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Analogs for the c-Theorem for Four Dimensional Renormalisable Field Theories

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A perturbative analysis of products of composite operations on curved space is shown to give various consistency conditions which include a relation analogous to a generalisation of Zamolodchikov's c-theorem to four dimensional renormalisable field theories. A detailed BRS analysis is given for gauge theories to ensure independence of gauge fixing and calculations of the various new counterterms required by this analysis are undertaken to two loops. Although positivity of the metric on the space of couplings is not demonstrated in general it is shown to be valid for weak coupling. The change in the C function under renormalisation flow for gauge theories in the large N limit with appropriate numbers of fermions, so that there is a perturbatively accessible infra red stable fixed point, is evaluated. The analysis is extended to take account of the lower dimension operators occuring in scalar field theories and the weak coupling metric is calculated for quartic scalar field and Yukawa interactions. The form of the β function for the Yukawa coupling is shown to be constrained by the c-theorem demonstrated in this paper at two loops, in accord with previously calculated results.

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1. Introduction

Ever since 't Hooft showed the essential renormalisability of gauge field theories [1] there has been much detailed discussion of the formalism of renormalisation and also detailed calculations at two and more loops of the various renormalisation group β functions [2]. Nevertheless additional insight has recently been gained as a consequence of the intensive investigations of two dimensional conformal field theories. In particular Zamolodchikov's c-theorem [3] has provided an understanding of how conformal field theories under perturbation by some relevant operator may flow, as required by solving the renormalisation group, for increasing length scales towards some infra-red stable fixed point defining a new conformal field theory with reduced degrees of freedom. Zamolodchikov constructed a function of the couplings C which decreases monotonically under this flow and at the fixed points, where the β functions and hence the trace of the energy momentum tensor vanish, is equal to the Virasoro central charge of the associated conformal field theory. In essence perturbing a conformal field theory by superrenormalisable operators introduces masses into a previously massless theory and the infra-red stable fixed point then corresponds to the conformal field theory defined by the subset of the remaining massless fields (which therefore define the low energy effective field theory). Various calculations [4] have shown how conformal field theories are linked when they are perturbed by a single scalar operator $\mathcal{O}$ with coupling g and dimension $2-\epsilon$, $\epsilon \ll 1$, $S_I = g\int \mathcal{O}$, so that the associated β function $\beta(g) \approx -\epsilon g + bg^2$ has a perturbatively accessible fixed point $g_* \approx \epsilon/b$.

Cardy [5] has also suggested that the essential ideas of the c-theorem may be generalisable to four dimensional field theories although Zamolodchikov's proof is no longer applicable. Just as in two dimensions C may be related to terms in the trace of the energy momentum tensor on curved space. In two dimensions C is defined by the coefficient of the scalar curvature while in four dimensions it is natural to consider the corresponding coefficient of the topological Euler density. With an appropriate normalisation for free fields then

$$C_0 = 62n_V + 11n_F + n_S \ , \tag{1.1}$$

for n_V massless vectors, n_F Dirac fermions and n_S scalars [6](extensions to spin $\frac{3}{2}$ [7] and 2 [8] have also been calculated but these do not correspond to renormalisable field theories, nevertheless all contributions are positive). In four dimensions the possible conditions required to demonstrate that C is strictly decreasing under renormalisation flow are no longer simple to demonstrate [5,9]. Nevertheless Cardy [5] conjectured a potential application to QCD for a $SU(N)$ gauge group and f fermions in the fundamental representation with a single coupling of g. According to conventional wisdom, assuming confinement, the infra-red limit may be restricted to f^2-1 massless Goldstone bosons corresponding to the spontaneous breakdown of $SU(f)\times SU(f)$ chiral symmetry and if $C \to C_*$ then from (1.1) we may expect

$$C_0 = 62(N^2-1) + 11fN, \qquad C_* = f^2 - 1 \ . \tag{1.2}$$

So long as $11N - 2f > 0$, which is necessary for asymptotic freedom so that $g = 0$ is a UV stable fixed point, $C_0 > C_*$.

Although attractive this picture is essentially nonperturbative. Recently [10] a version of the c-theorem has been derived within the context of renormalisable field theories and the conventional perturbation expansion, both for the two dimensional and also the four dimensional cases. This depends on a careful analysis of the additional counterterms necessary to define products of composite operators and also allowing for a general curved space background. Related ideas were discussed some time ago for determining counterterms proportional to R^2 [11], where R is the scalar curvature, and were extended later to consideration of other possible counterterms [12]. The method described here in sections 2 and 3 ensures that we can obtain a complete analysis of all possible relations of this kind which include equations corresponding to the c-theorem.

Of course these results are restricted to perturbative calculation. In order to discuss a potential infra-red stable field point we consider briefly the situation where the number of fermion flavours f is such that $11N-2f$ although positive remains bounded for large N. In this case the two loop contribution to the β function is $\mathrm{O}(N^2)$ and there is a zero, $\beta(g_*)=0$, such that $g_*^2 N=\mathrm{O}(1/N)$ as $N\to\infty$. Usually the large N limit, $g^2N=\mathrm{O}(1)$, is taken with f fixed and, subject to confinement, has been argued to be phenomenologically realistic, with narrow meson resonances and baryons as solitons [13]. However it is also possible to take $f=\mathrm{O}(N)$ and the field theory still simplifies by the leading term being given by planar graphs although resonances are no longer narrow [14]. As a special case this limit realises the situation where there is a perturbatively accessible infra-red stable fixed point [15] and for which we are then able to calculate the change in the C function from $g=0$ to $g=g_*$. Although beyond the scope of this paper it is plausible that for $f=\mathrm{O}(N)$ confinement no longer applies, so that physical states are then not only $SU(N)$ singlets, and that therefore there is a phase transition as a function of f for large N.

In this paper in the next section we discuss the basic framework for renormalisation involving composite operators and their products with an arbitrary curved space background using dimensional regularisation. This extends previous work by one of us and is at the basis of section 3 where various consistency conditions are derived. These relate counterterms depending on the curvature tensor formed from the spatial metric to those necessary to define products of composite operators on flat space. In section 4 the general formalism is applied to gauge theories coupled to fermions. A careful BRS analysis is given to ensure that the additional contributions necessary in our discussion depend only on the usual gauge invariant coupling g rather than the gauge fixing parameters. Section 5 contains calculations to two loops for a gauge field theory coupled to fermions and the change in the C function in going to the perturbative infra-red stable fixed point described above is calculated. In section 6 the analysis is extended to scalar field theories where there are complications due to the presence of operators with dimensions less than four. The metric on the space of couplings is computed for both quartic scalar and also in section 7 for Yukawa interactions, when the leading term occurs at three and two loops respectively. In both cases, as well as for a gauge theory, this metric is positive definite for weak coupling. In section 7 the relative coefficients of the different terms appearing in the two loop β function for the Yukawa coupling is shown to be partially determined by integrability conditions for the variation of C. Further remarks and other implications of the perturbative c-theorem obtained here are contained in a conclusion.

2. Renormalisation and Composite Operators

In general the definition of finite composite operators in quantum field theories requires further analysis beyond the usual treatment of renormalisation [2]. However if all couplings are allowed to be x dependent in the original field theory then finite local operators can be immediately obtained by functional differentiation of the renormalised quantum action.

To describe the basic general framework we assume an initial Lagrangian $\mathcal{L}(\phi, g)$ for a set of fields ϕ and with dimensionless couplings g^i (additional couplings with positive mass dimension will be discussed in relation to specific models later). For a renormalisable theory then $\mathcal{L}_{\rm o} = \mathcal{L}(\phi_{\rm o}, g_{\rm o})$ is assumed to be such that for suitable cut off dependent $g_{\rm o}^i(g),\ \phi_{\rm o}(\phi, g)$ then this gives a finite perturbative quantum field theory as the cut off is removed.

Here we use dimensional regularisation so that $\mathcal{L}$ is extended to be defined on a d dimensional space with metric $\gamma_{\mu\nu}$ so that

$$\mathcal{L}(\phi', g') = \mu^{-\varepsilon}\mathcal{L}(\phi, g) \ , \quad g'^i = \mu^{k^i\varepsilon} g^i \ , \quad \phi' = \mu^{\delta\varepsilon}\phi \ , \quad \varepsilon = 4 - d \ . \tag{2.1}$$

With minimal subtraction then

$$g_{\rm o}^i = \mu^{k^i\varepsilon}\big(g^i + L^i(g)\big) \ , \quad \phi_{\rm o} = \mu^{\delta\varepsilon} Z(g)\phi \ , \tag{2.2}$$

where $L^i(g)$ and $Z(g) - 1$ contain only poles in ε. The quantum action

$$S_{\rm o} = \int d^d x\, \sqrt{\gamma}\, \mathcal{L}_{\rm o} \equiv \int \mathcal{L}_{\rm o} \ , \tag{2.3}$$

is then supposed to define a finite theory on flat space, or $\gamma_{\mu\nu}$ constant, and also for g^i constant as usual. From (2.2) it is easy to derive in the standard fashion the renormalisation group equation.

$$\begin{gathered}\Big(\hat\beta^i \frac{\partial}{\partial g^i} - (\hat\gamma\phi)\cdot\frac{\partial}{\partial\phi} - \varepsilon\Big)\mathcal{L}_{\rm o} = 0 \ , \\ \mu\frac{d}{d\mu} g^i \Big|_{g_{\rm o}} = \hat\beta^i = -k^i g^i \varepsilon + \beta^i(g) \ , \quad \mu\frac{d}{d\mu} Z \Big|_{g_{\rm o}} = Z\gamma(g) \ , \quad \hat\gamma = \gamma + \delta\varepsilon \ ,\end{gathered} \tag{2.4}$$

(throughout the index on k^i is irrelevant for the summation convention).

