j}{}^{l}J^{(n-2)}(t) = -n\,\tilde{t}_{j}t^{l}J^{(n-2)}(t)\,, \qquad (7.19)$$

in accord with (2.13). The conditions (2.14) satisfied by taking

$$\mathcal{Q}_{J}{}^{i} = \big(\tfrac{1}{2}(n-2)\tilde{\partial}^{i}\ \ (n+1)t^{i}\big)\,, \qquad \mathcal{Q}_{\psi}{}^{ij} = t^{i}t^{j}\,, \qquad (7.20)$$

where $\mathcal{Q}_{J}{}^{i}\mathcal{D}_{J,ij}\,J^{(n-2)}(t) = 0$.

8. Ward Identities for $\mathcal{N}=3$

The preceding results can be used to obtain Ward identities for $\mathcal{N}=3$ superconformal symmetry in three and four point functions where $\varphi_r(x_r) \to \varphi^{(n_r)}(x_r,t_r)$, $\Delta_r = \frac{1}{2}n_r$. For the two point function then $SU(2)$ invariance dictates

$$\big\langle \varphi^{(n)}(x_1,t_1)\,\varphi^{(n)}(x_2,t_2)\big\rangle = \bigg(\frac{(t_1\tilde{t}_2)^2}{x_{12}{}^2}\bigg)^{\frac{1}{2}n}\,, \qquad (8.1)$$

letting $t_1\tilde{t}_2 = -t_2\tilde{t}_1 = t_1{}^i\tilde{t}_{2i}$. The three point function corresponding to (4.1), with $\Phi^{(n)}$ belonging to the $\frac{1}{2}n$ $SU(2)$ representation and $\Delta_{1,2} = \frac{1}{2}n_{1,2}$, becomes

$$\begin{aligned}&\big\langle \varphi^{(n_1)}(x_1,t_1)\,\varphi^{(n_2)}(x_2,t_2)\,\Phi^{(n)}_{\alpha_1\dots\alpha_{2s}}(x_3,t_3)\big\rangle \\ &\qquad = C_{12,\Phi}\,\mathcal{C}^{(s)}_{\Delta_1\Delta_2\,\Delta,\alpha_1\dots\alpha_{2s}}(x_1,x_2,x_3)\,f_{n_1n_2n}(t_1,t_2,t_3)\,,\end{aligned} \qquad (8.2)$$

with $f_{n_1n_2n_3}(t_1,t_2,t_3)$ homogeneous of degree n_r in t_r. $SU(2)$ invariance requires

$$\begin{aligned}f_{n_1n_2n_3}(t_1,t_2,t_3) &= (t_1\tilde{t}_2)^{\frac{1}{2}(n_1+n_2-n_3)}(t_2\tilde{t}_3)^{\frac{1}{2}(n_2+n_3-n_1)}(t_3\tilde{t}_1)^{\frac{1}{2}(n_3+n_1-n_2)} \\ &|n_1-n_2| \le n_3 \le n_1+n_2\,.\end{aligned} \qquad (8.3)$$

This satisfies

$$f_{n_1n_2n_3}(t_1,t_2,t_3) = (-1)^{\frac{1}{2}(n_1+n_2+n_3)}f_{n_2n_1n_3}(t_2,t_1,t_3) = f_{n_3n_1n_2}(t_3,t_1,t_2)\,, \qquad (8.4)$$

in accord with standard results for $SU(2)$ $3j$-symbols, and

$$\begin{aligned}L_{1,i}{}^{j}f_{n_1n_2n_3}(t_1,t_2,t_3) = {}& \tfrac{1}{2}n_1\,\tilde{t}_{2,i}t_2{}^{j}f_{n_1\,n_2-2\,n_3}(t_1,t_2,t_3) - \tfrac{1}{2}n_1\,\tilde{t}_{3,i}t_3{}^{j}f_{n_1n_2\,n_3-2}(t_1,t_2,t_3) \\ &+ \tfrac{1}{2}(n_2-n_3)\,\tilde{t}_{1,i}t_1{}^{j}f_{n_1-2\,n_2n_3}(t_1,t_2,t_3)\,.\end{aligned} \qquad (8.5)$$

For four point functions, as in (5.2), then the scalar $F(u,v,t_1,t_2,t_3,t_4)$ now becomes a homogeneous function of degree n_1,n_2,n_3,n_4 in t_1,t_2,t_3,t_4 in addition to depending on u,v. Using (7.15) and (7.16), (5.21) becomes

$$-\frac{2}{n_1}\begin{pmatrix} \tilde{t}_{1(i}\partial_{1j)} \\ -\partial_{1i}\,\partial_{1j} \end{pmatrix} F = \sum_{r=2}^{4} T_{1r} \begin{pmatrix} \overleftarrow{\partial}_{r(i}\tilde{t}_{rj)} \\ \tilde{t}_{r\,i}\,\tilde{t}_{rj} \end{pmatrix}, \tag{8.6}$$

where T_{1r} are 2×2 matrices since ψ_α as in (6.14) has two components. Also from (5.23)

$$\begin{aligned} -\frac{4}{{n_1}^2}\begin{pmatrix} \tilde{t}_{1(i}\partial_{1j)} \\ -\partial_{1i}\,\partial_{1j} \end{pmatrix} \frac{\partial}{\partial x} F = {}& \begin{pmatrix} \tilde{t}_{1i}\,\tilde{t}_{1j} \\ -\tilde{t}_{1(i}\partial_{1j)} \end{pmatrix} K_1 \\ & + \frac{1}{x}\, T_{13} \begin{pmatrix} \overleftarrow{\partial}_{3(i}\tilde{t}_{3j)} \\ \tilde{t}_{3i}\,\tilde{t}_{3j} \end{pmatrix} + \frac{1}{x(1-x)}\, T_{14} \begin{pmatrix} \overleftarrow{\partial}_{4(i}\tilde{t}_{4j)} \\ \tilde{t}_{4i}\,\tilde{t}_{4j} \end{pmatrix}. \end{aligned} \tag{8.7}$$

Since $t_1\tilde{t}_2\, t_3\tilde{t}_4 + t_3\tilde{t}_1\, t_2\tilde{t}_4 + t_2\tilde{t}_3\, t_1\tilde{t}_4 = 0$, there is one invariant which is homogeneous of degree zero in all four t_r,

$$\alpha = \frac{t_3\tilde{t}_1\, t_2\tilde{t}_4}{t_2\tilde{t}_1\, t_3\tilde{t}_4} = 1 - \frac{t_2\tilde{t}_3\, t_1\tilde{t}_4}{t_2\tilde{t}_1\, t_3\tilde{t}_4}\,. \tag{8.8}$$

Defining

$$f_{n_1n_2n_3n_4}(t_1,t_2,t_3,t_4) = (t_1\tilde{t}_2)^{n_1}(t_2\tilde{t}_3)^{n_3-e}(t_4\tilde{t}_2)^{n_4-e}(t_3\tilde{t}_4)^{e}\,, \tag{8.9}$$

for

$$e = \tfrac{1}{2}(n_1+n_3+n_4-n_2) = 0,1,2,\dots\,, \tag{8.10}$$

we may then write

$$\begin{aligned} F(u,v,t_1,t_2,t_3,t_4) &= f_{n_1\,n_2\,n_3\,n_4}(t_1,t_2,t_3,t_4)\,\mathcal{F}(x,\bar{x},\alpha)\,, \\ K_1(x,\bar{x},t_1,t_2,t_3,t_4) &= f_{n_1-2\,n_2\,n_3\,n_4}(t_1,t_2,t_3,t_4)\,\mathcal{K}(x,\bar{x},\alpha)\,, \end{aligned} \tag{8.11}$$

where, assuming $e \le n_3,n_4$, $\mathcal{F}(x,\bar{x},\alpha)$ is a polynomial in α of degree $\min(e,n_1)$, $\mathcal{K}(x,\bar{x},\alpha)$ similarly has degree $\min(e{-}1,n_1{-}2)$. Similarly, T_{13},T_{14} may also be expressed as

$$\begin{aligned} T_{13} &= \begin{pmatrix} f_{n_1\,n_2\,n_3\,n_4}(t_1,t_2,t_3,t_4)\,\mathcal{A}_3 & f_{n_1\,n_2\,n_3-2\,n_4}(t_1,t_2,t_3,t_4)\,\mathcal{B}_3 \\ f_{n_1-2\,n_2\,n_3\,n_4}(t_1,t_2,t_3,t_4)\,\mathcal{C}_3 & f_{n_1-2\,n_2\,n_3-2\,n_4}(t_1,t_2,t_3,t_4)\,\mathcal{D}_3 \end{pmatrix}, \\ T_{14} &= \begin{pmatrix} f_{n_1\,n_2\,n_3\,n_4}(t_1,t_2,t_3,t_4)\,\mathcal{A}_4 & f_{n_1\,n_2\,n_3\,n_4-2}(t_1,t_2,t_3,t_4)\,\mathcal{B}_4 \\ f_{n_1-2\,n_2\,n_3\,n_4}(t_1,t_2,t_3,t_4)\,\mathcal{C}_4 & f_{n_1-2\,n_2\,n_3\,n_4-2}(t_1,t_2,t_3,t_4)\,\mathcal{D}_4 \end{pmatrix}, \end{aligned} \tag{8.12}$$

with $\mathcal{A}_3,\mathcal{A}_4$ polynomials of degree $\min(e,n_1)$, $\mathcal{B}_3,\mathcal{B}_4$ of degree $\min(e{-}1,n_1)$, $\mathcal{C}_3,\mathcal{C}_4$ of degree $\min(e{-}1,n_1{-}2)$ and $\mathcal{D}_3,\mathcal{D}_4$ of degree $\min(e{-}2,n_1{-}2)$.

The various i, j components in (8.7) give three independent equations which may be obtained by contracting with the basis formed by $t_2{}^i t_2{}^j$, $t_1{}^{(i} t_2{}^{j)}$ and $t_1{}^i t_1{}^j$. Using

$$\begin{aligned} t_1 \cdot \partial_3 \alpha &= -\frac{t_1 \tilde{t}_2}{t_3 \tilde{t}_2}\, \alpha(1-\alpha)\,, \qquad & t_2 \cdot \partial_3 \alpha &= -\frac{t_4 \tilde{t}_2}{t_3 \tilde{t}_4}\,(1-\alpha)\,, \\ t_1 \cdot \partial_4 \alpha &= \frac{t_1 \tilde{t}_2}{t_4 \tilde{t}_2}\, \alpha(1-\alpha)\,, \qquad & t_2 \cdot \partial_4 \alpha &= -\frac{t_3 \tilde{t}_2}{t_3 \tilde{t}_4}\, \alpha\,, \end{aligned} \tag{8.13}$$

then (8.7) reduces to

$$\begin{aligned} \frac{4}{n_1{}^2}\, \frac{\partial}{\partial x} \mathcal{F}' &= \mathcal{K} + \frac{1}{x}\, \mathcal{Y}_3 + \frac{1}{x(1-x)}\, \mathcal{Y}_4\,, \\ \frac{2}{n_1}\, \frac{\partial}{\partial x} \mathcal{F} &= \frac{1}{x}\big(-\tfrac{1}{2} n_3\, \mathcal{A}_3 + \alpha\, \mathcal{Y}_3\big) - \frac{1}{x(1-x)}\big(\tfrac{1}{2} n_4\, \mathcal{A}_4 + (1-\alpha)\, \mathcal{Y}_4\big)\,, \\ 0 &= \frac{\alpha}{x}\big(- n_3\, \mathcal{A}_3 + \alpha\, \mathcal{Y}_3\big) + \frac{1-\alpha}{x(1-x)}\big(n_4\, \mathcal{A}_4 + (1-\alpha)\, \mathcal{Y}_4\big)\,, \end{aligned} \tag{8.14}$$

and

$$\begin{aligned} \frac{4}{n_1{}^2}\, \frac{\partial}{\partial x} \mathcal{F}'' &= \mathcal{K}' + \frac{1}{x}\, \mathcal{Z}_3 + \frac{1}{x(1-x)}\, \mathcal{Z}_4\,, \\ \frac{4}{n_1{}^2}(n_1-1)\, \frac{\partial}{\partial x} \mathcal{F}' &= \tfrac{1}{2}(n_1-2)\, \mathcal{K} \\ &\quad + \frac{1}{x}\big(-\tfrac{1}{2} n_3\, \mathcal{C}_3 + \alpha\, \mathcal{Z}_3\big) - \frac{1}{x(1-x)}\big(\tfrac{1}{2} n_4\, \mathcal{C}_4 + (1-\alpha)\, \mathcal{Z}_4\big)\,, \\ \frac{4}{n_1}(n_1-1)\, \frac{\partial}{\partial x} \mathcal{F} &= \frac{\alpha}{x}\big(- n_3\, \mathcal{C}_3 + \alpha\, \mathcal{Z}_3\big) + \frac{1-\alpha}{x(1-x)}\big(n_4\, \mathcal{C}_4 + (1-\alpha)\, \mathcal{Z}_4\big)\,, \end{aligned} \tag{8.15}$$

with primes denoting differentiation with respect to α and where

$$\begin{aligned} \mathcal{Y}_3 &= e\, \mathcal{A}_3 + (1-\alpha)\, \mathcal{A}_3' + \mathcal{B}_3\,, \qquad & \mathcal{Y}_4 &= -e\, \mathcal{A}_4 + \alpha\, \mathcal{A}_4' + \mathcal{B}_4\,, \\ \mathcal{Z}_3 &= (e-1)\, \mathcal{C}_3 + (1-\alpha)\, \mathcal{C}_3' + \mathcal{D}_3\,, \qquad & \mathcal{Z}_4 &= -(e-1)\, \mathcal{C}_4 + \alpha\, \mathcal{C}_4' + \mathcal{D}_4\,. \end{aligned} \tag{8.16}$$

Furthermore by contracting (8.6) with $t_2{}^i t_2{}^j$, which eliminates T_2, we get

$$\frac{2}{n_1}\, \mathcal{F}' = \mathcal{Y}_3 + \mathcal{Y}_4\,, \qquad \frac{2}{n_1}\, \mathcal{F}'' = \mathcal{Z}_3 + \mathcal{Z}_4\,. \tag{8.17}$$

The remaining four equations obtained by contracting (8.6) with $t_1{}^{(i} t_2{}^{j)}$, $t_1{}^i t_1{}^j$ should serve to determine the four components in T_2. It is easy to see that by combining (8.17) with (8.14) and (8.15) that we must have

$$\mathcal{Y}_3' = \mathcal{Z}_3\,, \qquad \mathcal{Y}_4' = \mathcal{Z}_4\,. \tag{8.18}$$

In addition to (8.14), (8.15) and (8.17) there are corresponding equations for $x \to \bar{x}$.