If the metric is generalised to describe an arbitrary curved space $\gamma_{\mu\nu}(x)$ and also the couplings are extended to arbitrary $g^i(x)$ then, apart from introducing appropriate covariant derivatives so that $\mathcal{L}$ is still a scalar, the form of the dependence of $\mathcal{L}$ on g^i and correspondingly of $\mathcal{L}_{\rm o}$ on $g_{\rm o}^i$ remains unchanged. However additional counterterms are required depending on the curvature tensor, assuming manifest coordinate invariance is maintained throughout, and also on $\partial_\mu g^i$. Thus it is necessary to extend $\mathcal{L}_{\rm o} \to \tilde{\mathcal{L}}_{\rm o}$. If we consider, for simplicity, only those necessary counterterms independent of the fields ϕ then by power counting and discarding total derivatives we may take [10]

$$
\begin{aligned}
\tilde{\mathcal{L}}_{\rm o} &= \mathcal{L}_{\rm o} - \mu^{-\varepsilon}\lambda\cdot\mathcal{R}\ , \quad \lambda = (a,b,c,\mathcal{E}_i,\mathcal{F}_{ij},\mathcal{G}_{ij},\Lambda)\ , \\
\lambda\cdot\mathcal{R} &= a\,F + b\,G + c\,H^2 + \mathcal{E}_i\partial_\mu g^i\,\partial^\mu H + \tfrac{1}{2}\mathcal{F}_{ij}\partial_\mu g^i\partial^\mu g^j\,H + \tfrac{1}{2}\mathcal{G}_{ij}\partial_\mu g^i\partial_\nu g^j\,G^{\mu\nu} + \Lambda\ , \\
\Lambda &= \tfrac{1}{2}\mathcal{A}_{ij}\,\nabla^2 g^i\nabla^2 g^j + \tfrac{1}{2}\mathcal{B}_{ijk}\,\partial_\mu g^i\partial^\mu g^j\nabla^2 g^k + \tfrac{1}{4}\mathcal{C}_{ijk\ell}\,\partial_\mu g^i\partial^\mu g^j\partial_\nu g^k\partial^\nu g^\ell\ .
\end{aligned} \tag{2.5}
$$

F, G and H^2 are the additional purely metric counterterms required in the extension from flat space and at least at one loop they have been much discussed. With $G_{\mu\nu}$ they are here defined by

$$
\begin{aligned}
F &= R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - \frac{4}{d-2}R^{\alpha\beta}R_{\alpha\beta} + \frac{2}{(d-2)(d-1)}R^2\ , \\
G &= \frac{2}{(d-3)(d-2)}\big(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2\big)\ , \\
H &= \frac{1}{d-1}R\ , \quad G_{\mu\nu} = \frac{2}{d-2}\big(R_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R\big)\ .
\end{aligned} \tag{2.6}
$$

The d-dependent factors introduced in (2.6) are not essential but simplify the subsequent analysis. F is the square of the conformal Weyl tensor while G is the Euler density,

$$
\int_{S^4} d^4x\sqrt{\gamma}\,G = 64\pi^2\ .
$$

Under reparameterisations on the space of couplings it is clear that $\mathcal{E}_i, \mathcal{F}_{ij}, \mathcal{G}_{ij}$ and $\mathcal{A}_{ij}$ are tensors while, although Λ is a scalar, $\mathcal{B}_{ijk}$ and $\mathcal{C}_{ijk\ell}$ transform with additional inhomogeneous pieces (this could be avoided by introducing a connection Γ^i_{jk} and replacing $\nabla^2 g^i$ in (2.5) by $\nabla^2 g^i + \Gamma^i_{jk}\partial_\mu g^j\partial^\mu g^k$).

λ in (2.5) contains just poles in ε and we assume that (2.4) now becomes

$$
\Big(\hat{\beta}^i\frac{\partial}{\partial g^i} - (\hat{\gamma}\phi)\cdot\frac{\partial}{\partial\phi} - \varepsilon\Big)\tilde{\mathcal{L}}_{\rm o} = \mu^{-\varepsilon}\beta_\lambda\cdot\mathcal{R}\ . \tag{2.7}
$$

This requires

$$
\begin{aligned}
&\Big(\varepsilon - \hat{\beta}^i\frac{\partial}{\partial g^i}\Big)\lambda\cdot\mathcal{R} = \beta_\lambda\cdot\mathcal{R}\ , \quad \beta_\lambda = (\beta_a,\beta_b,\beta_c,\chi^e_i,\chi^f_{ij},\chi^g_{ij},\beta_\Lambda)\ , \\
&\beta_\Lambda = \tfrac{1}{2}\chi^a_{ij}\,\nabla^2 g^i\nabla^2 g^j + \tfrac{1}{2}\chi^b_{ijk}\,\partial_\mu g^i\partial^\mu g^j\nabla^2 g^k + \tfrac{1}{4}\chi^c_{ijk\ell}\,\partial_\mu g^i\partial^\mu g^j\partial_\nu g^k\partial^\nu g^\ell\ ,
\end{aligned} \tag{2.8}
$$

where $h^i\partial/\partial g^i\partial_\mu g^j = \partial_\mu h^j$. The terms on the r.h.s. of (2.7) may be removed, and a homogeneous renormalisation group equation restored, by introducing additional couplings for each term in $\lambda\cdot\mathcal{R}$. However these new couplings perform no further role and are therefore not introduced here.

For our purposes β_a is irrelevant but at one loop or for a free theory

$$\beta_b^{(1)} = \frac{1}{64\pi^2}\frac{1}{90}\big(62n_V + n_S + 11n_F\big) \ , \quad \beta_c^{(1)} = 0 \ , \tag{2.9}$$

for n_V massless vectors, n_F Dirac fermions and n_S scalars, which corresponds to (1.1) if $C_0 = 360 \times 16\pi^2\beta_b^{(1)}$. As particular cases of (2.8) we find

$$\Big(\varepsilon - \hat\beta^\ell \frac{\partial}{\partial g^\ell}\Big)\mathcal{A}_{ij} - \partial_i\hat\beta^k\mathcal{A}_{kj} - \partial_j\hat\beta^k\mathcal{A}_{ik} = \chi^a_{ij} \ , \tag{2.10a}$$

$$\Big(\varepsilon - \hat\beta^\ell \frac{\partial}{\partial g^\ell}\Big)\mathcal{B}_{ijk} - \partial_i\hat\beta^\ell\mathcal{B}_{\ell jk} - \partial_j\hat\beta^\ell\mathcal{B}_{i\ell k} - \partial_k\hat\beta^\ell\mathcal{B}_{ij\ell} - 2\partial_i\partial_j\hat\beta^\ell\mathcal{A}_{\ell k} = \chi^b_{ijk} \ , \tag{2.10b}$$

with similar equations as (2.10a) holding for $\chi^f_{ij}, \ \chi^g_{ij}$ in terms of $\mathcal{F}_{ij}, \ \mathcal{G}_{ij}$.

As a consequence of the extension to an arbitrary metric $\gamma_{\mu\nu}(x)$ and also $g^i(x)$ it is straightforward to define a finite energy momentum tensor and also the complete set of finite local dimension four scalar operators specified by $\mathcal{L}$ by

$$T_{\mu\nu}(x) = 2\frac{\delta}{\delta\gamma^{\mu\nu}(x)}\tilde{S}_{\rm o} \ , \quad [\mathcal{O}_i(x)] = \frac{\delta}{\delta g^i(x)}\tilde{S}_{\rm o} \ . \tag{2.11}$$

From (2.11) $[\mathcal{O}_i]$ has the generic form

$$h^i[\mathcal{O}_i] = h^i\frac{\partial}{\partial g^i}\tilde{\mathcal{L}}_{\rm o} - \nabla_\mu J_h^\mu \ , \tag{2.12}$$

where J_h^μ arises from counterterms containing $\partial_\mu g^i$ and is necessary for finiteness even when $\partial_\mu g^i$ is subsequently set to zero. From the particular ϕ independent counterterms in (2.5) $J_h^\mu|_{\partial_\nu g=0} = -h^i\mathcal{E}_i\partial^\mu H$. These operators have simple properties under changes of renormalisation scale since, defining the Callan-Symanzik operator by

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \int\Big(\hat\beta^i\frac{\delta}{\delta g^i} - (\hat\gamma\phi)\cdot\frac{\delta}{\delta\phi}\Big) \ , \tag{2.13}$$

then directly from the definitions (2.11) and (2.7) it follows that

$$\begin{aligned}\mathcal{D}T_{\mu\nu} &= 2\frac{\delta}{\delta\gamma^{\mu\nu}}\int\mu^{-\varepsilon}\beta_\lambda\cdot\mathcal{R} \ , \\ \mathcal{D}[\mathcal{O}_i] &= -\partial_i\hat\beta^j[\mathcal{O}_j] + \frac{\delta}{\delta g^i}\int\mu^{-\varepsilon}\beta_\lambda\cdot\mathcal{R} + (\partial_i\hat\gamma\,\phi)\cdot\frac{\delta}{\delta\phi}\tilde{S}_{\rm o} \ .\end{aligned} \tag{2.14}$$

Clearly from (2.14) $\partial_i\hat\beta^j$ is the anomalous dimension matrix for the operators $[\mathcal{O}_j]$ while the contribution of the last term vanishes on using the equations of motion.

Within this framework it is similarly straightforward to define operator products which are finite on insertion into correlation functions, for instance

$$\frac{\delta \tilde{S}_{\rm o}}{\delta g^i(x)}\,\frac{\delta \tilde{S}_{\rm o}}{\delta g^j(y)} - \frac{\delta^2 \tilde{S}_{\rm o}}{\delta g^i(x)\delta g^j(y)}\ , \tag{2.15a}$$

$$\begin{aligned}&\frac{\delta \tilde{S}_{\rm o}}{\delta g^i(x)}\,\frac{\delta \tilde{S}_{\rm o}}{\delta g^j(y)}\,\frac{\delta \tilde{S}_{\rm o}}{\delta g^k(z)} - \frac{\delta^2 \tilde{S}_{\rm o}}{\delta g^i(x)\delta g^j(y)}\,\frac{\delta \tilde{S}_{\rm o}}{\delta g^k(z)} \\ &\quad - \frac{\delta^2 \tilde{S}_{\rm o}}{\delta g^j(y)\delta g^k(z)}\,\frac{\delta \tilde{S}_{\rm o}}{\delta g^i(x)} - \frac{\delta^2 \tilde{S}_{\rm o}}{\delta g^k(z)\delta g^i(x)}\,\frac{\delta \tilde{S}_{\rm o}}{\delta g^j(y)} + \frac{\delta^3 \tilde{S}_{\rm o}}{\delta g^i(x)\delta g^j(y)\delta g^k(z)}\ .\end{aligned} \tag{2.15b}$$

Such operator products can be used to provide an alternative definition of $\mathcal{A}_{ij},\ \mathcal{B}_{ijk},\ldots$ appearing in (2.5). Restricting to flat space and letting $[\mathcal{O}_i]|_{\partial_\mu g=0} = [\mathcal{O}_i]^c = \partial_i h^a \mathcal{O}^{\rm o}_a$, with $\mathcal{O}^{\rm o}_a$ independent of g ($[\mathcal{O}_i]^c$ is the local composite operator as obtained in a conventional analysis) then

$$\begin{gathered}\frac{\delta}{\delta g^i(x)}[\mathcal{O}_j(y)]\Big|_{\partial_\mu g=0} = K_{ij}^k\,[\mathcal{O}_k(y)]^c \delta^d(x-y) - \mu^{-\varepsilon}\mathcal{A}_{ij}\partial^2\partial^2\delta^d(x-y)\ , \\ K_{ij}^k = \partial_i\partial_j h^a\,\frac{\partial g^k}{\partial h^a}\ .\end{gathered} \tag{2.16}$$

This shows that the correlation function $\langle[\mathcal{O}(x)_i]^c[\mathcal{O}(y)_i]^c\rangle_{S_{\rm o}}$ is divergent and requires a further subtraction. In terms of the Fourier transform $\Gamma_{ij}(-p,p)$ of this correlation function, assuming for simplicity that $\langle[\mathcal{O}_i]^c\rangle_{S_{\rm o}} = 0$, the finite amplitude within minimal subtraction is given by

$$\Gamma^R_{ij}(-p,p) = \Gamma_{ij}(-p,p) + \mu^{-\varepsilon}\mathcal{A}_{ij}(p^2)^2\ . \tag{2.17}$$

Similarly from (2.15b) for the three point correlation function $\Gamma_{ijk}(p,q,r),\ p+q+r=0$,

$$\begin{aligned}\Gamma^R_{ijk}(p,q,r) = {}& \Gamma_{ijk}(p,q,r) - K^\ell_{ij}\,\Gamma_{\ell k}(-r,r) - K^\ell_{jk}\,\Gamma_{\ell i}(-p,p) - K^\ell_{ki}\,\Gamma_{\ell j}(-q,q) \\ & - \mu^{-\varepsilon}\tfrac{1}{2}\big(\mathcal{B}_{ijk}\,(r^2)^2 + \mathcal{B}_{jki}\,(p^2)^2 + \mathcal{B}_{kij}\,(q^2)^2\big) \\ & - \mu^{-\varepsilon}\big(\bar{\mathcal{A}}_{ijk}\,p^2q^2 + \bar{\mathcal{A}}_{jki}\,q^2r^2 + \bar{\mathcal{A}}_{kij}\,r^2p^2\big)\ ,\end{aligned} \tag{2.18}$$

where

$$\bar{\mathcal{A}}_{ijk} = \mathcal{A}_{ij,k} - \tfrac{1}{2}\mathcal{B}_{ikj} - \tfrac{1}{2}\mathcal{B}_{jki}\ . \tag{2.19}$$

Applying $\mathcal{D}$ to (2.17) gives, using (2.10a),

$$\mathcal{D}\Gamma^R_{ij}(-p,p) = -\partial_i\hat{\beta}^k\Gamma^R_{kj}(-p,p) - \partial_j\hat{\beta}^k\Gamma^R_{ik}(-p,p) - \mu^{-\varepsilon}\chi^a_{ij}(p^2)^2\ . \tag{2.20}$$

Similarly from (2.18) and (2.10b)

$$\begin{aligned}\mathcal{D}\Gamma^R_{ijk}(p,q,r) = {}& -\partial_i\hat{\beta}^\ell\Gamma^R_{\ell jk}(p,q,r) - \partial_j\hat{\beta}^\ell\Gamma^R_{i\ell k}(p,q,r) - \partial_k\hat{\beta}^\ell\Gamma^R_{ij\ell}(p,q,r) \\ & + \partial_i\partial_j\hat{\beta}^\ell\,\Gamma^R_{\ell k}(-r,r) + \partial_j\partial_k\hat{\beta}^\ell\,\Gamma^R_{\ell i}(-p,p) + \partial_k\partial_i\hat{\beta}^\ell\,\Gamma^R_{\ell j}(-q,q) \\ & + \mu^{-\varepsilon}\tfrac{1}{2}\big(\chi^b_{ijk}\,(r^2)^2 + \chi^b_{jki}\,(p^2)^2 + \chi^b_{kij}\,(q^2)^2\big) \\ & + \mu^{-\varepsilon}\big(\bar{\chi}^a_{ijk}\,p^2q^2 + \bar{\chi}^a_{jki}\,q^2r^2 + \bar{\chi}^a_{kij}\,r^2p^2\big)\ ,\end{aligned} \tag{2.21}$$

where $\bar{\chi}^a$ is defined in terms of χ^a, χ^b just as $\bar{\mathcal{A}}$ in (2.19) and we have used

$$\Big(\varepsilon - \hat{\beta}^\ell \frac{\partial}{\partial g^\ell}\Big) h^a = 0 \quad \Rightarrow \quad \hat{\beta}^\ell \frac{\partial}{\partial g^\ell} K^k_{ij} + \partial_i \hat{\beta}^\ell K^k_{\ell j} + \partial_j \hat{\beta}^\ell K^k_{i\ell} - K^\ell_{ij} \partial_\ell \hat{\beta}^k = -\partial_i \partial_j \hat{\beta}^k \ . \quad (2.22)$$

Clearly χ^a_{ij}, χ^b_{ijk} play an essential role in determining the behaviour of Γ^R_{ij} and Γ^R_{ijk} or equivalently of operator products. Analogous relations for four point functions involving $\chi^c_{ijk\ell}$ can also be derived.

3. Consistency Relations

The trace of the energy momentum operator is a scalar operator of dimension four and should therefore be expandable in the basis of scalar operators $[\mathcal{O}_i]$. To derive such an expression for the trace we assume that under conformal rescaling of the metric and the fields

$$\delta\gamma^{\mu\nu} = 2\sigma\,\gamma^{\mu\nu} \ , \qquad \delta\phi = \sigma\,\Delta\phi \ , \tag{3.1}$$

where Δ is a matrix defining the canonical dimensions of the fields ϕ. Since $\delta\sqrt{\gamma} = -d\sigma\sqrt{\gamma}$ it follows that, neglecting dimensional couplings in $\tilde{\mathcal{L}}_{\rm o}$ so that for σ constant in (3.1) $\delta\tilde{\mathcal{L}}_{\rm o} = 4\sigma\tilde{\mathcal{L}}_{\rm o}$,

$$\gamma^{\mu\nu} T_{\mu\nu} = \varepsilon \tilde{\mathcal{L}}_{\rm o} + \nabla_\mu I^\mu - (\Delta\phi)\cdot \frac{\delta}{\delta\phi} S_{\rm o} \ . \tag{3.2}$$

I^μ arises from the appearance in $\tilde{\mathcal{L}}_{\rm o}$ of terms containing derivatives of $\gamma_{\mu\nu}$ such as the Riemann tensor. Assuming

$$(X\phi)\cdot \frac{\delta}{\delta\phi} S_{\rm o} = (X\phi)\cdot \frac{\partial}{\partial\phi} \mathcal{L}_{\rm o} - \nabla_\mu K^\mu_X \ , \tag{3.3}$$

and using (2.7) and (2.12) this becomes

$$\gamma^{\mu\nu} T_{\mu\nu} = \hat{\beta}^i [\mathcal{O}_i] - \mu^{-\varepsilon} \beta_\lambda \cdot \mathcal{R} + \mu^{-\varepsilon} \nabla_\mu Z^\mu + \nabla_\mu J^\mu_\Theta - (\Delta\phi + \hat{\gamma}\phi)\cdot \frac{\delta}{\delta\phi} S_{\rm o} \ , \tag{3.4}$$

where

$$I^\mu + J^\mu_{\hat{\beta}} - K^\mu_{\hat{\gamma}} = J^\mu_\Theta + \mu^{-\varepsilon}\big(Z^\mu + \nabla_\nu X^{\mu\nu}\big) \ , \quad X_{\mu\nu} = -X_{\nu\mu} \ . \tag{3.5}$$

J^μ_Θ is some potential finite operator current formed from ϕ (when it occurs such currents may be defined in terms of the variation of $\tilde{S}_{\rm o}$ with respect to an arbitrary gauge field A_μ which plays the role of an additional coupling). The remaining current Z^μ appearing in (3.4) is field independent and arises from the counterterms in (2.5). For an appropriate choice of $X_{\mu\nu}$ Z^μ must be finite as a consequence of all other terms in (3.4) being well defined. I^μ in (3.2) may be computed by using that under the conformal variation of the metric in (3.1)

$$\begin{gathered} \delta F = 4\sigma F \ , \quad \delta G = 4\sigma G - 8 G^{\alpha\beta} \nabla_\alpha \nabla_\beta \sigma \ , \quad \delta H = 2\sigma H + 2\nabla^2\sigma \ , \\ \delta G_{\mu\nu} = 2\big(\nabla_\mu \nabla_\nu \sigma - \gamma_{\mu\nu} \nabla^2 \sigma\big) \ , \quad \delta\nabla^2 = 2\sigma\nabla^2 - (d-2)\partial^\mu\sigma\partial_\mu \ , \end{gathered} \tag{3.6}$$

while the corresponding contributions to $J^\mu_{\hat\beta}$, defined by (2.12), are

$$u_i^\mu \hat\beta^i - \nabla^\mu (v_i \hat\beta^i) + 2 v_i \nabla^\mu \hat\beta^i \ , \quad u_i^\mu = \frac{\partial \tilde{\mathcal{L}}_{\rm o}}{\partial \partial_\mu g^i} \ , \quad v_i = \frac{\partial \tilde{\mathcal{L}}_{\rm o}}{\partial \nabla^2 g^i} \ . \tag{3.7}$$