In the extremal case, $e = 0$, then $\mathcal{F}' = 0$ and in (8.14) and (8.15) only $\mathcal{A}_3, \mathcal{A}_4$, as well as $\mathcal{F}$, are non zero. In consequence $\frac{\partial}{\partial x}\mathcal{F} = 0$ and with the corresponding equation when x $\to \bar{x}$ $\mathcal{F}$ is just a constant which is given in terms of the three point functions given in (8.2) by

$$\mathcal{F} = C_{n_1 n_2\, n_2 - n_1}\, C_{n_3 n_4\, n_3 + n_4}\,. \tag{8.19}$$

9. $\mathcal{N}=6$ Superconformal Algebra

For $\mathcal{N}=6$, with R-symmetry $SO(6)\simeq SU(4)/\mathbb{Z}_2$, the superconformal algebra can be written in a very similar form to the $\mathcal{N}=3$ case given in section 4 except that the 12 supercharges and their conformal partners are now

$$Q_{\alpha ij}=-Q_{\alpha ji}\,,\qquad S^{\alpha ij}=-S^{\alpha ji}\,,\qquad i,j=1,2,3,4\,. \tag{9.1}$$

Replacing (1.42) we now have

$$\big\{Q_{\alpha ij},Q_{\beta kl}\big\}=\varepsilon_{ijkl}\,P_{\alpha\beta}\,,\qquad \big\{S^{\alpha ij},S^{\beta kl}\big\}=-\varepsilon^{ijkl}\,K^{\alpha\beta}\,, \tag{9.2}$$

and, instead of (6.3), reflecting the antisymmetry in (9.1),

$$\big\{Q_{\alpha ij},S^{\beta kl}\big\}=2\,\delta_i^{[k}\delta_j^{l]}\big(M_\alpha{}^\beta+\delta_\alpha{}^\beta H\big)-4\,\delta_\alpha{}^\beta\,R_{[i}{}^{[k}\delta_{j]}{}^{l]}\,. \tag{9.3}$$

The other commutation relations are unchanged in form except that, with $R_i{}^j$ now generators for $SU(4)$ satisfying (6.4),

$$\begin{aligned}\big[R_i{}^j,Q_{\alpha kl}\big]&=\delta_k^j Q_{\alpha il}+\delta_l^j Q_{\alpha ki}-\tfrac12\delta_i^j Q_{\alpha kl}\,,\\ \big[R_i{}^j,S^{\alpha kl}\big]&=-\,\delta_i^k S^{\alpha jl}-\delta_i^l S^{\alpha kj}+\tfrac12\delta_i^j S^{\alpha kl}\,,\end{aligned} \tag{9.4}$$

and, instead of (6.1),

$$\bar Q_\alpha{}^{ij}=\tfrac12\,\varepsilon^{ijkl}\,Q_{\alpha kl}\,,\qquad \bar S^\alpha{}_{ij}=\tfrac12\,\varepsilon_{ijkl}\,S^{\alpha kl}\,. \tag{9.5}$$

The construction of unitary positive energy representations is similar to the $\mathcal{N}=3$ case although the highest weight states are now labelled $\big|\Delta,s,[r_1,r_2,r_3]\big\rangle_{\rm hw}$ where we require $R_i{}^j\big|\Delta,s,[r_1,r_2,r_3]\big\rangle_{\rm hw}=0$ for $j>i$ and

$$H_i\big|\Delta,s,[r_1,r_2,r_3]\big\rangle_{\rm hw}=r_i\big|\Delta,s,[r_1,r_2,r_3]\big\rangle_{\rm hw}\,, \tag{9.6}$$

with H_i a basis for the $SU(4)$ Cartan generators so that r_i are integers and $[r_1,r_2,r_3]$ are then the Dynkin labels for the corresponding $SU(4)$ representation. Here

$$\begin{aligned}R_1{}^1&=\tfrac14(3H_1+2H_2+H_3)\,,\qquad R_2{}^2=\tfrac14(-H_1+2H_2+H_3)\,,\\ R_3{}^3&=-\,\tfrac14(H_1+2H_2-H_3)\,,\quad R_4{}^4=-\tfrac14(H_1+2H_2+3H_3)\,.\end{aligned} \tag{9.7}$$

Acting on $\big|\Delta,s,[r_1,r_2,r_3]\big\rangle_{\rm hw}$ $R_i{}^j$ for $j<i$ generates superconformal primary states forming a basis for the associated $SU(4)$ representation space of dimension $d(r_1,r_2,r_3)=$

$\frac{1}{12}(r_1+1)(r_2+1)(r_3+1)(r_1+r_2+2)(r_2+r_3+2)(r_1+r_2+r_3+3)$. The 12 supercharges $Q_{\alpha ij}$ then generate conformal primary states with dimension $2^{12}(2s+1)d(r_1,r_2,r_3)$.

When the highest weight state is annihilated by one or more supercharges truncated representations are obtained. For $s=0$, so that $M_\alpha{}^\beta\big|\Delta,0,[r_1,r_2,r_3]\big\rangle_{\rm hw}=0$, here we consider

$$
\begin{aligned}
Q_{\alpha 12}\big|\Delta,0,[r_1,r_2,r_3]\big\rangle_{\rm hw}=0 \quad &\Rightarrow \quad \big(H-R_1{}^1-R_2{}^2\big)\big|\Delta,0,[r_1,r_2,r_3]\big\rangle_{\rm hw}=0\,,\\
Q_{\alpha 13}\big|\Delta,0,[r_1,r_2,r_3]\big\rangle_{\rm hw}=0 \quad &\Rightarrow \quad \big(H-R_1{}^1-R_3{}^3\big)\big|\Delta,0,[r_1,r_2,r_3]\big\rangle_{\rm hw}=0\,,
\end{aligned} \tag{9.8}
$$

where the constraints arise from (9.3). Together the two BPS conditions in (9.8) require

$$
\Delta=\tfrac{1}{2}(r_1+r_3)\,, \qquad r_2=0\,. \tag{9.9}
$$

We may also impose in addition the further conditions $Q_{\alpha 14}\big|\Delta,0,[r_1,r_2,r_3]\big\rangle_{\rm hw}=0$ or $Q_{\alpha 23}\big|\Delta,0,[r_1,r_2,r_3]\big\rangle_{\rm hw}=0$ which imply respectively $r_3=0$ or $r_1=0$. Labelling conformal primary multiplets by $[r_1,r_2,r_3]_s$ the action of the remaining supercharges on superconformal primary states satisfying (9.8) give

$$
\begin{aligned}
[q,0,r]_0 \xrightarrow{\;Q\;} \;& \begin{matrix}[q{+}1,0,r{-}1]_{\frac12},[q{-}1,0,r{+}1]_{\frac12}\\ [q{-}1,1,r{-}1]_{\frac12}\end{matrix}\\
\xrightarrow{\;Q^2\;} \;& \begin{matrix}[q,0,r]_{1,0},[q,1,r{-}2]_{1,0},[q{-}2,1,r]_{1,0},[q{-}1,0,r{-}1]_1\\ [q{+}2,0,r{-}2]_0,[q{-}2,0,r{+}2]_0,[q{-}2,2,r{-}2]_0\end{matrix}\\
\xrightarrow{\;Q^3\;} \;&
\end{aligned} \tag{9.10}
$$

$[q,0,r]$ $SU(4)$ representations may be defined in terms of the representation space formed by symmetric traceless (q,r) tensors, $T^{j_1\dots j_r}_{i_1\dots i_q}=T^{(j_1\dots j_r)}_{(i_1\dots i_q)}$, $T^{j_1\dots j_{r-1}\,i}_{i_1\dots i_{q-1}\,i}=0$. Equivalently we may consider scalar homogeneous functions $T^{(q,r)}(t,\bar t)$, of degree (q,r), of contravariant and covariant 4-vectors t^i and $\bar t_j$, satisfying $t^i\bar t_i=0$, so that

$$
T^{(q,r)}(\lambda\, t,\mu\,\bar t)=\lambda^q\mu^r T^{(q,r)}(t,\bar t)\,. \tag{9.11}
$$

Acting on $T^{(q,r)}(t,\bar t)$ derivatives $\partial_i,\bar\partial^i$ are defined to give homogeneous functions of degree $(q-1,r),(q,r-1)$. A precise definition, taking into account the constraint $t^i\bar t_i=0$, is given in appendix C. Applied to $T^{(q,r)}(t,\bar t)=T^{j_1\dots j_r}_{i_1\dots i_q}\,t^{i_1}\dots t^{i_q}\,\bar t_{j_1}\dots\bar t_{j_r}$, with $T^{j_1\dots j_r}_{i_1\dots i_q}$ symmetric and traceless, $\partial_i,\bar\partial^i$ are defined just as expected without considering the condition $t^i\bar t_i=0$ but in general, without the traceless condition on $T^{j_1\dots j_r}_{i_1\dots i_q}$, there are additional contributions which are proportional to $\bar t_i, t^i$ respectively. Using the definition given in appendix C then, for arbitrary homogeneous functions $T^{(q,r)}(t,\bar t)$, derivatives have the properties

$$
\begin{gathered}
\big[\partial_i,\partial_j\big]T^{(q,r)}(t,\bar t)=\big[\partial_i,\bar\partial^j\big]T^{(q,r)}(t,\bar t)=\big[\bar\partial^i,\bar\partial^j\big]T^{(q,r)}(t,\bar t)=0\,,\\
\bar\partial^i\partial_i T^{(q,r)}(t,\bar t)=0\,, \qquad t^i\partial_i T^{(q,r)}(t,\bar t)=q\,T^{(q,r)}(t,\bar t)\,, \quad \bar t_i\bar\partial^i T^{(q,r)}(t,\bar t)=r\,T^{(q,r)}(t,\bar t)\,.
\end{gathered} \tag{9.12}
$$

General $SU(4)$ representations with Dynkin labels $[q,p,r]$ are also be expressible in a similar fashion in terms of homogeneous functions which are covariant symmetric tensors of rank p,

$$T^{(q+p,r)}_{i_1\dots i_p}(t,\bar{t}) = T^{(q+p,r)}_{(i_1\dots i_p)}(t,\bar{t})\,, \tag{9.13}$$

subject to the conditions[3]

$$t^i T^{(q+p,r)}_{i\, i_1\dots i_{p-1}}(t,\bar{t}) = \bar{\partial}^i T^{(q+p,r)}_{i\, i_1\dots i_{p-1}}(t,\bar{t}) = 0\,. \tag{9.14}$$

Of course $[q,p,r]$ representations may be equivalently defined in terms of symmetric contravariant tensors $T^{(q,r+p)i_1\dots i_p}(t,\bar{t})$ satisfying the analogous conditions to (9.14). For $p=1$ the connection between the contravariant and covariant expressions is given by

$$(q+2)\,T^{(q,r+1)i}(t,\bar{t}) = \varepsilon^{ijkl}\bar{t}_j\partial_k T^{(q+1,r)}_l(t,\bar{t})\,. \tag{9.15}$$

It is easy to see that this satisfies $\bar{t}_i T^{(q,r+1)i}(t,\bar{t}) = \partial_i T^{(q,r+1)i}(t,\bar{t}) = 0$ and also, subject to the conditions (9.14), $\varepsilon_{ijkl}t^j\bar{\partial}^k T^{(q,r+1)l}(t,\bar{t}) = (r+2)T^{(q+1,r)}_i(t,\bar{t})$.