Hence we obtain

$$\begin{aligned} Z_\mu = {}& 8G_{\mu\nu}\,\partial^\nu b - 4\partial_\mu(Hc) + 2H\,\mathcal{E}_i\partial_\mu g^i - \partial_\mu H\,\mathcal{E}_i\hat\beta^i - G_{\mu\nu}\,\mathcal{G}_{ij}\hat\beta^i\partial^\nu g^j \\ & - H\,\mathcal{F}_{ij}\hat\beta^i\partial_\mu g^j + 2\nabla_\mu\nabla_\nu\big(\mathcal{E}_i\partial^\nu g^i\big) - \partial_\mu\big((\mathcal{F}_{ij}-\mathcal{G}_{ij})\partial_\nu g^i\partial^\nu g^j\big) \\ & - \nabla_\nu\big(\mathcal{G}_{ij}\partial_\mu g^i\partial^\nu g^j\big) - (2-\varepsilon)\mathcal{A}_{ij}\partial_\mu g^i\nabla^2 g^j + \partial_\mu\big(\mathcal{A}_{ij}\hat\beta^i\nabla^2 g^j\big) - 2\mathcal{A}_{ij}\partial_\mu\hat\beta^i\nabla^2 g^j \\ & - \tfrac12(2-\varepsilon)\mathcal{B}_{ijk}\,\partial_\nu g^i\partial^\nu g^j\partial_\mu g^k + \tfrac12\nabla_\mu\big(\mathcal{B}_{ijk}\partial_\nu g^i\partial^\nu g^j\hat\beta^k\big) \\ & - \mathcal{B}_{ijk}\big(\hat\beta^i\partial_\mu g^j\nabla^2 g^k + \partial_\nu g^i\partial^\nu g^j\partial_\mu\hat\beta^k\big) - \mathcal{C}_{ijk\ell}\,\hat\beta^i\partial_\mu g^j\partial^\nu g^k\partial_\nu g^\ell + \nabla^\nu(\mathcal{Q}_{ij}\partial_\mu g^i\partial_\nu g^j)\,, \end{aligned} \tag{3.8}$$

assuming the appropriate contribution to the $X_{\mu\nu}$ has the form

$$X_{\mu\nu} = -\mathcal{Q}_{ij}\partial_\mu g^i\partial_\nu g^j \ , \quad \mathcal{Q}_{ij} = -\mathcal{Q}_{ji} \ .$$

The finiteness of Z_μ may be disentangled, using when necessary results such as (2.10a,b), into separate finiteness conditions for

$$8\partial_i b - \mathcal{G}_{ij}\,\hat\beta^j \ , \tag{3.9a}$$
$$4c + \mathcal{E}_i\,\hat\beta^i \ , \tag{3.9b}$$
$$4\partial_i c + (\mathcal{F}_{ij} + \mathcal{A}_{ij})\hat\beta^j \ , \tag{3.9c}$$
$$2\mathcal{E}_i + \mathcal{A}_{ij}\,\hat\beta^j \ , \tag{3.9d}$$
$$\mathcal{G}_{ij} + 2\mathcal{A}_{ij} - \bar{\mathcal{A}}_{ijk}\,\hat\beta^k \tag{3.9e}$$
$$\mathcal{F}_{ij} + \mathcal{A}_{ij} + \tfrac12\varepsilon\mathcal{A}_{ij} - \big(\bar{\mathcal{A}}_{ijk} - \bar{\mathcal{A}}_{k(ij)}\big)\hat\beta^k \ , \tag{3.9f}$$
$$\Lambda_{ij} - \Lambda_{ji} \ , \quad \Lambda_{ij} = \mathcal{A}_{ki}\,\partial_j\hat\beta^k + \tfrac12\mathcal{B}_{kji}\,\hat\beta^k \ , \tag{3.9g}$$
$$(2-\varepsilon)\big(\bar{\mathcal{A}}_{k(ij)} - \tfrac12\bar{\mathcal{A}}_{ijk}\big) + \partial_j\Lambda_{ik} + \partial_i\Lambda_{jk} - \partial_k\Lambda_{(ij)} - \mathcal{B}_{ij\ell}\,\partial_k\hat\beta^\ell - \mathcal{C}_{ijk\ell}\,\hat\beta^\ell \ , \tag{3.9h}$$

where $\bar{\mathcal{A}}_{ijk}$ is given by (2.19). Both $\bar{\mathcal{A}}_{ijk}$ and Λ_{ij} transform as tensors under redefinitions of couplings. These conditions are not completely independent since contracting (3.9f) with $\hat\beta^j$ and using (3.9g) and (2.10a) gives (3.9c) with c expressed in terms of (3.9b) and (3.9d) up to finite terms.

The essential consequence of (3.9a,b,c,d,e,f) is that all counterterms required in (2.5) for a curved space background may be defined in terms of $\mathcal{A}_{ij}$ and $\mathcal{B}_{ijk}$ required for flat space only. These results were partially obtained previously [11,12] by restricting to $\partial_\mu g = 0$ but by considering that

$$\tilde{\mathcal{L}}_{\rm o} = \mathcal{L}_{\rm o} - \mu^{-\varepsilon}(aF + bG + cH^2)$$

defines a finite quantum field theory for arbitrary metric $\gamma_{\mu\nu}(x)$. In this case we consider the finite operator products, analogous to (2.15a,b),

$$\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma(x)}\,\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma(y)} - \frac{\delta^2\tilde{S}_{\rm o}}{\delta\sigma(x)\delta\sigma(y)}\ , \tag{3.10a}$$

$$\begin{aligned}\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma(x)}\,\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma(y)}\,\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma(z)} - \frac{\delta^2\tilde{S}_{\rm o}}{\delta\sigma(x)\delta\sigma(y)}\,\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma(z)}& \\ -\frac{\delta^2\tilde{S}_{\rm o}}{\delta\sigma(y)\delta\sigma(z)}\,\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma(x)} - \frac{\delta^2\tilde{S}_{\rm o}}{\delta\sigma(z)\delta\sigma(x)}\,\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma(y)} + \frac{\delta^3\tilde{S}_{\rm o}}{\delta\sigma(x)\delta\sigma(y)\delta\sigma(z)}&\ ,\end{aligned} \tag{3.10b}$$

where $\delta/\delta\sigma$ denotes the response to a conformal rescaling as in (3.1) with subsequently $\sigma = 0$. Thus, if for simplicity we suppose $\delta\mathcal{L}_{\rm o} = 4\sigma\mathcal{L}_{\rm o}$ and neglecting equation of motion terms,

$$\begin{aligned}\frac{\delta\tilde{S}_{\rm o}}{\delta\sigma} &\simeq \varepsilon\tilde{\mathcal{L}}_{\rm o} - 4\mu^{-\varepsilon}\,c\nabla^2 H = \hat{\Theta}\ ,\\ \frac{\delta^2\tilde{S}_{\rm o}}{\delta\sigma(x)\delta\sigma(y)} &\simeq 4\delta^d(x,y)\hat{\Theta} - 8\mu^{-\varepsilon}c\,\nabla^2\nabla^2\delta^d(x,y) + 8\varepsilon\mu^{-\varepsilon}b\,G^{\mu\nu}\nabla_\mu\nabla_\nu\delta^d(x,y)\\ &\quad - 4(2+\varepsilon)\mu^{-\varepsilon}c\,H\nabla^2\delta^d(x,y) - 4(4+\varepsilon)\mu^{-\varepsilon}c\,\partial_\mu H\partial^\mu\delta^d(x,y)\big)\ .\end{aligned}$$

Restricting to flat space then (3.10a,b) allow definition of the finite correlation functions

$$\Gamma^R_{\hat{\Theta}\hat{\Theta}}(-p,p) = \Gamma_{\hat{\Theta}\hat{\Theta}}(-p,p) + 8\mu^{-\varepsilon}c\,(p^2)^2\ , \tag{3.11a}$$

$$\begin{aligned}\Gamma^R_{\hat{\Theta}\hat{\Theta}\hat{\Theta}}(p,q,r) = {}& \Gamma_{\hat{\Theta}\hat{\Theta}\hat{\Theta}}(p,q,r) - \varepsilon\big(\Gamma_{\hat{\Theta}\hat{\Theta}}(-p,p) + \Gamma_{\hat{\Theta}\hat{\Theta}}(-q,q) + \Gamma_{\hat{\Theta}\hat{\Theta}}(-r,r)\big)\\ &+ 4\mu^{-\varepsilon}\big(\varepsilon b + (2-\varepsilon)c\big)\big((p^2)^2 + (q^2)^2 + (r^2)^2\big)\\ &- 8\mu^{-\varepsilon}\big(\varepsilon b + 2c\big)\big(p^2q^2 + q^2r^2 + r^2p^2\big)\ ,\end{aligned} \tag{3.11b}$$

where $\Gamma_{\hat{\Theta}\dots}$ are correlation functions of $\hat{\Theta}|_{\gamma_{\mu\nu}=\delta_{\mu\nu}} = \hat{\Theta}_0$ after Fourier transform. Using $\hat{\Theta}_0 = \hat{\beta}^i[\mathcal{O}_i]^c$ then comparing (3.11a) and (2.17) we require

$$8c \sim \mathcal{A}_{ij}\,\hat{\beta}^i\hat{\beta}^j\ , \tag{3.12}$$

with $\sim$ representing equality up to finite terms. Similarly from (2.18) and (3.11b) using

$$\hat{\beta}^i\,K^k_{ij} = \varepsilon\delta^k_j - \partial_j\hat{\beta}^k\ , \tag{3.13}$$

which can be obtained analogously to (2.22), gives

$$8\big(\varepsilon b + (2-\varepsilon)c\big) \sim 8\big(\hat{\beta}^i\partial_i b + 2c - \hat{\beta}^i\partial_i c\big) \sim -2\Lambda_{ij}\,\hat{\beta}^i\hat{\beta}^j\ , \tag{3.14a}$$

$$8\big(\varepsilon b + 2c\big) \sim 8\big(\hat{\beta}^i\partial_i b + 2c\big) \sim \bar{\mathcal{A}}_{ijk}\,\hat{\beta}^i\hat{\beta}^j\hat{\beta}^k\ . \tag{3.14b}$$

It is straightforward to see that (3.12) and (3.14a,b) are direct corollaries of finiteness of expressions (3.9a,...f), using

$$\bar{\mathcal{A}}_{ijk}\,\hat{\beta}^k \sim \varepsilon\mathcal{A}_{ij} - 2\Lambda_{(ij)}\ , \quad \bar{\mathcal{A}}_{jki}\,\hat{\beta}^j\hat{\beta}^k \sim \partial_i(\mathcal{A}_{jk}\,\hat{\beta}^j\hat{\beta}^k) - 2\Lambda_{ij}\,\hat{\beta}^j\ .$$

The conditions that (3.9a,...h) contain no poles in ε can be reexpressed in terms of equations involving the β functions in (2.8). From (3.9a)

$$8\partial_i b - \mathcal{G}_{ij}\,\hat{\beta}^j = W_i\ , \qquad W_i = \mathcal{G}^1_{ij}\,k^j g^j\ , \tag{3.15}$$

where $\mathcal{G}^1_{ij}$ denotes the residue of the simple ε pole in $\mathcal{G}_{ij}$. Using

$$(\varepsilon - \hat{\beta}^k\partial_k)\big(8\partial_i b - \mathcal{G}_{ij}\,\hat{\beta}^j\big) - \partial_i\hat{\beta}^k\big(8\partial_k b - \mathcal{G}_{kj}\,\hat{\beta}^j\big) = \partial_i\beta_b - \chi^g_{ij}\,\hat{\beta}^j\ ,$$

we obtain

$$8\partial_i\beta_b = \chi^g_{ij}\beta^j - \beta^j\partial_j W_i - \partial_i\beta^j W_j\ , \tag{3.16a}$$

$$\zeta W_i \equiv (1 + k^j g^j\partial_j)W_i + W_i k^i = \chi^g_{ij}\,k^j g^j\ , \tag{3.16b}$$

were ζ is easily seen to be an operator counting the number of loops and (3.16b) is equivalent to the definition of W_i in (3.15). Defining

$$\tilde{\beta}_b = \beta_b + \tfrac{1}{8}W_i\beta^i\ ,$$

(3.16) gives

$$8\partial_i\,\tilde{\beta}_b = \chi^g_{ij}\,\beta^j + (\partial_i W_j - \partial_j W_i)\beta^j\ , \tag{3.17a}$$

$$8\beta^i\partial_i\,\tilde{\beta}_b = \chi^g_{ij}\,\beta^i\beta^j\ . \tag{3.17b}$$

These equations are similar to those of Zamolodchikov [3] for two dimensional field theories and it is natural to define

$$C = 360 \times 16\pi^2\tilde{\beta}_b\ . \tag{3.18}$$

(3.17a) shows that $\tilde{\beta}_b$ is stationary at critical points where $\beta^i = 0$, $\tilde{\beta}_b = \beta_b$ and C is the natural analogue of the Virasoro central charge while (3.17b) demonstrates that if χ^g_{ij} is positive then the renormalisation flow of C or $\tilde{\beta}_b$ is monotonic. For a free theory $C = C_0$ as given by (1.1). As expressed in (3.17a,b) the equations are covariant under redefinitions of couplings and so should be valid in other regularisation schemes although the precise simple definition of W_i is no longer appropriate. Any arbitrariness, such as provided by the d dependent factors in (2.6), in the definition of b leads to a contribution to W_i of the form $\partial_i B$ which cancels in (3.17a).

For a single coupling g (3.16a,b) can be easily integrated to give

$$\beta_b(g) = \beta_b^0 - \frac{1}{8}\int_0^g dg'\, g'^{\frac{k+1}{k}} W(g')\, \frac{\partial}{\partial g'}\left(\frac{\beta(g')}{g'^{\frac{k+1}{k}}}\right) \ , \tag{3.19}$$

where β_b^0 is given by (2.9). In general, $\beta(g) \propto g^{\frac{k+1}{k}}$ as $g \to 0$ so if W is first non zero at n loops β_b has corrections at $n+2$ loops.

It is also possible to derive expressions for β_c from (3.9b,d) or (3.12), where

$$8c - \mathcal{A}_{ij}\,\hat\beta^i\hat\beta^j = X + \varepsilon Y \ , \quad Y = -\mathcal{A}^1_{ij}\,k^i g^i k^j g^j \ . \tag{3.20}$$

Just as in going from (3.15) to (3.16a,b) we obtain

$$8\beta_c = \chi^a_{ij}\,\beta^i\beta^j - \beta^i\partial_i X \ , \tag{3.21a}$$

$$\zeta Y = -\chi^a_{ij}\,k^i g^i k^j g^j \ , \quad \zeta X = 2\chi^a_{ij}\,\beta^i k^j g^j + \beta^i\partial_i Y \ . \tag{3.21b}$$

Since from (3.11a)

$$\mathcal{D}\Gamma^R_{\hat\Theta\hat\Theta}(-p,p) = -8\mu^{-\varepsilon}\beta_c(p^2)^2$$

the results (3.21a,b) can also be obtained by using (2.20) and writing

$$\Gamma^R_{\hat\Theta\hat\Theta}(-p,p) - \hat\beta^i\hat\beta^j\Gamma^R_{ij}(-p,p) = \mu^{-\varepsilon}(X+\varepsilon Y)(p^2)^2 \ .$$

For a single coupling

$$8\beta_c(g) = \frac{1}{k^2}\,\frac{\beta(g)}{g^{\frac{k+1}{k}}}\int_0^g dg'\, g'^{\frac{2}{k}}\, Y(g')\frac{\partial}{\partial g'}\left(\frac{\beta(g')}{g'^{\frac{k+1}{k}}}\right) \ , \tag{3.22}$$

so that for Y non zero at n loops β_c is non zero at $n+3$ loops.

A similar discussion is possible for the other expressions in (3.9), for example from (3.9c)

$$\chi^g_{ij} + 2\chi^a_{ij} - \bar\chi^a_{ijk}\,\beta^k = -\beta^k\partial_k V_{ij} - \gamma_i{}^k V_{kj} - \gamma_j{}^k V_{ik} \ , \quad \zeta V_{ij} = \bar\chi^a_{ijk}\,k^k g^k \ . \tag{3.23}$$

This relates χ^g_{ij}, which played a crucial role above, to flat space quantities.

4. Gauge Theories

The discussion of composite operators in gauge theories and the derivation of a relation for the trace of the energy momentum tensor has by now a relatively long history [16]. Nevertheless the extension of the framework described in the last two sections to this case appears to us to elucidate some points.

For simplicity we restrict to a simple gauge group G and as always the fundamental gauge field Lagrangian is

$$\mathcal{L}(A) = \tfrac{1}{4}\,F^{\mu\nu}\!\cdot F_{\mu\nu} \ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \times A_\nu \ , \tag{4.1}$$

$(X{\cdot}Y = X_a Y_a\ ,\ (X\times Y)_a = f_{abc}X_bY_c)$. Fermions will be incorporated later. The quantum action is then given by

$$\begin{aligned}
\mathcal{L}^q(\phi,J,g,\xi,\rho) &= \frac{1}{g^2}\mathcal{L}(A) + \tilde{K}^\mu{\cdot}D'_\mu(A,g)c - \tfrac{1}{2}g\,L{\cdot}(c\times c) - \xi\,\tfrac{1}{2}b{\cdot}b + (G^{T\mu}b){\cdot}a_\mu\ , \\
A_\mu = A^c_\mu + g\,a_\mu\ , &\quad D'_\mu(A,g) = \partial_\mu + A_\mu\times\ + v_\mu\ , \quad v_\mu = \frac{1}{g}\partial_\mu g\ , \\
\tilde{K}^\mu = K^\mu + G^{T\mu}\bar{c}\ , &\quad \phi = (a_\mu,\,c,\,b,\,\bar{c})\ , \quad J = (K^\mu,\,L)\ .
\end{aligned} \tag{4.2}$$

a_μ is the quantum gauge field with A^c_μ a fixed background, c, $\bar{c}$ the ghosts, anti-ghosts and b the usual gauge fixing auxiliary field. K^μ and L are external sources which are important in the BRS analysis later. G^T_μ denotes the operator transpose of G_μ so that, after elimination of b, the gauge fixing term in $\mathcal{L}^q$ becomes $\mathcal{L}^q_{g.f.} = \frac{1}{2}G^\mu a_\mu{\cdot}G^\nu a_\nu/\xi$. We choose the background gauge covariant form

$$G^\mu a_\mu = \nabla^\mu a_\mu + A^{c\mu}\times a_\mu - \rho^\mu a_\mu\ , \quad G^T_\mu = -D^c_\mu - \rho_\mu\ , \tag{4.3}$$

where D^c_μ is the background gauge covariant derivative. ξ, ρ_μ are thus gauge parameters and along with g are assumed to be arbitrary functions of x, the vector ρ_μ plays a necessary role in the subsequent analysis. The Feynman gauge used later for calculation is obtained by taking

$$\xi = 1\ , \qquad \rho_\mu = v_\mu\ . \tag{4.4}$$

$\mathcal{L}^q$ enjoys the usual BRS invariance, $s\mathcal{L}^q = 0$, under

$$sa_\mu = D'_\mu c\ , \quad sc = -\tfrac{1}{2}g\,c\times c\ , \quad s\bar{c} = -b\ , \quad sb = 0\ , \quad s^2 = 0\ . \tag{4.5}$$

Corresponding to (2.1) we take

$$g' = \mu^{\frac{1}{2}\varepsilon}g\ , \quad \phi' = \mu^{-\frac{1}{2}\varepsilon}\phi\ , \quad J' = \mu^{-\frac{1}{2}\varepsilon}J\ , \quad \xi' = \xi\ , \quad \rho'_\mu = \rho_\mu\ .$$