For application to the truncated supermultiplets given in (9.10) we therefore consider superconformal primary fields

$$\varphi^{(q,r)}(x,t,\bar{t})\,, \qquad \Delta = \tfrac{1}{2}(q+r)\,, \tag{9.16}$$

and, at the first level, fermion fields

$$\chi_\alpha^{(q+1,r-1)}(x,t,\bar{t})\,, \quad \bar{\chi}_{\dot\alpha}^{(q-1,r+1)}(x,t,\bar{t})\,, \quad \psi_{\alpha,i}^{(q,r-1)}(x,t,\bar{t})\,, \qquad \Delta = \tfrac{1}{2}(q+r+1)\,, \tag{9.17}$$

satisfying, from (9.14) for $p=1$,

$$t^i\psi_{\alpha,i}^{(q,r-1)}(t,\bar{t}) = \bar{\partial}^i\psi_{\alpha,i}^{(q,r-1)}(t,\bar{t}) = 0\,, \tag{9.18}$$

as required for the representation with Dynkin labels $[q-1,1,r-1]$. Equivalently we may take, as related by (9.15), $\psi_{\alpha,i}^{(q,r-1)}(t,\bar{t}) \to \bar{\psi}_{\dot\alpha}^{(q-1,r)i}(t,\bar{t})$ satisfying the corresponding conditions to (9.18). Superconformal transformations on $\varphi^{(q,r)}$ then take the form

$$\begin{aligned}\delta\varphi^{(q,r)}(t,\bar{t}) = {}& \hat{\epsilon}^{\,\alpha\, ij}\,\bar{t}_i\partial_j\chi_\alpha^{(q+1,r-1)}(t,\bar{t}) + \hat{\bar{\epsilon}}{}^{\dot\alpha}_{ij}\, t^i\bar{\partial}^j\bar{\chi}_{\dot\alpha}^{(q-1,r+1)}(t,\bar{t}) \\ & + \hat{\epsilon}^{\,\alpha\, ij}\,\bar{t}_i\,\psi_{\alpha,j}^{(q,r-1)}(t,\bar{t})\,.\end{aligned} \tag{9.19}$$

---

[3] The dimension formula follows from

$$\begin{aligned}d(q,p,r) = {}& \tfrac{1}{6}(p+1)(p+2)(p+3)\,d(q+p,0,r) \\ & - \tfrac{1}{6}p(p+1)(p+2)\,\big(d(q+p+1,0,r) + d(q+p,0,r-1)\big) \\ & + \tfrac{1}{6}(p-1)p(p+1)\,d(q+p+1,0,r-1)\,,\end{aligned}$$

where the second and third terms correspond to the constraint conditions and the last term removes overcounting.

The corresponding superconformal transformations of the fermion fields involving $\varphi^{(q,r)}$ are then

$$\begin{aligned}
\delta\chi_\alpha^{(q+1,r-1)}(t,\bar t) &= a\,\hat{\bar\epsilon}{}^{\beta}_{kl}\,t^k\bar\partial^l\,i\partial_{\beta\alpha}\varphi^{(q,r)}(t,\bar t) + a(q+r)\,\eta_{\alpha\,kl}\,t^k\bar\partial^l\,\varphi^{(q,r)}(t,\bar t)\,,\\
\delta\bar\chi_\alpha^{(q-1,r+1)}(t,\bar t) &= b\,\hat\epsilon^{\beta\,kl}\,\bar t_k\partial_l\,i\partial_{\beta\alpha}\varphi^{(q,r)}(t,\bar t) + b(q+r)\,\bar\eta_\alpha{}^{kl}\,\bar t_k\partial_l\,\varphi^{(q,r)}(t,\bar t)\,,\\
\delta\psi_{\alpha,j}^{(q,r-1)}(t,\bar t) &= c\,\hat{\bar\epsilon}{}^{\beta}_{kl}\,\big(q\,\delta_j{}^k - t^k\partial_j\big)\bar\partial^l\,i\partial_{\beta\alpha}\varphi^{(q,r)}(t,\bar t)\\
&\quad + c(q+r)\,\eta_{\alpha\,kl}\,\big(q\,\delta_j{}^k - t^k\partial_j\big)\bar\partial^l\,\varphi^{(q,r)}(t,\bar t)\,,
\end{aligned} \tag{9.20}$$

where the form of $\delta\psi_{\alpha,j}^{(q,r-1)}$ is dictated by the requirement that it satisfy (9.18). In (9.19) and (9.20)

$$\hat{\bar\epsilon}{}^{\alpha}_{ij} = \tfrac{1}{2}\,\varepsilon_{ijkl}\,\hat\epsilon^{\alpha kl}\,, \qquad \bar\eta_\alpha{}^{kl} = \tfrac{1}{2}\,\varepsilon^{klij}\,\eta_{\alpha\,ij}\,. \tag{9.21}$$

In (9.20) a, b, c are coefficients which are determined by requiring consistency with the superconformal algebra. If we consider the commutator of two transformations then from (9.19) and (9.20)

$$\begin{aligned}
(\delta_2\delta_1-\delta_1\delta_2)\varphi^{(q,r)}(t,\bar t) = U^{\alpha\beta,ij}{}_{kl}\big(&(a-c)\,\bar t_i\partial_j\,t^k\bar\partial^l + b\,t^k\bar\partial^l\,\bar t_i\partial_j\\
&+ (q+1)c\,\delta_j{}^k\,\bar t_i\bar\partial^l\,\big)\,i\partial_{\beta\alpha}\varphi^{(q,r)}(t,\bar t)\\
+ (q+r)\,W^{ij}{}_{kl}\big(&(a-c)\,\bar t_i\partial_j\,t^k\bar\partial^l + (q+1)c\,\delta_j{}^k\,\bar t_i\bar\partial^l\,\big)\,\varphi^{(q,r)}(t,\bar t)\\
+ (q+r)\,W_{kl}{}^{ij}\,&b\,t^k\bar\partial^l\,\bar t_i\partial_j\,\varphi^{(q,r)}(t,\bar t)\,,
\end{aligned} \tag{9.22}$$

where we have used the definitions (7.7) with

$$U^{\alpha\beta,ij}{}_{kl} = \tfrac{1}{2}\varepsilon_{klmn}\,U^{\alpha\beta,ij,mn}\,, \quad W^{ij}{}_{kl} = \tfrac{1}{2}\varepsilon_{klmn}W^{ij,kl}\,, \quad W_{kl}{}^{ij} = \tfrac{1}{2}\varepsilon_{klmn}W^{mn,kl}\,. \tag{9.23}$$

These satisfy the critical identities

$$U^{\alpha\beta,ik}{}_{jk} = \tfrac{1}{4}\,\delta_j{}^i\,U^{\alpha\beta,kl}{}_{kl}\,, \tag{9.24}$$

which depends on the symmetry relation for $U^{\alpha\beta,ij,mn}$ in (7.7), and

$$W_{kl}{}^{ij} = \tfrac{1}{4}\,\varepsilon^{ijmn}\varepsilon_{klpq}\,W^{mn}{}_{pq} = W^{ij}{}_{kl} - 4\,\delta_{[l}{}^{[j}\,W^{i]m}{}_{k]m} + \delta_{[k}{}^{i}\delta_{l]}{}^{j}\,W^{mn}{}_{mn}\,. \tag{9.25}$$

With the aid of (9.25) and taking

$$c = a + b\,, \tag{9.26}$$

we may use

$$\big[\bar t_{[i}\partial_{j]}, t^{[k}\bar\partial^{l]}\big] = -\delta_{[j}{}^{[l}\,\bar t_{i]}\bar\partial^{k]} + \delta_{[j}{}^{[l}\,t^{k]}\partial_{i]}\,, \tag{9.27}$$

to obtain

$$
\begin{aligned}
(\delta_2\delta_1-\delta_1\delta_2)\varphi^{(q,r)}(t,\bar t) = {}& -U^{\alpha\beta,il}{}_{kl}\big((qc+a)\,\bar t_i\bar\partial^k+b\,t^k\partial_i\big)\,i\partial_{\beta\alpha}\varphi^{(q,r)}(t,\bar t)\\
&-(q+r)\,W^{il}{}_{kl}\big((qc+a)\,\bar t_i\bar\partial^k+b\,t^k\partial_i\big)\,\varphi^{(q,r)}(t,\bar t)\\
&-4(q+r)b\,W^{il}{}_{kl}\,t^{[k}\bar\partial^{j]}\,\bar t_{[i}\partial_{j]}\,\varphi^{(q,r)}(t,\bar t)\\
&+\tfrac12(q+r)b\,W^{kl}{}_{kl}\,t^{[i}\bar\partial^{j]}\,\bar t_{[i}\partial_{j]}\varphi^{(q,r)}(t,\bar t)\,.
\end{aligned}
\tag{9.28}
$$

Since

$$
4t^{[k}\bar\partial^{j]}\,\bar t_{[i}\partial_{j]}\,\varphi^{(q,r)}(t,\bar t)=-q\,\bar t_i\bar\partial^k\varphi^{(q,r)}(t,\bar t)-(r+2)\,t^k\partial_i\varphi^{(q,r)}(t,\bar t)-q\,\delta_i^k\,\varphi^{(q,r)}(t,\bar t)\,,\tag{9.29}
$$

and using (9.24) we may obtain the final result

$$
\begin{aligned}
&(\delta_2\delta_1-\delta_1\delta_2)\varphi^{(q,r)}(t,\bar t)\\
&=-\tfrac14\big((q+1)ra+q(r+1)b\big)\Big(U^{\alpha\beta,kl}{}_{kl}\,i\partial_{\beta\alpha}\varphi^{(q,r)}(t,\bar t)+(q+r)\,W^{kl}{}_{kl}\,\varphi^{(q,r)}(t,\bar t)\Big)\\
&\quad-(q+r)\,W^{il}{}_{kl}\Big((q+1)a\,\big(\bar t_i\bar\partial^k-\tfrac14 r\,\delta_i^k\big)-(r+1)b\,\big(t^k\partial_i-\tfrac14 q\,\delta_i^k\big)\Big)\,\varphi^{(q,r)}(t,\bar t)\,.
\end{aligned}
\tag{9.30}
$$

For (9.30) to be compatible with closure of the superconformal algebra we require $(q+1)a=(r+1)b$. Assuming

$$
a=-\frac{4}{(q+1)(q+r)}\,,\qquad b=-\frac{4}{(r+1)(q+r)}\,,\tag{9.31}
$$

then (9.30) takes the form

$$
\begin{aligned}
(\delta_2\delta_1-\delta_1\delta_2)\varphi^{(q,r)}(t,\bar t)&=v^{\alpha\beta}\,i\partial_{\alpha\beta}\varphi^{(q,r)}(t,\bar t)+4w_j{}^i\,\big(L_i{}^j+\tfrac14(q+r)\,\delta_i^j\big)\,\varphi^{(q,r)}(t,\bar t)\,,\\
v^{\alpha\beta}&=\hat\epsilon_1^{\alpha\,ij}\,\hat{\bar\epsilon}_2{}^{\alpha}{}_{ij}-\hat\epsilon_2^{\beta\,ij}\,\hat{\bar\epsilon}_1{}^{\beta}{}_{ij}\,,\qquad w_j{}^i=\hat\epsilon_1^{\alpha\,ik}\,\eta_{2\,\alpha\,jk}-\hat\epsilon_2^{\alpha\,ik}\,\eta_{1\,\alpha\,jk}\,,
\end{aligned}
\tag{9.32}
$$

for

$$
L_i{}^j\varphi^{(q,r)}(t,\bar t)=-\big(t^j\partial_i-\bar t_i\bar\partial^j-\tfrac14(q-r)\,\delta_i^j\big)\varphi^{(q,r)}(t,\bar t)\,.\tag{9.33}
$$

This is in accord with (9.2) and (9.3) but now

$$
\big[R_i{}^j,\varphi^{(q,r)}(x,t,\bar t)\big]=-L_i{}^j\,\varphi^{(q,r)}(x,t,\bar t)\,.\tag{9.34}
$$

The transformation of the fermion fields also include at the next level contributions from vector fields $J_{\alpha\beta}=J_{\beta\alpha}$ which in accordance with the representations required in (9.10) are expressible in terms of the following homogeneous functions of $t,\bar t$,

$$
J^{(q,r)}_{\alpha\beta}(t,\bar t)\,,\quad J^{(q-1,r-1)}_{\alpha\beta}(t,\bar t)\,,\quad V^{(q+1,r-2)}_{\alpha\beta,i}(t,\bar t)\,,\quad V^{(q-1,r)}_{\alpha\beta,i}(t,\bar t)\,,\qquad \Delta=\tfrac12(q+r+2)\,,\tag{9.35}
$$

satisfying

$$t^i V^{(q+1,r-2)}_{\alpha\beta,i}(t,\bar t) = \bar\partial^i V^{(q+1,r-2)}_{\alpha\beta,i}(t,\bar t) = t^i V^{(q-1,r)}_{\alpha\beta,i}(t,\bar t) = \bar\partial^i V^{(q-1,r)}_{\alpha\beta,i}(t,\bar t) = 0\,. \tag{9.36}$$