Assuming no anomalies the renormalised quantum action is constrained by

$$\int\biggl(\frac{\delta S^q_{\rm o}}{\delta f}{\cdot}\frac{\delta S^q_{\rm o}}{\delta J} - b{\cdot}\frac{\delta S^q_{\rm o}}{\delta \bar{c}}\biggr) = 0\ , \quad f = (a_\mu, c)\ , \tag{4.6a}$$

$$\frac{\delta}{\delta b}S^q_{\rm o} = G^\mu a_\mu - \xi\,b\ . \tag{4.6b}$$

(4.6a) is equivalent to the nilpotence of the functional operator

$$\mathcal{Q}^{S^q_{\rm o}} = \int\biggl(\frac{\delta S^q_{\rm o}}{\delta J}{\cdot}\frac{\delta}{\delta f} + \frac{\delta S^q_{\rm o}}{\delta f}{\cdot}\frac{\delta}{\delta J} - b{\cdot}\frac{\delta}{\delta \bar{c}}\biggr)\ . \tag{4.7}$$

These results are obtained order by order from the BRS symmetry (4.5), if $S^q_{{\rm o},n}$ defines a finite theory to n loops and satisfies (4.6a,b) then at $n+1$ loops $\Gamma^{(n+1)}_{\rm div}$ is local and it follows that $\mathcal{Q}^{S^q}\Gamma^{(n+1)}_{\rm div} = 0$. Hence the $n+1$ loop divergence may be cancelled by a suitable choice of $S^q_{{\rm o},n+1}$ also obeying (4.6a,b). Inserting a general ansatz consistent with power counting for $\mathcal{L}^q_{\rm o}$ into (4.6a,b), assuming rotational invariance and background gauge invariance are preserved, gives

$$\begin{aligned} \mathcal{L}^q_{\rm o} &= \mu^{-\varepsilon}\bigg(\frac{Z_g}{g^2}\mathcal{L}(A_{\rm o}) + \tilde K^\mu{\cdot}D'_{{\rm o}\mu}c - \tfrac12 Z_\gamma g\, L{\cdot}(c\times c) - \xi\,\tfrac12 b{\cdot}b + (G^{T\mu}b){\cdot}a_\mu\bigg) \ , \\ A_{{\rm o}\mu} &= A^c_\mu + Z_\beta g\, a_\mu \ , \quad D'_{{\rm o}\mu} = \frac{Z_\gamma}{Z_\beta}\Big(D^c_\mu + Z_\beta g\, a_\mu\times\, + \frac{1}{Z_\gamma g}\partial_\mu(Z_\gamma g)\Big) \ . \end{aligned} \tag{4.8}$$

The novel feature here is the form of $D'_{{\rm o}\mu}$. In addition, to take account of the counterterms required for curved space, we write

$$\tilde{\mathcal{L}}^q_{\rm o} = \mathcal{L}^q_{\rm o} - \mu^{-\varepsilon}\lambda(g)\cdot\mathcal{R} \ , \tag{4.9}$$

with the additional terms just as in (2.5) but restricted to the single coupling g. It is crucial that we are able to use the fundamental theorem that if α_i is any gauge parameter

$$\frac{\partial}{\partial\alpha_i}\tilde S^q_{\rm o} = \mathcal{Q}^{S^q_{\rm o}}X^i \ , \tag{4.10}$$

where X^i is the integral of a local invariant function of ghost number -1 [19]. In this case

$$\begin{aligned} \frac{\delta}{\delta\rho^\mu}S^q_{\rm o} = \mathcal{Q}^{S^q_{\rm o}}\mu^{-\varepsilon}\bar c{\cdot}a_\mu \ , \quad & \frac{\delta}{\delta\xi}S^q_{\rm o} = \mathcal{Q}^{S^q_{\rm o}}\mu^{-\varepsilon}\big(\beta\,\tilde K^\mu{\cdot}a_\mu - \gamma\, L{\cdot}c\big) \ , \\ & \frac{\partial}{\partial\xi}Z_\beta = \beta Z_\beta \ , \ \frac{\partial}{\partial\xi}Z_\gamma = \gamma Z_\gamma \ , \end{aligned} \tag{4.11}$$

so that even for arbitrary $\xi(x)$ it appears just in Z_β, Z_γ and the additional terms in (4.9) depend only on g.

From (4.8) it is easy to see that

$$\begin{aligned} \mathcal{L}^q_{\rm o} &= \mathcal{L}^q(\phi_{\rm o}, J_{\rm o}, g_{\rm o}, \xi_{\rm o}, \rho_{\rm o}) \ , \\ g_{\rm o} &= \mu^{\frac12\varepsilon}Z_g^{-\frac12}g \ , \qquad \xi_{\rm o} = Z_a^2\xi \ , \qquad \rho_{{\rm o}\mu} = \rho_\mu + \frac{1}{Z_a}\partial_\mu Z_a \ , \\ a_{{\rm o}\mu} &= \mu^{-\frac12\varepsilon}Z_a a_\mu \ , \qquad c_{\rm o} = \mu^{-\frac12\varepsilon}Z_c c \ , \quad b_{\rm o} = \mu^{-\frac12\varepsilon}\frac{1}{Z_a}b \ , \quad \bar c_{\rm o} = \mu^{-\frac12\varepsilon}\frac{1}{Z_a}\bar c \ , \\ K^\mu_{\rm o} &= \mu^{-\frac12\varepsilon}\frac{1}{Z_a}K^\mu \ , \quad L_{\rm o} = \mu^{-\frac12\varepsilon}\frac{1}{Z_c}L \ , \qquad Z_a = Z_\beta Z_g^{\frac12} \ , \quad Z_c = Z_\gamma Z_g^{\frac12} \ . \end{aligned} \tag{4.12}$$

Usually there is an arbitrariness in the multiplicative renormalisation of $c, \bar c$ as a consequence of the $U(1)$ ghost number symmetry but this is fixed due to the x dependence of g, ξ. These renormalisation factors are such that $\mathcal{Q}^{S^q_{\rm o}}$ remains invariant under

$\phi \to \phi_{\rm o}$, $J \to J_{\rm o}$. It is now straightforward to define β functions for each coupling and also two essential ξ dependent anomalous dimensions, as in (2.4),

$$
\begin{gathered}
\hat\beta^g = -\tfrac{1}{2}\varepsilon g + \beta^g(g) \ , \quad \hat\beta^\xi = \beta^\xi = -2\gamma_a \xi \ , \quad \hat\beta^\rho_\mu = \beta^\rho_\mu = -\partial_\mu \gamma_a \ , \\
\mu\frac{d}{d\mu} Z_a \Big|_{g_{\rm o},\xi_{\rm o}} = Z_a\gamma_a \ , \quad \mu\frac{d}{d\mu} Z_c \Big|_{g_{\rm o},\xi_{\rm o}} = Z_c\gamma_c \ .
\end{gathered}
\tag{4.13}
$$

Replacing (2.13) we may write

$$
\begin{gathered}
\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \int \bigg(\hat\beta^g \frac{\delta}{\delta g} + \beta^\xi \frac{\delta}{\delta \xi} + \beta^\rho_\mu \frac{\delta}{\delta\rho_\mu} - (\hat\gamma\phi)\cdot\frac{\delta}{\delta\phi} - (\hat\gamma J)\cdot \frac{\delta}{\delta J} \bigg) \ , \\
\hat\gamma\phi = (\gamma_a a_\mu, \, \gamma_c c, \, -\gamma_a b, \, -\gamma_a \bar c) - \tfrac{1}{2}\varepsilon\phi \ , \quad \hat\gamma J = -(\gamma_a K^\mu, \, \gamma_c L) - \tfrac{1}{2}\varepsilon J \ ,
\end{gathered}
\tag{4.14}
$$

so that $\mathcal{D}\tilde S^q_{\rm o} = \mu^{-\varepsilon}\int \beta_\lambda \cdot \mathcal{R}$ as in (2.7).

It is straightforward to now define finite local composite operators, setting $J=0$ for convenience on the r.h.s.,

$$
\begin{aligned}
g\frac{\delta}{\delta g} S^q_{\rm o} = {} & - \frac{2}{g^2}\big[\tfrac{1}{4} F^{\mu\nu}\cdot F_{\mu\nu}\big] \\
& - \big[(G^{T\mu}\bar c)\cdot D'_\mu c + (G^{T\mu} b)\cdot a_\mu\big] + a_\mu \cdot \frac{\delta}{\delta a_\mu} S^q_{\rm o} + c\cdot\frac{\delta}{\delta c} S^q_{\rm o} \ , && (4.15a) \\
\frac{\delta}{\delta\xi} S^q_{\rm o} = {} & - \big[\tfrac{1}{2} b\cdot b\big] \ , && (4.15b) \\
\frac{\delta}{\delta\rho^\mu} S^q_{\rm o} = {} & - \big[\bar c\cdot D'_\mu c + a_\mu\cdot b\big] \ , && (4.15c) \\
\nabla^\mu \frac{\delta}{\delta\rho^\mu} S^q_{\rm o} = {} & \big[(G^{T\mu}\bar c)\cdot D'_\mu c + (G^{T\mu} b)\cdot a_\mu\big] - \big[b\cdot G^\mu a_\mu\big] - \bar c\cdot \frac{\delta}{\delta\bar c} S^q_{\rm o} \ , && (4.15d) \\
a_\mu\cdot\frac{\delta}{\delta a_\mu} S^q_{\rm o} = {} & \big[a_\mu\cdot D_\nu\Big(\frac{1}{g}F^{\mu\nu}\Big) + g\,(G^{T\mu}\bar c)\cdot a_\mu \times c + (G^{T\mu} b)\cdot a_\mu\big] \ , && (4.15e) \\
c\cdot\frac{\delta}{\delta c} S^q_{\rm o} = {} & - \big[(D''_\mu G^{T\mu}\bar c)\cdot c\big] \ , \qquad D''_\mu = D_\mu - v_\mu \ , && (4.15f) \\
\bar c\cdot\frac{\delta}{\delta\bar c} S^q_{\rm o} = {} & \big[\bar c\cdot G^\mu D'_\mu c\big] \ , && (4.15g) \\
b\cdot\frac{\delta}{\delta b} S^q_{\rm o} = {} & \big[-\xi\, b\cdot b + b\cdot G^\mu a_\mu\big] \ . && (4.15h)
\end{aligned}
$$