For $q = r = 1$ $V^{(q+1,r-2)}_{\alpha\beta,i}, V^{(q-1,r)}_{\alpha\beta,i}$ are absent. The additional terms in the superconformal transformations involving the vector fields in (9.35) then have the form

$$\begin{aligned}
\delta'\chi^{(q+1,r-1)}_\alpha(t,\bar t) &= a\,\hat{\bar\epsilon}{}^\beta_{kl}\,t^k\bar\partial^l\,J^{(q,r)}_{\beta\alpha}(t,\bar t) + a'\,\hat\epsilon^{\,\beta\,kl}\,\bar t_k\,V^{(q+1,r-2)}_{\beta\alpha,l}(t,\bar t)\,,\\
\delta'\bar\chi^{(q-1,r+1)}_\alpha(t,\bar t) &= b\,\hat\epsilon^{\,\beta\,kl}\,\bar t_k\partial_l\,J^{(q,r)}_{\beta\alpha}(t,\bar t) + \hat\epsilon^{\,\beta\,kl}\,\bar t_k\,V^{(q-1,r)}_{\beta\alpha,l}(t,\bar t)\,,\\
\delta'\psi^{(q,r-1)}_{\alpha,j}(t,\bar t) &= c\,\hat{\bar\epsilon}{}^\beta_{kl}\,\big(q\,\delta_j{}^k - t^k\partial_j\big)\bar\partial^l\,J^{(q,r)}_{\beta\alpha}(t,\bar t)\\
&\quad + \hat{\bar\epsilon}{}^\beta_{kl}\,\big((r+1)\,\delta_j{}^k t^l + \bar t_j t^k\bar\partial^l\big)J^{(q-1,r-1)}_{\beta\alpha}(t,\bar t)\\
&\quad + \hat\epsilon^{\,\beta\,kl}\,\big((q+1)(r+1)\,\bar t_k\partial_l V^{(q+1,r-2)}_{\beta\alpha,j}(t,\bar t) + q\,\bar t_j\partial_k\,V^{(q+1,r-2)}_{\beta\alpha,l}(t,\bar t)\\
&\qquad\qquad + (r+1)\,\bar t_k\partial_j\,V^{(q+1,r-2)}_{\beta\alpha,l}(t,\bar t)\big)\\
&\quad - \hat{\bar\epsilon}{}^\beta_{kl}\,t^k\bar\partial^l\,V^{(q-1,r)}_{\beta\alpha,j}(t,\bar t)\,.
\end{aligned} \tag{9.37}$$

The detailed form in the expression for $\delta'\psi^{(q,r-1)}_{\alpha,j}$ is determined by compatibility with (9.18). For the commutator calculated using (9.37) there are contributions involving the vector fields in (9.35),

$$\begin{aligned}
(\delta_2{}'\delta_1 - \delta_1{}'\delta_2)\varphi^{(q,r)}(t,\bar t) = {}& U^{\alpha\beta,ij}{}_{kl}\big((a-c)\,\bar t_i\partial_j\,t^k\bar\partial^l + b\,t^k\bar\partial^l\,\bar t_i\partial_j\\
&\qquad + (q+1)c\,\delta_j{}^k\,\bar t_i\bar\partial^l\,\big)\,J^{(q,r)}_{\beta\alpha}(t,\bar t)\\
&+ (r+1)\,U^{\alpha\beta,ij}{}_{jl}\,\bar t_i t^l\,J^{(q-1,r-1)}_{\beta\alpha}(t,\bar t)\\
&+ U^{\alpha\beta,ij}{}_{kl}\,\big(t^k\bar\partial^l\,\bar t_i - \bar t_i\,t^k\bar\partial^l\big)\,V^{(q-1,r)}_{\beta\alpha,j}(t,\bar t)\\
&+ \big(a' + (q+2)(r+1)\big)\,U^{\alpha\beta,ij,kl}\,\bar t_i\bar t_k\partial_j V^{(q+1,r-2)}_{\beta\alpha,l}(t,\bar t)\,,
\end{aligned} \tag{9.38}$$

in the last line using the symmetry in (7.7). The superconformal algebra requires that this is zero. Using (9.24) this is easily achieved by taking

$$a = q(r+1)\,,\quad b = -(q+1)r\,,\quad c = q-r\,,\qquad a' = -(q+2)(r+1)\,. \tag{9.39}$$

These results for $\mathcal{N} = 6$ superconformal symmetry transformations starting from a superconformal primary $\varphi^{(q,r)}$ can be recast in terms of the general formalism of section three by writing the fermion fields in (9.17) as a vector

$$\psi_\alpha(x,t,\bar t) = \begin{pmatrix} \chi^{(q+1,r-1)}_\alpha(x,t,\bar t)\\ \bar\chi^{(q-1,r+1)}_\alpha(x,t,\bar t)\\ \psi^{(q,r-1)}_{\alpha,m}(x,t,\bar t)\end{pmatrix}\,, \tag{9.40}$$

and for the vector fields in (9.35)

$$J_{\alpha\beta}(x,t,\bar t) = \begin{pmatrix} J^{(q,r)}_{\alpha\beta}(x,t,\bar t) \\ J^{(q-1,r-1)}_{\alpha\beta}(x,t,\bar t) \\ V^{(q-1,r)}_{\alpha\beta,n}(x,t,\bar t) \\ V^{(q+1,r-2)}_{\alpha\beta,n}(x,t,\bar t) \end{pmatrix} . \tag{9.41}$$

With the notation in (9.40) and (9.41) then for (2.10) and (2.11) we require

$$\mathcal{D}_{\psi,ij}\psi_\alpha(t,\bar t) = \big(\bar t_{[i}\partial_{j]} \quad \tfrac12 \varepsilon_{ijkl}\, t^k \bar\partial^l \quad \bar t_{[i}\delta_{j]}{}^m\big)\psi_\alpha(t,\bar t)\,, \tag{9.42}$$

and

$$\begin{aligned} &\mathcal{D}_{\varphi,ij}\varphi^{(q,r)}(t,\bar t) \\ &= -\frac{4}{(q+1)(r+1)(q+r)} \begin{pmatrix} (r+1)\,\tfrac12\varepsilon_{ijkl}t^k\bar\partial^l \\ (q+1)\,\bar t_{[i}\partial_{j]} \\ (q+r+2)\,\tfrac12\varepsilon_{ijkl}\big(q\,\delta_m{}^k - t^k\partial_m\big)\bar\partial^l \end{pmatrix} \varphi^{(q,r)}(t,\bar t)\,, \end{aligned} \tag{9.43}$$

and also

$$\begin{aligned} &\mathcal{D}_{J,ij}J_{\alpha\beta}(t,\bar t) \\ &= \left(\begin{matrix} q(r+1)\,\tfrac12\varepsilon_{ijkl}t^k\bar\partial^l & 0 \\ -(q+1)r\,\bar t_{[i}\partial_{j]} & 0 \\ (q-r)\,\tfrac12\varepsilon_{ijkl}\big(q\,\delta_m{}^k - t^k\partial_m\big)\bar\partial^l & \tfrac12\varepsilon_{ijkl}\big((r+1)\,\delta_m{}^k t^l + \bar t_m t^k\bar\partial^l\big) \end{matrix} \right. \\ &\qquad \left. \begin{matrix} 0 & -(q+2)(r+1)\,\bar t_{[i}\delta_{j]}{}^n \\ \bar t_{[i}\delta_{j]}{}^n & 0 \\ -\tfrac12\varepsilon_{ijkl}t^k\bar\partial^l\,\delta_m{}^n & (q+1)(r+1)\,\bar t_{[i}\partial_{j]}\delta_m{}^n + q\,\bar t_m\partial_{[i}\delta_{j]}{}^n + (r+1)\,\bar t_{[i}\delta_{j]}{}^n\partial_m \end{matrix} \right) J_{\alpha\beta}(t,\bar t)\,. \end{aligned} \tag{9.44}$$

With these definitions and

$$\bar{\mathcal{D}}_\varphi{}^{kl} = \tfrac12\,\varepsilon^{klmn}\mathcal{D}_{\varphi,mn}\,, \qquad \bar{\mathcal{D}}_J{}^{kl} = \tfrac12\,\varepsilon^{klmn}\mathcal{D}_{J,mn}\,, \tag{9.45}$$

using (9.33),

$$\mathcal{D}_{\psi,ij}\bar{\mathcal{D}}_\varphi{}^{kl}\varphi^{(q,r)}(t,\bar t) = \Big(\delta_{[i}{}^k\delta_{j]}{}^l + \frac{4}{q+r}\,\delta_{[i}{}^{[k}L_{j]}{}^{l]}\Big)\varphi^{(q,r)}(t,\bar t)\,, \tag{9.46}$$

as required by (2.12), and furthermore

$$\mathcal{D}_{\psi,ij}\bar{\mathcal{D}}_J{}^{kl}J_{\alpha\beta}(t,\bar t) = 2\,\delta_{[i}{}^{[k}M_{j]}{}^{l]}J_{\alpha\beta}(t,\bar t)\,, \tag{9.47}$$

where

$$M_j{}^l = \tfrac12(r+1)\begin{pmatrix} -(q+1)\big(q\,\bar t_j\bar\partial^l + r\,t^l\partial_j - \tfrac12 rq\,\delta_j{}^l\big) & -\bar t_j t^l \\ \delta_j{}^n t^l & (q+1)\,\varepsilon^{lnpq}\,\bar t_j\bar t_p\partial_q \end{pmatrix} . \tag{9.48}$$

10. Ward Identities for $\mathcal{N}=6$

The basic two point function for the superconformal primary fields is now

$$\big\langle \varphi^{(q,r)}(x_1,t_1,\bar t_1)\,\varphi^{(r,q)}(x_2,t_2,\bar t_2)\big\rangle = \frac{(t_1\cdot\bar t_2)^q\ (t_2\cdot\bar t_1)^r}{({x_{12}}^2)^{\frac12(q+r)}}\,. \tag{10.1}$$

The three point function in general is of the form

$$\begin{aligned}&\big\langle \varphi^{(q_1,r_1)}(x_1,t_1,\bar t_1)\,\varphi^{(q_2,r_2)}(x_2,t_2,\bar t_2)\,\varphi^{(q_3,r_3)}(x_3,t_3,\bar t_3)\big\rangle\\ &\quad = \frac{C}{({x_{12}}^2)^{\frac12(q_1+q_2-r_3)}\,({x_{23}}^2)^{\frac12(q_2+q_3-r_1)}\,({x_{13}}^2)^{\frac12(q_1+q_3-r_2)}}\,,\end{aligned} \tag{10.2}$$

where this is non zero only if

$$q_1+q_2+q_3=r_1+r_2+r_3\,, \tag{10.3}$$

and then in general

$$\begin{aligned}C&=(t_1\cdot\bar t_2)^{q_1}(t_2\cdot\bar t_1)^{q_2-r_3}(t_2\cdot\bar t_3)^{r_3}(t_3\cdot\bar t_2)^{r_2-q_1}(t_3\cdot\bar t_1)^{r_1+r_3-q_2}\mathcal{C}(\lambda)\\ \lambda&=\frac{t_2\cdot\bar t_1\ t_1\cdot\bar t_3\ t_3\cdot\bar t_2}{t_1\cdot\bar t_2\ t_2\cdot\bar t_3\ t_3\cdot\bar t_1}\,,\end{aligned} \tag{10.4}$$

with $\mathcal{C}(\lambda)$ of the form

$$\mathcal{C}(\lambda)=\sum_{0,r_3-q_2,q_1-r_2\le n\le q_1,r_3,r_1+r_3-q_2} c_n\,\lambda^n\,. \tag{10.5}$$

The sum over n reflects the multiplicity $N_{[q,0,r]}$ of representations with Dynkin labels $[q,0,r]$ that may appear in the tensor product $[q_1,0,r_1]\otimes[q_2,0,r_2]$. For a non zero result in (10.5) it is necessary that $q_1+q_2-r_3, q_1+q_3-r_2, q_2+q_3-r_1\ge 0$ and, with $q-r=q_1+q_2-r_1-r_2$, we have then $N_{[q,0,r]}=\min(q_1,q_2,q,r_1,r_2,r,q_1+q_2-q,r+q_1-r_2,r+q_2-r_1)+1$.

For simplicity we consider here the three point function (10.2) for the case $q_1=r_1=1$ when

$$C=\begin{cases} c\,t_1\cdot\bar t_2\,t_2\cdot\bar t_1\,(t_2\cdot\bar t_3)^{q_2-1}(t_3\cdot\bar t_2)^{r_2-1}\,, & q_3=r_2-1\,,\ r_3=q_2-1\,,\\ c\,t_1\cdot\bar t_2\,t_3\cdot\bar t_1\,(t_2\cdot\bar t_3)^{q_2}(t_3\cdot\bar t_2)^{r_2-1} & \\ \ +\tilde c\,t_1\cdot\bar t_3\,t_2\cdot\bar t_1\,(t_2\cdot\bar t_3)^{q_2-1}(t_3\cdot\bar t_2)^{r_2}\,, & q_3=r_2\,,\ r_3=q_2\,,\\ c\,t_1\cdot\bar t_3\,t_3\cdot\bar t_1\,(t_2\cdot\bar t_3)^{q_2}(t_3\cdot\bar t_2)^{r_2}\,, & q_3=r_2+1\,,\ r_3=q_2+1\,,\end{cases} \tag{10.6}$$

and $q_2+q_3=r_2+r_3$. We focus then on the superconformal identities obtained by considering

$$\delta\big\langle \psi_m^{(1,0)}\,\varphi^{(q_2,r_2)}\,\varphi^{(q_3,r_3)}\big\rangle=0\,, \tag{10.7}$$

which leads to, by combining $(3.12a,b)$ with the result (9.16) for Δ,

$$(1+q_2-r_3)\,\bar{\mathcal{D}}^{ij}_{\varphi_1,m}C = \bar{\mathcal{D}}^{ij}_{J_1,m}\,K_{1,23} - P_{12,m}\overleftarrow{\bar{\mathcal{D}}}{}^{ij}_{\psi_2}\,. \tag{10.8}$$

In this case from (9.43) and (10.6)

$$\begin{aligned}\bar{\mathcal{D}}^{ij}_{\varphi_1,m}C &= -2(\delta_m{}^{[i} - t_1{}^{[i}\partial_{1m})\bar{\partial}_1{}^{j]}\,C\\ &= \begin{cases} -4c\,\delta_{[m}{}^{[i}\,t_2{}^{j]}\,\bar{t}_{2,k]}\,t_1{}^k\,(t_2\cdot\bar{t}_3)^{q_2-1}(t_3\cdot\bar{t}_2)^{r_2-1}\,, & q_3=r_2-1\,,\ r_3=q_2-1\,,\\ -4c\,\delta_{[m}{}^{[i}\,t_3{}^{j]}\,\bar{t}_{2,k]}\,t_1{}^k\,(t_2\cdot\bar{t}_3)^{q_2}(t_3\cdot\bar{t}_2)^{r_2-1} & \\ \quad -4\tilde{c}\,\delta_{[m}{}^{[i}\,t_2{}^{j]}\,\bar{t}_{3,k]}\,t_1{}^k\,(t_2\cdot\bar{t}_3)^{q_2-1}(t_3\cdot\bar{t}_2)^{r_2} & \\ \quad +(c+\tilde{c})\,\delta_m{}^{[i}\,t_1{}^{j]}\,(t_2\cdot\bar{t}_3)^{q_2}(t_3\cdot\bar{t}_2)^{r_2}\,, & q_3=r_2\,,\ r_3=q_2\,,\\ -4c\,\delta_{[m}{}^{[i}\,t_3{}^{j]}\,\bar{t}_{3,k]}\,t_1{}^k\,(t_2\cdot\bar{t}_3)^{q_2}(t_3\cdot\bar{t}_2)^{r_2}\,, & q_3=r_2+1\,,\ r_3=q_2+1\,.\end{cases}\end{aligned} \tag{10.9}$$

With the restriction to $q_1=r_1=1$ only $J^{(0,0)}_{\alpha\beta}$ contributes so that in (10.8)

$$K_{1,23} = \kappa\,(t_2\cdot\bar{t}_3)^{q_2}(t_3\cdot\bar{t}_2)^{r_2}\,,\qquad q_3=r_2\,,\ r_3=q_2\,, \tag{10.10}$$

and from (9.44)

$$\bar{\mathcal{D}}^{ij}_{J_1,m}K_{1,23} = 2\kappa\,\delta_m{}^{[i}\,t_1{}^{j]}\,(t_2\cdot\bar{t}_3)^{q_2}(t_3\cdot\bar{t}_2)^{r_2}\,. \tag{10.11}$$

Finally writing

$$\begin{aligned}P_{12,m} = \big(&\langle\psi^{(1,0)}_m\chi^{(q_2+1,r_2-1)}\varphi^{(q_3,r_3)}\rangle\ \ \langle\psi^{(1,0)}_m\bar{\chi}^{(q_2-1,r_2+1)}\varphi^{(q_3,r_3)}\rangle\\ &\langle\psi^{(1,0)}_m\bar{\psi}^{(q_2-1,r_2)n}\varphi^{(q_3,r_3)}\rangle\big)\,,\end{aligned} \tag{10.12}$$

then

$$\begin{aligned}P_{12,m}\overleftarrow{\bar{\mathcal{D}}}{}^{ij}_{\psi_2} = {}&\tfrac{1}{2}\varepsilon^{ijkl}\bar{t}_{2,k}\partial_{2,l}\,\langle\psi^{(1,0)}_m\chi^{(q_2+1,r_2-1)}\varphi^{(q_3,r_3)}\rangle\\ &+t_2{}^{[i}\bar{\partial}_2{}^{j]}\langle\psi^{(1,0)}_m\bar{\chi}^{(q_2-1,r_2+1)}\varphi^{(q_3,r_3)}\rangle + t_2{}^{[i}\langle\psi^{(1,0)}_m\bar{\psi}^{(q_2-1,r_2)j]}\varphi^{(q_3,r_3)}\rangle\,.\end{aligned} \tag{10.13}$$

Requiring $t_2{}^m P_{12,m}=0$, the non zero contributions in (10.12) arise for

$$\begin{aligned}\langle\psi^{(1,0)}_m\chi^{(q_2+1,r_2-1)}\varphi^{(r_2,q_2)}\rangle &= \alpha\,\tfrac{1}{2}\varepsilon_{mnkl}\,t_1{}^n t_2{}^k t_3{}^l\,(t_2\cdot\bar{t}_3)^{q_2}(t_3\cdot\bar{t}_2)^{r_2-1}\,,\\ \langle\psi^{(1,0)}_m\bar{\chi}^{(q_2-1,r_2+1)}\varphi^{(r_2,q_2)}\rangle &= \tilde{\alpha}\,\bar{t}_{2,[m}\bar{t}_{3,k]}\,t_1{}^k\,(t_2\cdot\bar{t}_3)^{q_2-1}(t_3\cdot\bar{t}_2)^{r_2}\,,\end{aligned} \tag{10.14}$$

and

$$\begin{aligned}\langle\psi^{(1,0)}_m\bar{\psi}^{(q_2-1,r_2)n}\varphi^{(r_2-1,q_2-1)}\rangle &= \gamma\Big(\delta_{[m}{}^n\,\bar{t}_{2,k]}\,t_1{}^k\,(t_2\cdot\bar{t}_3)^{q_2-1}(t_3\cdot\bar{t}_2)^{r_2-1}\\ &\qquad -\frac{q_2-1}{q_2+1}\,t_2{}^n\,\bar{t}_{3,[m}\,\bar{t}_{2,k]}\,t_1{}^k\,(t_2\cdot\bar{t}_3)^{q_2-2}(t_3\cdot\bar{t}_2)^{r_2-1}\Big)\,,\\ \langle\psi^{(1,0)}_m\bar{\psi}^{(q_2-1,r_2)n}\varphi^{(r_2,q_2)}\rangle &= \gamma\,\delta_{[m}{}^{[n}\,\bar{t}_{3,k]}\,t_3{}^{l]}\,\bar{t}_{2,l}\,t_1{}^k\,(t_2\cdot\bar{t}_3)^{q_2-1}(t_3\cdot\bar{t}_2)^{r_2-1}\,.\end{aligned} \tag{10.15}$$

satisfying also $\bar t_{2,n}\langle\psi_m^{(1,0)}\bar\psi^{(q_2-1,r_2)n}\varphi^{(q_3,r_3)}\rangle = \partial_{2,n}\langle\psi_m^{(1,0)}\bar\psi^{(q_2-1,r_2)n}\varphi^{(q_3,r_3)}\rangle = 0$. In (10.13) with the explicit expressions in (10.14)

$$\begin{aligned}
&\tfrac12\varepsilon^{ijkl}\bar t_{2,k}\partial_{2,l}\,\langle\psi_m^{(1,0)}\chi^{(q_2+1,r_2-1)}\varphi^{(r_2,q_2)}\rangle\\
&= \alpha q_2\big(t_2{}^{[i}t_3{}^{j]}\,\bar t_{2,[m}\bar t_{3,k]}\,t_1{}^k\,(t_2\cdot\bar t_3)^{q_2-1}(t_3\cdot\bar t_2)^{r_2-1} + \delta_{[m}{}^{[i}t_2{}^{j]}\,\bar t_{3,k]}\,t_1{}^k\,(t_2\cdot\bar t_3)^{q_2-1}(t_3\cdot\bar t_2)^{r_2}\big)\\
&\quad + \alpha(q_2+1)\big(\delta_{[m}{}^{[i}\,t_3{}^{j]}\,\bar t_{2,k]}\,t_1{}^k\,(t_2\cdot\bar t_3)^{q_2}(t_3\cdot\bar t_2)^{r_2-1} - \tfrac12\,\delta_m{}^{[i}\,t_1{}^{j]}\,(t_2\cdot\bar t_3)^{q_2}(t_3\cdot\bar t_2)^{r_2}\big)\,,\\
&t_2{}^{[i}\bar\partial_2{}^{j]}\langle\psi_m^{(1,0)}\bar\chi^{(q_2-1,r_2+1)}\varphi^{(r_2,q_2)}\rangle\\
&= \tilde\alpha\big(q_3\,t_2{}^{[i}t_3{}^{j]}\,\bar t_{2,[m}\bar t_{3,k]}\,t_1{}^k\,(t_2\cdot\bar t_3)^{q_2-1}(t_3\cdot\bar t_2)^{r_2-1} - \delta_{[m}{}^{[i}t_2{}^{j]}\,\bar t_{3,k]}\,t_1{}^k\,(t_2\cdot\bar t_3)^{q_2-1}(t_3\cdot\bar t_2)^{r_2}\big)\,.
\end{aligned}\tag{10.16}$$

The superconformal identity (10.8) is trivial when $r_3 = q_2+1$ in that both sides are zero. When $r_3 = q_2-1$ the identity just gives $8c=\gamma$. For $r_3=q_2$ we get

$$(q_2+1)\alpha = 4c\,,\quad (q_3+1)\tilde\alpha = -4\tilde c\,,\quad \gamma = \frac{8q_2\,c}{q_2+1} - \frac{8q_3\,\tilde c}{q_3+1}\,,\quad 2\kappa = \tilde c - c\,. \tag{10.17}$$

To extend the discussion to four point functions we consider the simplest case where $q_r = r_r = 1$, so that $\Delta_r = 1$, for each $r=1,2,3,4$. The invariant function F appearing in (5.2) is conveniently written in the form

$$\begin{aligned}
&F(u,v;t,\bar t) = \sum_{r,s=2,3,4} f_{rs}(u,v)\,t_1\cdot\bar t_r\,t_s\cdot\bar t_1\,F_{rs}(t,\bar t)\,,\\
&F_{22} = t_3\cdot\bar t_4\,t_4\cdot\bar t_3\,,\ F_{33} = t_2\cdot\bar t_4\,t_4\cdot\bar t_2\,,\ F_{23} = t_2\cdot\bar t_4\,t_4\cdot\bar t_3\,,\ F_{32} = t_3\cdot\bar t_4\,t_4\cdot\bar t_3\,,\ \text{etc.}\,.
\end{aligned}\tag{10.18}$$

In a similar fashion to (10.12) $T_{12}(x,\bar x;t,\bar t)$, given by (5.20), is expressed as

$$T_{12,m} = \big(\langle\psi_m^{(1,0)}\chi^{(2,0)}\varphi^{(1,1)}\varphi^{(1,1)}\rangle\ \ \langle\psi_m^{(1,0)}\bar\chi^{(0,2)}\varphi^{(1,1)}\varphi^{(1,1)}\rangle\ \ \langle\psi_m^{(1,0)}\bar\psi^{(0,1))n}\varphi^{(1,1)}\varphi^{(1,1)}\rangle\big)\,, \tag{10.19}$$

where

$$\begin{aligned}
&\langle\psi_m^{(1,0)}\bar\chi^{(0,2)}\varphi^{(1,1)}\varphi^{(1,1)}\rangle = \sum_{2\le r<s\le 4} a_{2,rs}\,\bar t_{r,[m}\bar t_{s,n]}t_1{}^n\,A_{2,rs}\,,\\
&A_{2,34} = t_4\cdot\bar t_2\,t_3\cdot\bar t_2\,,\quad A_{2,24} = t_4\cdot\bar t_3\,t_3\cdot\bar t_2\,,\quad A_{2,23} = t_3\cdot\bar t_4\,t_4\cdot\bar t_2\,,
\end{aligned}\tag{10.20}$$

and

$$\begin{aligned}
&\langle\psi_m^{(1,0)}\chi^{(2,0)}\varphi^{(1,1)}\varphi^{(1,1)}\rangle = \tfrac12\sum_{2\le r<s\le 4} b_{2,rs}\,\varepsilon_{mnkl}t_r{}^k t_s{}^l\,t_1{}^n\,B_{2,rs}\,,\\
&B_{2,34} = t_2\cdot\bar t_3\,t_2\cdot\bar t_4\,,\quad B_{2,24} = t_3\cdot\bar t_4\,t_2\cdot\bar t_3\,,\quad B_{2,23} = t_4\cdot\bar t_3\,t_2\cdot\bar t_4\,,
\end{aligned}\tag{10.21}$$

and

$$\langle \psi_m^{(1,0)} \bar\psi^{(0,1)n} \varphi^{(1,1)} \varphi^{(1,1)} \rangle = 2d_2\, \delta_{[m}{}^{[n} \bar t_{3,k]} t_4{}^{l]}\, t_1{}^k \bar t_{2,l}\, t_4 \cdot \bar t_3 + 2\tilde d_2\, \delta_{[m}{}^{[n} \bar t_{4,k]} t_3{}^{l]}\, t_1{}^k \bar t_{2,l}\, t_3 \cdot \bar t_4 \\ + c_2\, \delta_{[m}{}^{n} \bar t_{2,k]}\, t_1{}^k\, t_3 \cdot \bar t_4\, t_4 \cdot \bar t_3 + 2e_2\, \bar t_{3,[m} \bar t_{4,k]}\, t_3{}^{[n} t_4{}^{l]}\, t_1{}^k \bar t_{2,l}\,. \tag{10.22}$$

The remaining contribution to the Ward identities arises from $K_1(x,\bar x; t, \bar t)$ which is here written as

$$K_1 = k_1\, t_3 \cdot \bar t_2\, t_2 \cdot \bar t_4\, t_4 \cdot \bar t_3 + \tilde k_1\, t_2 \cdot \bar t_3\, t_3 \cdot \bar t_4\, t_4 \cdot \bar t_2\,. \tag{10.23}$$