The l.h.s. of (4.15a,...h) provide a definition for each local operator appearing on the r.h.s., (4.15e,...h) are equation of motion operators (of course it is trivial to eliminate b by setting $b = G^\mu a_\mu/\xi$) while (4.15b,c,d) are given by variations of a gauge parameter and as a consequence of (4.10) or (4.11) do not contribute to matrix elements between physical states. However the basis defined by the l.h.s. of (4.15a,...h) is more convenient from the point of view of computing the operator mixing under renormalisation. Thus from (2.14) and using (4.13)

$$
\begin{gathered}
\mathcal{D}\begin{pmatrix}[\mathcal{O}_g]\\ [\mathcal{O}_\xi]\\ [\mathcal{O}_\rho]\end{pmatrix} = -\begin{pmatrix}\hat\beta^g_{,g} & -2\,\gamma_{a,g}\xi & \gamma_{a,g}\\ 0 & -2(\gamma_a\xi)_{,\xi} & \gamma_{a,\xi}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}[\mathcal{O}_g]\\ [\mathcal{O}_\xi]\\ [\mathcal{O}_\rho]\end{pmatrix} + \begin{pmatrix}\hat\gamma_{,g}\phi\\ \hat\gamma_{,\xi}\phi\\ 0\end{pmatrix}\cdot\frac{\delta}{\delta\phi}S^q_{\rm o}\ ,\\
[\mathcal{O}_g] = \frac{\delta}{\delta g}S^q_{\rm o}\ ,\quad [\mathcal{O}_\xi] = \frac{\delta}{\delta\xi}S^q_{\rm o}\ ,\quad [\mathcal{O}_\rho] = \nabla^\mu\frac{\delta}{\delta\rho^\mu}S^q_{\rm o}\ .
\end{gathered}
\tag{4.16}
$$

Within this framework it is simple to obtain an expression for the trace of the energy momentum tensor as in section 3. Corresponding to (3.2)

$$
\begin{gathered}
\gamma^{\mu\nu}T_{\mu\nu} = \varepsilon\tilde{\mathcal{L}}^q_{\rm o} + \nabla_\mu I^\mu - 2\nabla^\mu\frac{\delta}{\delta\rho^\mu}S^q_{\rm o} - \Big((\Delta\phi)\cdot\frac{\delta}{\delta\phi} + (\Delta J)\cdot\frac{\delta}{\delta J}\Big)S^q_{\rm o}\ ,\\
\Delta\phi = 2(0,\,0,\,b,\,\bar c)\ ,\qquad \Delta J = (2K^\mu,\,4L)\ ,
\end{gathered}
\tag{4.17}
$$

and

$$
\begin{aligned}
\varepsilon\mathcal{L}^q_{\rm o} &= \Big(\hat\beta^g\frac{\partial}{\partial g} + \beta^\xi\frac{\partial}{\partial\xi} + \beta^\rho_\mu\frac{\partial}{\partial\rho_\mu} - (\hat\gamma\phi)\cdot\frac{\partial}{\partial\phi} - (\hat\gamma J)\cdot\frac{\partial}{\partial J}\Big)\mathcal{L}^q_{\rm o}\\
&= \Big(\hat\beta^g\frac{\delta}{\delta g} + \beta^\xi\frac{\delta}{\delta\xi} - (\hat\gamma\phi)\cdot\frac{\delta}{\delta\phi} - (\hat\gamma J)\cdot\frac{\delta}{\delta J}\Big)S^q_{\rm o} + \big(\gamma_a + \tfrac{1}{2}\varepsilon\big)\nabla^\mu\frac{\delta}{\delta\rho^\mu}S^q_{\rm o}\ .
\end{aligned}
\tag{4.18}
$$

Up to terms which vanish for physical matrix elements and for $J=0$,

$$
\gamma^{\mu\nu}T_{\mu\nu} \simeq \hat\beta^g\frac{\delta}{\delta g}\tilde S^q_{\rm o} + \mu^{-\varepsilon}\big(-\beta_\lambda\cdot\mathcal{R} + \nabla_\mu Z^\mu\big)\ ,\quad \hat\beta^g\frac{\delta}{\delta g}\tilde S^q_{\rm o} \simeq -\frac{2\hat\beta^g}{g^3}\big[\tfrac14\,F^{\mu\nu}\cdot F_{\mu\nu}\big]\ ,\tag{4.19}
$$

where $\beta_\lambda\cdot\mathcal{R}$ and Z^μ are just as discussed in section 3 with the same requirement of finiteness of Z^μ. With this definition it is trivial that $\mathcal{D}\gamma^{\mu\nu}T_{\mu\nu} = \mu^{-\varepsilon}\delta_\sigma\int\beta_\lambda\cdot\mathcal{R}$.

It is also simple to extend this treatment to allow for fermions coupled to the quantum gauge field so that $\mathcal{L}^q$ now includes, as well as (4.2),

$$
\begin{aligned}
\mathcal{L}^q_\psi &= \bar\psi\big(\gamma^\mu\overleftrightarrow{D}^\psi_\mu + M\big)\psi - g\,\eta t\cdot c\psi - g\,\bar\psi t\cdot c\bar\eta\ ,\\
\overleftrightarrow{D}^\psi_\mu &= \tfrac12\nabla_\mu - \tfrac12\overleftarrow{\nabla}_\mu + A^\psi_\mu + t\cdot A_\mu\ ,\quad [t_a\,,A^\psi_\mu] = [t_a\,,M] = 0\ .
\end{aligned}
\tag{4.20}
$$

t_a are matrix generators of G in the representation defined by ψ, $[t_a\,,t_b] = f_{abc}\,t_c$, $t_a^\dagger = -t_a$. M is a general mass matrix while A^ψ_μ is an external vector field coupled to a gauge singlet fermion current. $A^\psi_\mu = A^\psi_{\mu\alpha}T_\alpha$ may be regarded as belonging to the Lie algebra of a group $\hat G$ defined by the maximal set of generators $T_\alpha = -T_\alpha^\dagger$, $[T_\alpha\,,T_\beta] = F_{\alpha\beta\gamma}T_\gamma$ commuting with t_a. η, $\bar\eta$ are the sources for the BRS variations of ψ, $\bar\psi$, since $s\psi = -g\,t\cdot c\psi$, $s\bar\psi = -g\,\bar\psi t\cdot c$, so that previous formulae may be extended to include fermions by taking $\phi = (a_\mu,\,c,\,b,\,\bar c,\,\psi,\,\bar\psi)$, $J = (K^\mu,\,L,\,\eta,\,\bar\eta)$ and $f = (a_\mu,\,c,\,\psi,\,\bar\psi)$. $\mathcal{L}_\psi$ is invariant under local $\hat G$ gauge transformations if

$$
\delta A^\psi_\mu = D^\psi_\mu\omega = \partial_\mu\omega + [A^\psi_\mu,\,\omega]\ ,\quad \delta M = [M,\,\omega]\ ,\quad \delta\psi = -\omega\psi\ ,\quad \delta\bar\psi = \bar\psi\omega\ ,\tag{4.21}
$$

for $\omega = \omega_\alpha T_\alpha$.

By BRS symmetry and also local $\hat{G}$ invariance $\mathcal{L}^q_{\rm o}$ is still of the same form (4.12) with in addition

$$\begin{gathered}\psi_{\rm o} = \mu^{-\frac{1}{2}\varepsilon} Z_\psi \psi \ , \quad \bar\psi_{\rm o} = \mu^{-\frac{1}{2}\varepsilon} \bar\psi \bar Z_\psi \ , \quad \eta_{\rm o} = \mu^{-\frac{1}{2}\varepsilon} \eta \frac{1}{Z_\psi} \ , \quad \bar\eta_{\rm o} = \mu^{-\frac{1}{2}\varepsilon} \frac{1}{\bar Z_\psi} \bar\eta \ , \\ M_{\rm o} = Z_m M \ , \qquad [t_a, \, Z_\psi] = [t_a, \, Z_m] = 0 \ , \quad \bar Z_\psi = Z_\psi^\dagger \end{gathered} \tag{4.22}$$

$Z_m(g)$ is independent of the gauge parameters, in contrast to $Z_\psi(g,\xi)$. In general we may expect to require $A^\psi_\mu \to A^\psi_{{\rm o}\mu} = A^\psi_\mu + N\partial_\mu g$ where $N^\dagger = -N$, $[t_a, \, N] = 0$ but in this case for a single coupling g, when N and also Z_ψ are scalars formed from t_a so that $[T_\alpha, \, N] = 0$ and $[T_\alpha, \, Z_\psi] = 0$, it follows that $N = 0$. Besides A^ψ_μ, M is also allowed to have an arbitrary x dependence so further counterterms are necessary and are of the general form

$$\begin{aligned} \tilde{\mathcal{L}}^q_{\rm o} &= \mathcal{L}^q_{\rm o} - \mu^{-\varepsilon}\big(\lambda(g)\cdot\mathcal{R} + \lambda_m(g)\cdot\mathcal{M} + \tfrac14 {\rm tr}(K(g)F^\psi_{\mu\nu}F^{\psi\mu\nu}) + {\rm tr}(\Lambda(g)M^4)\big) \ , \\ \lambda_m\cdot\mathcal{M} &= \tfrac12\big(H\,{\rm tr}(h\,M^2) + {\rm tr}(r\,D^\psi_\mu M D^{\psi\mu}M) + 2\partial_\mu g\,{\rm tr}(s\,MD^{\psi\mu}M) + \partial_\mu g\partial^\mu g\,{\rm tr}(t\,M^2)\big) \ , \\ \lambda_m &= (h, \, r, \, s, \, t) \ , \qquad F^\psi_{\mu\nu} = \partial_\mu A^\psi_\nu - \partial_\nu A^\psi_\mu + [A^\psi_\mu, \, A^\psi_\nu] \ . \end{aligned} \tag{4.23}$$