For expansion of the Ward identities it is convenient to define the basis

$$(D_{rs})_m^{ij} = \delta_{[m}{}^{[i} \bar t_{r,n]} t_s{}^{j]}\, t_1{}^n\, F_{rs}\,, \quad (E_{rsu})_m^{ij} = t_r{}^{[i} t_s{}^{j]}\, \bar t_{r,[m} \bar t_{u,n]} t_1{}^n\, t_u \cdot \bar t_s\,, \quad r,s,u = 2,3,4\,, \\ U_m^{ij} = \delta_{[m}{}^{[i} \delta_{n]}{}^{j]}\, t_1{}^n\, t_3 \cdot \bar t_2\, t_2 \cdot \bar t_4\, t_4 \cdot \bar t_3\,, \quad \tilde U_m^{ij} = \delta_{[m}{}^{[i} \delta_{n]}{}^{j]}\, t_1{}^n\, t_2 \cdot \bar t_3\, t_3 \cdot \bar t_4\, t_4 \cdot \bar t_2\,. \tag{10.24}$$

These are not independent since[4]

$$\sum_{r\neq s\neq u} E_{rsu} + \sum_{r\neq s} D_{rs} - \sum_r D_{rr} - \tfrac12 (U + \tilde U) = 0\,. \tag{10.25}$$

In terms of the basis in (10.24) then

$$\bar{\mathcal D}_{\varphi_1} F = -4 \sum_{r,s} f_{rs}\, D_{rs} + f\, U + \tilde f\, \tilde U\,, \quad f = f_{23} + f_{34} + f_{42}\,,\ \tilde f = f_{32} + f_{43} + f_{24}\,, \tag{10.26}$$

and

$$\bar{\mathcal D}_{J_1} K = 2k_1\, U + 2\tilde k_1\, \tilde U\,. \tag{10.27}$$

For the fermion contributions arising from (10.20), (10.21) and (10.22)

$$\begin{aligned} &\bar{\mathcal D}_{\bar\chi_2} \langle \psi_m^{(1,0)} \bar\chi^{(0,2)} \varphi^{(1,1)} \varphi^{(1,1)} \rangle \\ &\quad = a_{2,23}(-D_{32} + E_{243}) + a_{2,24}(-D_{42} + E_{234}) + a_{2,34}(E_{423} - E_{234})\,, \\ &\bar{\mathcal D}_{\chi_2} \langle \psi_m^{(1,0)} \chi^{(2,0)} \varphi^{(1,1)} \varphi^{(1,1)} \rangle \\ &\quad = b_{2,23}(2D_{23} + D_{42} - D_{22} + E_{234} - U) + b_{2,24}(2D_{24} + D_{32} - D_{22} + E_{243} - \tilde U) \\ &\qquad + b_{2,34}(-D_{23} + D_{24} - D_{34} + D_{43} + D_{33} - D_{44} - E_{342} + E_{432} + \tfrac12 (U - \tilde U))\,, \\ &\bar{\mathcal D}_{\psi_2,n} \langle \psi_m^{(1,0)} \bar\psi^{(0,1)n} \varphi^{(1,1)} \varphi^{(1,1)} \rangle \\ &\quad = -\, c_2\, D_{22} - d_2(D_{32} + E_{243}) - \tilde d_2(D_{42} + E_{234}) - e_2(E_{324} + E_{423})\,. \end{aligned} \tag{10.28}$$

[4] Define $X^{ijk} = \varepsilon_{rsu}\, t_r{}^i t_s{}^j t_u{}^k$, for $r,s,u = 2,3,4$ and $\varepsilon_{234} = 1$, and $\bar X_{ijk} = \varepsilon_{rsu}\, \bar t_{r,i} \bar t_{s,j} \bar t_{u,k}$. Then $0 = 10\, \delta_m{}^{[i} \delta_n{}^j\, X^{klp]}\, \bar X_{klp} = \delta_m{}^{[i} \delta_n{}^{j]}\, X^{klp}\, \bar X_{klp} - 6\, \delta_{[m}{}^{[i}\, X^{j]kl}\, \bar X_{n]kl} + 3\, X^{ijk} \bar X_{mnk}$ where $X^{ijk} \bar X_{mnk} t_1{}^n = -4 \sum_{r\neq s\neq t} (E_{rsu})_m^{ij}$, $\delta_{[m}{}^{[i}\, X^{j]kl} \bar X_{n]kl} t_1{}^n = -2 \sum_r (D_{rr})_m^{ij} + 2 \sum_{r\neq s} (D_{rs})_m^{ij}$ and $\delta_m{}^{[i} \delta_n{}^{j]}\, X^{klp}\, \bar X_{klp}\, t_1{}^n = 6(U + \tilde U)_m^{ij}$.

The sum of the three contributions in (10.28) then corresponds to $T_{12}\overleftarrow{\overline{\mathcal{D}}}_{\psi_2}$ for T_{12} as in (10.19). The results for $T_{13}\overleftarrow{\overline{\mathcal{D}}}_{\psi_3}$ and $T_{14}\overleftarrow{\overline{\mathcal{D}}}_{\psi_4}$ are obtained by cyclic permutation of 2,3,4.

These results may then be applied to the two four point Ward identity equations (5.21) and (5.23). For each T_{1r}, $r = 2, 3, 4$, there are 10 independent coefficients and each Ward identity has 17 terms given by the basis (10.24). To handle the constraint (10.25) in each identity it is natural to introduce an associated Lagrange multiplier. The identity (5.23) then can be taken as determining T_{12} with 6 additional relations involving the coefficients in T_{13} and T_{14}. The 17 additional equations, depending on a Lagrange multiplier, arising from (5.23) for T_{13} and T_{14} form in fact a linearly dependent set and so there is a necessary constraint on the f_{rs} for a solution. To write this in a succinct form it is convenient to define

$$\begin{aligned} a &= \frac{1}{u}\big(f_{22} + \tfrac{1}{2}(f_{23} + f_{32} + f_{42} + f_{24} - f_{34} - f_{43})\big)\,, \\ b &= \frac{1}{u}\big(f_{33} + \tfrac{1}{2}(f_{34} + f_{43} + f_{23} + f_{32} - f_{24} - f_{42})\big)\,, \\ c &= \frac{1}{u}\big(f_{44} + \tfrac{1}{2}(f_{42} + f_{24} + f_{34} + f_{43} - f_{23} - f_{32})\big)\,, \end{aligned} \tag{10.29}$$

and then we must require

$$x\frac{\partial}{\partial x}(x\,a) + \frac{\partial}{\partial x}b + (1-x)\frac{\partial}{\partial x}\big((1-x)\,c\big) = 0\,. \tag{10.30}$$

Of course there is also the conjugate equation obtained for $x \to \bar{x}$. The Ward identities further give

$$\frac{1}{2u}(k_1 - \tilde{k}_1) = \frac{\partial}{\partial x}\big(x\,a - (1-x)\,c\big) \tag{10.31}$$

but $k_1 + k_2$ is undetermined.

Appendix A. Superconformal Group in Three Dimensions

The three dimensional conformal generators $P_{\alpha\beta}, M_\alpha{}^\beta, H, K^{\alpha\beta}$ can be assembled as a 4×4 matrix

$$\mathcal{M}_A{}^B = \begin{pmatrix} M_\alpha{}^\beta + \delta_\alpha{}^\beta H & P_{\alpha\beta'} \\ K^{\alpha'\beta} & -M_{\beta'}{}^{\alpha'} - \delta_{\beta'}{}^{\alpha'} H \end{pmatrix}, \qquad A = (\alpha, \alpha')\,,\ B = (\beta, \beta')\,, \tag{A.1}$$

which satisfies

$$J^{BC}\mathcal{M}_C{}^D J_{DA} = \mathcal{M}_A{}^B\,, \qquad J^{AB} = \begin{pmatrix} 0 & \delta_{\beta'}{}^\alpha \\ -\delta_{\alpha'}{}^\beta & 0 \end{pmatrix}, \quad J_{AB} = \begin{pmatrix} 0 & \delta_\alpha{}^{\beta'} \\ -\delta_\beta{}^{\alpha'} & 0 \end{pmatrix}. \tag{A.2}$$

The commutation relations (1.13) are equivalent to

$$\big[\mathcal{M}_A{}^B, \mathcal{M}_C{}^D\big] = \delta_C{}^B \mathcal{M}_A{}^D - \delta_A{}^D \mathcal{M}_C{}^B - J_{AC}\, J^{BE}\mathcal{M}_E{}^D - J^{BD}\,\mathcal{M}_A{}^E J_{EC}\,, \tag{A.3}$$

which is just the Lie algebra for $Sp(4)$.

Defining

$$\mathcal{Q}_A = \begin{pmatrix} Q_\alpha \\ -\bar{S}^{\alpha'} \end{pmatrix}, \qquad \bar{\mathcal{Q}}^B = \begin{pmatrix} S^\beta & \bar{Q}_{\beta'} \end{pmatrix} = (\mathcal{Q}_A)^T J^{AB}\,, \tag{A.4}$$

then (1.42) and (1.43) are equivalent to

$$\big\{\mathcal{Q}_A, \bar{\mathcal{Q}}^B\big\} = 2\,\mathcal{M}_A{}^B\,\mathbb{I} - 2\,\delta_A{}^B\,\rho_I R_I\,. \tag{A.5}$$

Furthermore (1.45) becomes

$$\big[\mathcal{M}_A{}^B, \mathcal{Q}_C\big] = \delta_C{}^B\,\mathcal{Q}_A - J_{AC} J^{BD}\,\mathcal{Q}_D\,. \tag{A.6}$$

With R_I a generator for $SO(\mathcal{N})$ the full three dimensional superconformal algebra becomes $OSp(\mathcal{N}|4)$.

Appendix B. Action of Derivatives

Here we specify more precisely derivatives with respect to d-vectors t^i and $\bar{t}_i$ which are compatible with the condition $t^i\bar{t}_i = 0$. For the case of interest in the text $d = 4$. For a collection of arbitrary vectors $a_{m,i}$ and $\bar{a}_n{}^j$ a basis of homogeneous functions is given by

$$T^{(q,r)}(t,\bar{t}) = \textstyle\prod_m (a_m\cdot t)^{q_m}\,\prod_n(\bar{a}_n\cdot\bar{t})^{r_n}\,. \tag{B.1}$$

where

$$q = \textstyle\sum_m q_m\,, \qquad r = \sum_n r_n\,. \tag{B.2}$$

Acting on this basis derivatives are defined by

$$
\begin{aligned}
&\partial_i\Big(\textstyle\prod_m (a_m\cdot t)^{q_m}\,\prod_n(\bar a_n\cdot\bar t)^{r_n}\Big)\\
&=\sum_p a_{p,i}\,q_p\,(a_p\cdot t)^{q_p-1}\textstyle\prod_{m\neq p}(a_m\cdot t)^{q_m}\,\prod_n(\bar a_n\cdot\bar t)^{r_n}\\
&\quad-\bar t_i\,\frac{1}{q+r+d-2}\sum_{p,q} q_p r_q\,a_p\cdot\bar a_q\,(a_p\cdot t)^{q_p-1}(\bar a_q\cdot\bar t)^{r_q-1}\textstyle\prod_{m\neq p}(a_m\cdot t)^{q_m}\,\prod_{n\neq q}(\bar a_n\cdot\bar t)^{r_n}\,,\\
&\bar\partial^i\Big(\textstyle\prod_m (a_m\cdot t)^{q_m}\,\prod_n(\bar a_n\cdot\bar t)^{r_n}\Big)\\
&=\sum_q \bar a_q{}^i\,r_q\,(\bar a_q\cdot t)^{r_q-1}\textstyle\prod_{m}(a_m\cdot t)^{q_m}\,\prod_{n\neq q}(\bar a_n\cdot\bar t)^{r_n}\\
&\quad-t^i\,\frac{1}{q+r+d-2}\sum_{p,q} q_p r_q\,a_p\cdot\bar a_q\,(a_p\cdot t)^{q_p-1}(\bar a_q\cdot\bar t)^{r_q-1}\textstyle\prod_{m\neq p}(a_m\cdot t)^{q_m}\,\prod_{n\neq q}(\bar a_n\cdot\bar t)^{r_n}\,.
\end{aligned}
\tag{B.3}
$$

It is straightforward to verify that these definitions imply (9.12). In (B.3) the second term in the results for ∂_i or $\bar\partial^i$ is necessary to account for $t^i\bar t_i=0$, it is of course absent when $a_p\cdot\bar a_q=0$. For $\bar t_{[i}\partial_{j]}$ and $t^{[i}\bar\partial^{\,j]}$ only the first term in (B.3) contributes which is the result expected naively.

With the definitions (B.3) we have

$$
\begin{aligned}
\big[\partial_i,\bar t_j\big]T^{(q,r)}(t,\bar t) &= -\frac{1}{q+r+d-1}\,\bar t_i\partial_j T^{(q,r)}(t,\bar t)\,,\\
\big[\partial_i,t^j\big]T^{(q,r)}(t,\bar t) &= \bigg(\delta_i^j-\frac{1}{q+r+d-1}\,\bar t_i\bar\partial^j\bigg)T^{(q,r)}(t,\bar t)\,.
\end{aligned}
\tag{B.4}
$$

together with corresponding results for $t\leftrightarrow\bar t$, $\partial\leftrightarrow\bar\partial$. Since $t^i\partial_i T^{(q,r)}(t,\bar t)=q\,T^{(q,r)}(t,\bar t)$ and $\bar t_i\bar\partial^i T^{(q,r)}(t,\bar t)=r\,T^{(q,r)}(t,\bar t)$ as a consequence of (B.4),

$$
\begin{aligned}
\partial_i\big(t^i T^{(q,r)}(t,\bar t)\big) &= \frac{(q+d-1)(q+r+d)}{q+r+d-1}\,T^{(q,r)}(t,\bar t)\,,\\
\bar\partial^i\big(\bar t_i T^{(q,r)}(t,\bar t)\big) &= \frac{(r+d-1)(q+r+d)}{q+r+d-1}\,T^{(q,r)}(t,\bar t)\,.
\end{aligned}
\tag{B.5}
$$

Using (B.4) with (9.12) we have

$$
\bar\partial^j\,t^{[k}\partial_j\bar\partial^{\,l]}T^{(q,r)}(t,\bar t)=0\,,\qquad \partial_j\,\bar t_{[k}\bar\partial^j\partial_{l]}T^{(q,r)}(t,\bar t)=0\,,
\tag{B.6}
$$

which is used in obtaining (9.20).

Further results that are relevant in (9.37), when $d = 4$, are

$$\begin{aligned}
\bar\partial^j \big(\bar t_j t^{[k}\bar\partial^{l]} T^{(q-1,r-1)}(t,\bar t)\big) &= (r+1)\Big(1+\frac{1}{q+r+1}\Big)\, t^{[k}\bar\partial^{l]} T^{(q-1,r-1)}(t,\bar t)\,,\\
\bar\partial^j \big(\delta_j{}^{[k} t^{l]} T^{(q-1,r-1)}(t,\bar t)\big) &= -\Big(1+\frac{1}{q+r+1}\Big)\, t^{[k}\bar\partial^{l]} T^{(q-1,r-1)}(t,\bar t)\,,\\
\bar\partial^j\, t^{[k}\bar\partial^{l]} T_j^{(q-1,r)}(t,\bar t) &= t^{[k}\bar\partial^{l]}\, \bar\partial^j T_j^{(q-1,r)}(t,\bar t)\,,\\
t^j\, t^{[k}\bar\partial^{l]} T_j^{(q-1,r)}(t,\bar t) &= t^{[k}\bar\partial^{l]} \big(t^j T_j^{(q-1,r)}(t,\bar t)\big)\,,\\
t^j\, \bar t_{[k}\partial_{l]} T_j^{(q+1,r-2)}(t,\bar t) &= \bar t_{[k}\partial_{l]} \big(t^j T_j^{(q+1,r-2)}(t,\bar t)\big) - \bar t_{[k} T_{l]}{}^{(q+1,r-2)}(t,\bar t)\,,\\
\bar\partial^j \big(\bar t_{[k}\partial_{l]} T_j^{(q+1,r-2)}(t,\bar t)\big) &= \bar t_{[k}\partial_{l]} \big(\bar\partial^j T_j^{(q+1,r-2)}(t,\bar t)\big) - \partial_{[k} T_{l]}{}^{(q+1,r-2)}(t,\bar t)\,,\\
\bar\partial^j \big(\bar t_j \partial_{[k} T_{l]}{}^{(q+1,r-2)}(t,\bar t)\big) &= (r+1)\Big(1+\frac{1}{q+r+1}\Big)\, \partial_{[k} T_{l]}{}^{(q+1,r-2)}(t,\bar t)\,,\\
\bar\partial^j \big(\bar t_{[k}\partial_j T_{l]}{}^{(q+1,r-2)}(t,\bar t)\big) &= \Big(1-\frac{q}{q+r+1}\Big)\, \partial_{[k} T_{l]}{}^{(q+1,r-2)}(t,\bar t)\,.
\end{aligned} \tag{B.7}$$

Generators for $SU(d)$ acting on these homogeneous tensors are given by

$$L_i{}^j T^{(q,r)}(t,\bar t) = -\Big(t^j\partial_i - \bar t_i\bar\partial^j - \frac{1}{d}(q-r)\,\delta_i{}^j\Big) T^{(q,r)}(t,\bar t)\,, \tag{B.8}$$

which reduces to (9.33) for $d = 4$. The commutation relations are

$$\big[L_i{}^j, L_k{}^l\big] = \delta_k{}^j L_i{}^l - \delta_i{}^l L_k{}^j\,, \tag{B.9}$$

and for the Casimir operator

$$L_j{}^i L_i{}^j T^{(q,r)}(t,\bar t) = \Big(q(q+d-1) + r(r+d-1) - \frac{1}{d}(q-r)^2\Big) T^{(q,r)}(t,\bar t)\,. \tag{B.10}$$

For $d = 4$ this agrees with (2.7) if $r_2 = 0$.

As an illustration of these results we may construct a vector $T_i^{(q+1,r)}(t,\bar t)$ satisfying (9.14) for $p = 1$,

$$T_i^{(q+1,r)}(t,\bar t) = V_{ij}^{(q,r)}(t,\bar t)\, t^j - \frac{1}{r+d-2}\,\bar t_i\, t^j \bar\partial^k V_{kj}^{(q,r)}(t,\bar t)\,, \quad V_{ij}^{(p,q)} = -V_{ji}^{(p,q)}\,. \tag{B.11}$$

This trivially satisfies $t^i T_i^{(q+1,r)}(t,\bar t) = 0$ while $\bar\partial^i T_i^{(q+1,r)}(t,\bar t) = 0$ follows from (B.4) and (B.5) for any $V_{ij}^{(q,r)}(t,\bar t)$. For $T^{(q,r)}(t,\bar t)$ given by (B.1) then a natural choice is to take $V_{ij}^{(q,r)}(t,\bar t) = a_{p,[i} a_{q,j]}\, T^{(q,r)}(t,\bar t)$ for any $p < q$. If $V_{ij}^{(q,r)}(t,\bar t) = a_{p,[i}\bar t_{j]}\, T^{(q,r-1)}(t,\bar t)$ then $T_i^{(q+1,r)}(t,\bar t) = 0$. There is a similar construction for the conjugate vector

$$T^{(q,r+1)i}(t,\bar t) = \bar V^{(q,r)ij}(t,\bar t)\,\bar t_j - \frac{1}{q+d-2}\, t^i\, \bar t_j \partial_k \bar V^{(q,r)kj}(t,\bar t)\,. \tag{B.12}$$

For $d=4$ (B.12) and (B.11) are related as in (9.15) if

$$\bar{V}^{(q,r)ij}(t,\bar{t}) = -\tfrac{1}{2}\varepsilon^{ijkl}V_{kl}^{(q,r)}(t,\bar{t})\,. \tag{B.13}$$

Appendix C. Four Dimensional Superconformal Ward Identities

In many respects the analysis of superconformal Ward identities in three dimensions is similar to that in four dimensions. Here we outline a simplified discussion of previous results obtained earlier for $\mathcal{N}=2$ and $\mathcal{N}=4$ four dimensional theories.

For $\mathcal{N}=2$, with $SU(2)$ R-symmetry, there are supercharges $Q_{\alpha i}$ and $\bar{Q}_{\dot\alpha}{}^i$ and associated superconformal Killing spinors $\hat\epsilon^\alpha{}_i(x) = \epsilon^\alpha{}_i - i\,\bar\eta_{\dot\alpha i}\tilde{\mathrm{x}}^{\dot\alpha\alpha}$, $\hat{\bar\epsilon}^{\dot\alpha i}(x) = \bar\epsilon^{\dot\alpha i} + i\,\tilde{\mathrm{x}}^{\dot\alpha\alpha}\eta_\alpha{}^i$, for $i=1,2$. As in section 6, in (7.1), we introduce auxiliary spinor variables t^i, and then the relevant superconformal transformations of a general $\frac{1}{2}$-BPS superconformal primary $\varphi^{(n)}(t)$ can be expressed as

$$\begin{aligned}
\delta_{\hat\epsilon}\varphi^{(n)}(t) &= \hat\epsilon^\alpha{}_i\, t^i\, \psi_\alpha{}^{(n-1)}(t) + \bar\psi_{\dot\alpha}{}^{(n-1)}(t)\,\tilde{t}_i\,\hat{\bar\epsilon}^{\dot\alpha i}\,,\\
\delta_{\hat\epsilon}\psi_\alpha^{(n-1)}(t) &= \frac{1}{n}\frac{\partial}{\partial t^i}\, i\partial_{\alpha\dot\alpha}\varphi^{(n)}(t)\,\hat{\bar\epsilon}^{\dot\alpha i} + 2\,\frac{\partial}{\partial t^i}\varphi^{(n)}(t)\,\eta_\alpha{}^i + J_{\alpha\dot\alpha}^{(n-2)}(t)\,\tilde{t}_i\,\hat{\bar\epsilon}^{\dot\alpha i}\,.
\end{aligned} \tag{C.1}$$

with definitions as in (7.3) and $\Delta_\varphi = n$. Following a similar discussion to that in section 4, with analogous definitions, the superconformal Ward identies for a four point function for four $\frac{1}{2}$-BPS fields take the form, from the corresponding equations to (5.21) and (5.23),

$$\begin{aligned}
\frac{\partial}{\partial t_1{}^i}\,F &= T_2\,\tilde{t}_{2i} + T_3\,\tilde{t}_{3i} + T_4\,\tilde{t}_{4i}\,,\\
\frac{1}{n_1}\,\frac{\partial}{\partial t_1{}^i}\,\frac{\partial}{\partial x}F &= -\,K\,\tilde{t}_{1i} + \frac{1}{x}\,T_3\,\tilde{t}_{3i} + \frac{1}{x(1-x)}\,T_4\,\tilde{t}_{4i}\,,
\end{aligned} \tag{C.2}$$

with also an associated equation for $\frac{\partial}{\partial\bar x}F$. By contracting these with $t_2{}^i$ and $t_1{}^i$ respectively we may easily obtain

$$\bigg(x\frac{\partial}{\partial x} - \frac{t_1\tilde{t}_3}{t_2\tilde{t}_3}\, t_2{}^i\frac{\partial}{\partial t_1{}^i}\bigg)F = \bigg(\frac{1}{1-x} + \frac{\alpha}{1-\alpha}\bigg)\, t_1\tilde{t}_4\, T_4\,, \tag{C.3}$$

with α defined as in (8.8). Using also (8.9) we may write

$$\begin{aligned}
F(u,v,t_1,t_2,t_3,t_4) &= f_{n_1n_2n_3n_4}(t_1,t_2,t_3,t_4)\,\mathcal{F}(x,\bar x,\alpha)\,,\\
t_1\tilde{t}_4\, T_4(x,\bar x,t_1,t_2,t_3,t_4) &= f_{n_1n_2n_3n_4}(t_1,t_2,t_3,t_4)\,\mathcal{T}_4(x,\bar x,\alpha)\,,
\end{aligned} \tag{C.4}$$

where $\mathcal{F}(x,\bar x,\alpha) = \mathcal{F}(\bar x,x,\alpha)$. Then (C.3) requires

$$\bigg(x\frac{\partial}{\partial x} - \alpha\frac{\partial}{\partial\alpha}\bigg)\mathcal{F}(x,\bar x,\alpha) = \frac{1-\alpha x}{(1-x)(1-\alpha)}\,\mathcal{T}_4(x,\bar x,\alpha)\,. \tag{C.5}$$

There is also a corresponding conjugate equation with $x \leftrightarrow \bar{x}$. The non trivial content of (C.5) is the factor $1-\alpha x$ on the right hand side, with $\mathcal{T}_4(x,\bar{x},\alpha)$ also required to contain a factor $1-\alpha$. The solution of the supconformal identities for $\mathcal{F}(x,\bar{x},\alpha)$ is straightforward, since (C.5) and its conjugate imply

$$\mathcal{F}(x,\bar{x},1/x) = f(\bar{x}) \,, \qquad \mathcal{F}(x,\bar{x},1/\bar{x}) = f(x) \,, \tag{C.6}$$

where we impose symmetry under $x \leftrightarrow \bar{x}$. This ensures that $\mathcal{F}(x,\bar{x},\alpha)$ can be expressed in terms of the single variable function f up to terms which vanish as either $\alpha x = 1$ or $\alpha\bar{x} = 1$ giving,

$$\mathcal{F}(x,\bar{x},\alpha) = x\bar{x}\, \frac{(\alpha - 1/x)f(x) - (\alpha - 1/\bar{x})f(\bar{x})}{x-\bar{x}} + (\alpha x - 1)(\alpha\bar{x} - 1)\,\mathcal{H}(x,\bar{x},\alpha) \,, \tag{C.7}$$

with $\mathcal{H}(x,\bar{x},\alpha)$ a polynomial in α with degree reduced by two. If $e=1$, since $\mathcal{F}$ is then just linear in α, $\mathcal{H}=0$.

The discussion for $\mathcal{N}=4$ in four dimensions is similar but more intricate. The supercharges and superconformal spinors are just as in the $\mathcal{N}=2$ case but with indices $i=1,2,3,4$. In this case for $\frac{1}{2}$-BPS operators whose superconformal primaries belong to $[0,p,0]$ representations of the $SU(4)$ R-symmetry group it is natural to introduce auxiliary six vectors t_r which are null, $t^2=0$. The superconformal primary field is then expressible in terms of homogenous functions $\varphi^{(p)}(x,t)$ and $\Delta = p$. The superconformal transformations which extend (1.18) to $\mathcal{N}=4$ are then

$$\begin{aligned} \delta\varphi^{(p)}(x,t) ={}& -\hat{\epsilon}(x)\,\gamma\cdot t\,\psi^{(p-1)}(x,t) + \bar{\psi}^{(p-1)}(x,t)\,\bar{\gamma}\cdot t\,\hat{\bar{\epsilon}}(x)\,, \\ \delta\psi_\alpha{}^{(p-1)}(x,t) ={}& \frac{1}{p}\,\bar{\gamma}\cdot\frac{\partial}{\partial t}\, i\partial_{\alpha\dot{\alpha}}\varphi^{(p)}(x,t)\,\hat{\bar{\epsilon}}^{\dot{\alpha}}(x) + 2\,\bar{\gamma}\cdot\frac{\partial}{\partial t}\varphi^{(p)}(x,t)\,\eta_\alpha \\ & + \left(1 + \frac{1}{2p+2}\,\bar{\gamma}\cdot t\,\gamma\cdot\frac{\partial}{\partial t}\right) J_{\alpha\dot{\alpha}\,r}{}^{(p-1)}(x,t)\,\bar{\gamma}_r\,\hat{\bar{\epsilon}}^{\dot{\alpha}}(x)\,. \end{aligned} \tag{C.8}$$

The descendant fields $\psi_{\alpha i}{}^{(p-1)}(x,t)$, $\bar{\psi}_{\dot{\alpha}}{}^{i\,(p-1)}(x,t)$ and $J_{\alpha\dot{\alpha}\,r}{}^{(p-1)}(x,t)$ in (C.8) satisfy the constraints

$$\begin{aligned} \gamma\cdot\frac{\partial}{\partial t}\,\psi_\alpha{}^{(p-1)}(x,t) = 0\,, \qquad & \bar{\psi}_{\dot{\alpha}}{}^{(p-1)}(x,t)\,\bar{\gamma}\cdot\frac{\overleftarrow{\partial}}{\partial t} = 0\,, \\ t_r J_{\alpha\dot{\alpha}\,r}{}^{(p-1)}(x,t) = 0\,, \qquad & \frac{\partial}{\partial t_r} J_{\alpha\dot{\alpha}\,r}{}^{(p-1)}(x,t) = 0\,, \end{aligned} \tag{C.9}$$

which ensure they belong to the representation spaces for the $[0,p-1,1]$, $[1,p-1,0]$ and $[1,p-1,1]$ $SU(4)$ representations. In (C.8) and (C.9) γ_r, $\bar{\gamma}_r$, $r=1,2,\ldots 6$, are 4×4 $SO(6)$ gamma matrices satisfying

$$\gamma_r{}^{ij} = -\gamma_r{}^{ji}\,, \quad \bar{\gamma}_{r\,ij} = \tfrac{1}{2}\,\varepsilon_{ijkl}\gamma_r{}^{kl}\,, \qquad \gamma_r\bar{\gamma}_s + \gamma_s\bar{\gamma}_r = -2\,\delta_{rs}1\,. \tag{C.10}$$

In consequence γ_r forms a basis for 4×4 antisymmetric matrices, a basis for symmetric matrices is given by $\gamma_{[r}\bar\gamma_s\gamma_{t]} = \frac{1}{6} i\,\varepsilon_{rstuvw}\gamma_u\bar\gamma_v\gamma_w = (\gamma_{[r}\bar\gamma_s\gamma_{t]})^T$ Since t_r is a null vectors derivatives with respect to t, as in (C.8) and (C.9), require special care along the lines of appendix C.

For the four point function for four $\frac{1}{2}$-BPS fields the corresponding expression to (C.2) for the superconformal Ward identities becomes

$$\bar\gamma\cdot\frac{\partial}{\partial t_1}F = T_2\,\bar\gamma\cdot t_2 + T_3\,\bar\gamma\cdot t_3 + T_4\,\bar\gamma\cdot t_4\,, \tag{C.11a}$$

$$\begin{aligned}\frac{1}{p_1}\,\bar\gamma\cdot\frac{\partial}{\partial t_1}\,\frac{\partial}{\partial x}F = {} & -\Big(1+\frac{1}{2p_1+2}\,\bar\gamma\cdot t_1\,\gamma\cdot\frac{\partial}{\partial t_1}\Big)\bar\gamma\cdot K \\ & + \frac{1}{x}\,T_3\,\bar\gamma\cdot t_3 + \frac{1}{x(1-x)}\,T_4\,\bar\gamma\cdot t_4\,,\end{aligned} \tag{C.11b}$$

To analyse these equations we note that, using (C.10) and the rules for differentiation with respect to a null vector t, for any homogeneous $f^{(p-1)}(t)$,

$$2(p+2)\,f^{(p-1)}(t) = -\bar\gamma\cdot\frac{\partial}{\partial t}\big(\gamma\cdot t\,f^{(p-1)}(t)\big) - \frac{p+2}{p+1}\,\bar\gamma\cdot t\,\gamma\cdot\frac{\partial}{\partial t}\,f^{(p-1)}(t)\,. \tag{C.12}$$

Since $\bar\gamma\cdot\frac{\partial}{\partial t_1}T_n = 0$, $n=2,3,4$, then

$$T_n\,\bar\gamma\cdot t_n = -\frac{1}{2(p_1+2)}\,\gamma\cdot\frac{\partial}{\partial t_1}\big(\gamma\cdot t_1\,T_n\,\bar\gamma\cdot t_n\big)\,, \tag{C.13}$$

and this allows us to write

$$T_n\,\bar\gamma\cdot t_n = \hat T_n + \frac{1}{2(p_1+1)}\,\bar\gamma\cdot t_1\,\gamma\cdot\frac{\partial}{\partial t_1}\,\hat T_n\,, \quad n=2,3,4\,, \tag{C.14}$$

where

$$\begin{aligned}\hat T_2 &= V_2\,\bar\gamma\cdot t_2 + W_2\,\bar\gamma\cdot t_3\,\gamma\cdot t_4\,\bar\gamma\cdot t_2\,,\\ \hat T_3 &= V_3\,\bar\gamma\cdot t_3 + W_3\,\bar\gamma\cdot t_4\,\gamma\cdot t_2\,\bar\gamma\cdot t_3\,,\\ \hat T_4 &= V_4\,\bar\gamma\cdot t_4 + W_4\,\bar\gamma\cdot t_2\,\gamma\cdot t_3\,\bar\gamma\cdot t_4\,.\end{aligned} \tag{C.15}$$

There are also similar expressions for the other terms in (C.11a,b),

$$\begin{aligned}\bar\gamma\cdot\frac{\partial}{\partial t_1}\,F &= \hat F + \frac{1}{2(p_1+1)}\,\bar\gamma\cdot t_1\,\gamma\cdot\frac{\partial}{\partial t_1}\,\hat F\,,\\ \Big(1+\frac{1}{2p_1+2}\,\bar\gamma\cdot t_1\,\gamma\cdot\frac{\partial}{\partial t_1}\Big)\bar\gamma\cdot K &= \hat K + \frac{1}{2(p_1+1)}\,\bar\gamma\cdot t_1\,\gamma\cdot\frac{\partial}{\partial t_1}\hat K\,,\end{aligned} \tag{C.16}$$

where

$$\begin{aligned}\hat F &= F_2\,\bar\gamma\cdot t_2 + F_3\,\bar\gamma\cdot t_3 + F_4\,\bar\gamma\cdot t_4\,,\\ \hat K &= K_2\,\bar\gamma\cdot t_2 + K_3\,\bar\gamma\cdot t_3 + K_4\,\bar\gamma\cdot t_4\,.\end{aligned} \tag{C.17}$$

By considering the trace of (C.16) with $\gamma \cdot t_1$ we must have

$$p_1\, F = t_1\cdot t_2\, F_2 + t_1\cdot t_3\, F_3 + t_1\cdot t_4\, F_4 \, , \qquad t_1\cdot t_2\, K_2 + t_1\cdot t_3\, K_3 + t_1\cdot t_4\, K_4 = 0 \, , \quad \text{(C.18)}$$

since $t_{1r}K_r = 0$. By using (C.14) and (C.16) in (C.11a, b) the identities reduce to

$$\hat{F} = \hat{T}_2 + \hat{T}_3 + \hat{T}_4 \, , \qquad \text{(C.19}a\text{)}$$

$$\frac{1}{p_1}\frac{\partial}{\partial x}\hat{F} = -\hat{K} + \frac{1}{x}\hat{T}_3 + \frac{1}{x(1-x)}\hat{T}_4 \, . \qquad \text{(C.19}b\text{)}$$

Hence we must have, to ensure that the symmetric terms in (C.19a, b) cancel,

$$W_2 + W_3 + W_4 = 0 \, , \qquad W_3 + \frac{1}{1-x}\, W_4 = 0 \, . \qquad \text{(C.20)}$$

We then have from (C.19a)

$$F_2 = V_2 + 2\, t_3\cdot t_4\, W_3 \, , \quad F_3 = V_3 + 2\, t_2\cdot t_4\, W_4 \, , \quad F_4 = V_4 + 2\, t_2\cdot t_3\, W_2 \, , \qquad \text{(C.21)}$$

and from (C.19b), or (C.11b),

$$\begin{aligned} x\frac{\partial}{\partial x}F = {} & \big(t_1\cdot t_3\, V_3 - (t_1\cdot t_4\; t_2\cdot t_3 + t_1\cdot t_3\; t_2\cdot t_4 - t_1\cdot t_2\; t_3\cdot t_4)\, W_3\big) \\ & + \frac{1}{1-x}\big(t_1\cdot t_4\, V_4 - (t_1\cdot t_2\; t_3\cdot t_4 + t_1\cdot t_4\; t_2\cdot t_3 - t_1\cdot t_3\; t_2\cdot t_4)\, W_4\big) \, . \end{aligned} \qquad \text{(C.22)}$$

Writing now, analogous to (C.4),

$$F(u,v,t_1,t_2,t_3,t_4) = f_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\; \mathcal{F}(x,\bar{x},\sigma,\tau) \, , \qquad \text{(C.23)}$$

with the definitions

$$\begin{aligned} & f_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4) = (t_1\cdot t_2)^{p_1}\,(t_3\cdot t_4)^{e}\,(t_2\cdot t_3)^{p_3-e}\,(t_2\cdot t_4)^{p_4-e} \, , \\ & e = \tfrac{1}{2}(p_1+p_3+p_4-p_2) \, , \qquad \sigma = \frac{t_1\cdot t_3\; t_2\cdot t_4}{t_1\cdot t_2\; t_3\cdot t_4} \, , \qquad \tau = \frac{t_1\cdot t_4\; t_2\cdot t_3}{t_1\cdot t_2\; t_3\cdot t_4} \, . \end{aligned} \qquad \text{(C.24)}$$

Also if, for $n = 2,3,4$,

$$t_1\cdot t_n\; F_n = f_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\; \mathcal{F}_n \, , \qquad \text{(C.25)}$$

then (C.16) and (C.17) require

$$\mathcal{F}_3 = \sigma\frac{\partial}{\partial\sigma}\,\mathcal{F} \, , \qquad \mathcal{F}_4 = \tau\frac{\partial}{\partial\tau}\,\mathcal{F} \, , \qquad \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 = p_1\, \mathcal{F} \, . \qquad \text{(C.26)}$$

Defining $V_n \to \mathcal{V}_n$ in a similar fashion to (C.25) and also

$$t_1 \cdot t_2 \; t_3 \cdot t_4 \; W_n = f_{p_1 p_2 p_3 p_4}(t_1, t_2, t_3, t_4) \, \mathcal{W}_n \,, \tag{C.27}$$

then from (C.20), (C.21) and (C.22) we may obtain

$$\begin{aligned} &\sigma \frac{\partial}{\partial \sigma} \mathcal{F} = \mathcal{V}_3 + 2\sigma \, \mathcal{W}_4 \,, \qquad \tau \frac{\partial}{\partial \tau} \mathcal{F} = \mathcal{V}_4 - 2\tau \, (\mathcal{W}_3 + \mathcal{W}_4) \,, \qquad \mathcal{W}_3 + \frac{1}{1-x} \, \mathcal{W}_4 = 0 \,, \\ &x \frac{\partial}{\partial x} \mathcal{F} = \mathcal{V}_3 - (\sigma + \tau - 1) \, \mathcal{W}_3 + \frac{1}{1-x} \big(\mathcal{V}_4 - (1 + \tau - \sigma) \mathcal{W}_4 \big) \,. \end{aligned} \tag{C.28}$$

To analyse these equations we introduce new variables, in a similar fashion to (5.18),

$$\sigma = \alpha \bar{\alpha} \,, \qquad \tau = (1 - \alpha)(1 - \bar{\alpha}) \,, \tag{C.29}$$

so that

$$\alpha \frac{\partial}{\partial \alpha} = \sigma \, \frac{\partial}{\partial \sigma} - \frac{\alpha}{1 - \alpha} \, \tau \, \frac{\partial}{\partial \tau} \,. \tag{C.30}$$

Hence from (C.28) we may obtain

$$\Big(x \frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial \alpha} \Big) \mathcal{F} = \frac{1 - \alpha x}{1 - x} \, \Big(\frac{1}{1 - \alpha} \, \mathcal{V}_4 - 2 \, \mathcal{W}_4 \Big) \,, \tag{C.31}$$

together with corresponding equations for $\alpha \to \bar{\alpha}$ and also $x \to \bar{x}$. Apart from terms involving single variable functions the solution of (C.31) and associated equations requires $\mathcal{F}(x, \bar{x}, \sigma, \tau) = (\alpha x - 1)(\alpha \bar{x} - 1)(\bar{\alpha} x - 1)(\bar{\alpha} \bar{x} - 1) \, \mathcal{H}(x, \bar{x}, \alpha, \bar{\alpha})$ where $\mathcal{H}(x, \bar{x}, \alpha, \bar{\alpha})$ is a symmetric polynomial in $\alpha, \bar{\alpha}$ with degree reduced by four.