This treatment enables the definition of further finite local composite operators by

$$\big[\mathcal{O}_m(X)\big] = X\cdot\frac{\delta}{\delta M}S^q_{\rm o} = \big[\bar\psi X\psi\big] \ , \quad X^\dagger = X \ , \quad J^\mu_\alpha = \frac{\delta}{\delta A^\psi_{\mu\alpha}} S^q_{\rm o} = \big[\bar\psi\gamma^\mu T_\alpha\psi\big] \ . \tag{4.24}$$

As a consequence of invariance under (4.21), for $\eta = \bar\eta = 0$,

$$\nabla_\mu J^\mu_\alpha + F_{\alpha\beta\gamma}\,A^\psi_{\mu\beta}\,J^\mu_\gamma = \big[\mathcal{O}_m([M, \, T_\alpha])\big] - (T_\alpha\psi)\cdot\frac{\delta}{\delta\psi}S^q_{\rm o} + (\bar\psi T_\alpha)\cdot\frac{\delta}{\delta\bar\psi}S^q_{\rm o} \ .$$

The renormalisation group may be extended to this case by defining anomalous dimensions γ_ψ, $\bar\gamma_\psi$ for the fermion fields as in (4.13) and also introducing a β function for M,

$$\beta^m = -\gamma_m(g)M \ , \qquad \mu\frac{d}{d\mu}Z_m\Big|_{g_{\rm o}} = Z_m\gamma_m \ . \tag{4.25}$$

For the new terms in (4.23)

$$\begin{gathered} \Big(\varepsilon - \hat\beta^g\frac{\partial}{\partial g} + (\gamma_m M)\cdot\frac{\partial}{\partial M}\Big)\lambda_m\cdot\mathcal{M} = \beta_m\cdot\mathcal{M} \ , \qquad \beta_m = (\beta_h, \, \kappa_r, \, \kappa_s, \, \kappa_t) \ , \\ \Big(\varepsilon - \hat\beta^g\frac{\partial}{\partial g}\Big)K = \beta_K \ , \qquad \Big(\varepsilon - \hat\beta^g\frac{\partial}{\partial g} + 4\gamma_m\Big)\Lambda_m = \beta^m_\Lambda \ . \end{gathered} \tag{4.26}$$

Using $\simeq$ to denote equality up to contributions of equation of motion operators and also operators obtained by variation of a gauge parameter we now have

$$\mathcal{D}\big[\mathcal{O}_m(X)\big] \simeq \big[\mathcal{O}_m(\mathcal{D}X + \gamma_m X)\big] \ , \quad \mathcal{D}\big[\mathcal{O}_g\big] \simeq -\hat\beta^g_{,g}\big[\mathcal{O}_g\big] + \big[\mathcal{O}_m(\gamma_{m,g}M)\big] \ , \quad \mathcal{D}J^\mu_\alpha = 0 \ . \tag{4.27}$$

The additional contributions in (4.23) may also be related to subtractions in correlation functions of composite operators. For instance if $\Gamma^{\mu\nu}_{\alpha\beta}(-p,p)$ denotes the Fourier transform of $\langle J^\mu_\alpha\, J^\nu_\beta\rangle$ on flat space then

$$\Gamma^{R\mu\nu}_{\alpha\beta}(-p,p) = \Gamma^{\mu\nu}_{\alpha\beta}(-p,p) - \mu^{-\varepsilon}\mathrm{tr}(KT_\alpha T_\beta)\big(p^2\delta^{\mu\nu} - p^\mu p^\nu\big) \ . \tag{4.28}$$

Hence, similarly to (2.20),

$$\mathcal{D}\Gamma^{R\mu\nu}_{\alpha\beta}(-p,p) = \mu^{-\varepsilon}\mathrm{tr}(\beta_K T_\alpha T_\beta)\big(p^2\delta^{\mu\nu} - p^\mu p^\nu\big) \ . \tag{4.29}$$

The trace of the energy momentum tensor may be obtained by extending the conformal transformation in (3.1) so that

$$\delta M = \sigma M \ , \quad \Delta\phi = (0,\, 0,\, 2b,\, 2\bar{c},\, \tfrac{3}{2}\psi,\, \tfrac{3}{2}\bar{\psi}) \ , \quad \Delta J = (2K^\mu,\, 4L,\, \tfrac{5}{2}\eta,\, \tfrac{5}{2}\bar{\eta}) \ , \tag{4.30}$$

ensuring still that $\delta\mathcal{L}^\psi = 4\sigma\mathcal{L}^\psi$. In this case (4.17) holds with $I^\mu \to I^\mu + I^\mu_m$ where

$$I^\mu_m = -\mu^{-\varepsilon}\big(-\partial^\mu\mathrm{tr}(h\,M^2) + \mathrm{tr}(r\,MD^{\psi\mu}M) + \partial^\mu g\,\mathrm{tr}(s\,M^2)\big) \ . \tag{4.31}$$

Replacing (4.19) we find the finite expression

$$\begin{aligned}\gamma^{\mu\nu}T_{\mu\nu} \simeq \hat{\beta}^g\big[\mathcal{O}_g\big] - \big[\mathcal{O}_m((1+\gamma_m)M)\big] - \mu^{-\varepsilon}\big(\beta_\lambda\cdot\mathcal{R} + \beta_m\cdot\mathcal{M} + \tfrac{1}{4}\mathrm{tr}(\beta_K F^\psi_{\mu\nu}F^{\psi\mu\nu}) + \mathrm{tr}(\beta_\Lambda M^4)\big)\\ + \mu^{-\varepsilon}\nabla_\mu\big(Z^\mu + Z^\mu_m\big) \ ,\end{aligned} \tag{4.32}$$

where

$$Z_{m\mu} = -\partial_\mu\mathrm{tr}(h\,M^2) + \mathrm{tr}\big((r - s\hat{\beta}^g + r\gamma_m)MD^\psi_\mu M\big) + \partial_\mu g\,\mathrm{tr}\big((s - t\hat{\beta}^g + s\gamma_m)M^2\big) \ . \tag{4.33}$$

This gives rise to new consistency conditions by requiring the finiteness of

$$2h - r + s\hat{\beta}^g - r\gamma_m \ , \tag{4.34a}$$
$$h' - s + t\hat{\beta}^g - s\gamma_m \ , \tag{4.34b}$$

which in turn are equivalent to the finite relations

$$2\beta_h - \kappa_r + \kappa_s\beta^g - \kappa_r\gamma_m = -\Big(\beta^g\frac{\partial}{\partial g} - 2\gamma_m\Big)S \ , \quad S = -\tfrac{1}{2}gs^1 \ , \tag{4.35a}$$
$$\beta_h' - \kappa_s + \kappa_t\beta^g - \kappa_s\gamma_m = -\Big(\beta^g\frac{\partial}{\partial g} - 2\gamma_m + \beta^{g\prime}\Big)T + \gamma_m' S \ , \quad T = -\tfrac{1}{2}gt^1 \tag{4.35b}$$

(4.35a) shows how β_h can be determined in terms of κ_r, κ_s which need only calculations restricted to flat space.

5. Calculations

The various renormalization functions introduced in the previous section are calculable in the usual perturbative loop expansion. To achieve this for the curvature-dependent terms we adopt a method of calculation developed by us earlier, within dimensional regularisation, which allows for the singular poles in ε to be found for an arbitrary spatial metric in a completely covariant fashion [18]. This depends on an expansion of the Green functions, defining the quantum field propagators in the presence of background classical fields and metric, based on the DeWitt [19] heat kernel expansion. The Green functions correspond to the differential operators acting on the quantum fields ϕ when the action is expanded to quadratic order in ϕ. From (4.2), eliminating b and setting $K_\mu,\ L$ to zero and in the gauge (4.4), these are

$$\begin{aligned}\Delta_{\rm a}^{\mu\nu} &= -\,D^{c2}\gamma^{\mu\nu} - 2F^{c\mu\nu}\times\ + Y_{\rm a}^{\mu\nu}\ ,\\ \Delta_{\rm gh} &= -\,D^{c2} + Y_{\rm gh}\,1\ ,\\ Y_{\rm a}^{\mu\nu} &= R^{\mu\nu} + \nabla^\mu v^\nu + \nabla^\nu v^\mu - \gamma^{\mu\nu}(\nabla^\sigma v_\sigma - v^\sigma v_\sigma)\ ,\quad Y_{\rm gh} = -\nabla^\sigma v_\sigma + v^\sigma v_\sigma\ ,\end{aligned} \tag{5.1}$$

where $Y^{\mu\nu}, Y_{\rm gh}$ are gauge singlets, $v_\mu = \partial_\mu g/g$ arising from the x dependent coupling g. For the fermion fields the appropriate second-order operator is

$$\begin{aligned}\Delta_\psi &= \big(-\gamma^\mu D^\psi_\mu + M\big)\big(\gamma^\nu D^\psi_\nu + M\big)\\ &= -\,D^{\psi 2} - \gamma^\mu D^\psi_\mu M - \tfrac{1}{2}\gamma^\mu\gamma^\nu F^\psi_{\mu\nu} + M^2 + \tfrac{1}{4}R\,1\ .\end{aligned} \tag{5.2}$$

At one loop then

$$\Gamma^{(1)} = -\ell{\rm n}\det\Delta_a + \ell{\rm n}\det\Delta_{\rm gh} + \tfrac{1}{2}\,\ell{\rm n}\det\Delta_\psi\ . \tag{5.3}$$

and the additional counterterms necessary for curved space and x dependent g are found from the standard formula for the coincident DeWitt coefficient a_2 as

$$\begin{aligned}\tilde S_{\rm o}^{(1)} = \frac{1}{16\pi^2\varepsilon}\int\bigg(&(n_V - 2n_F)\frac{1}{180}(3F-G) - \frac{1}{24}(2n_V - n_F)R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\\ &+\frac{n_V}{2}\,{\rm tr}\Big(\frac{1}{6}R - Y_{\rm a}\Big)^2 - n_V\,\Big(\frac{1}{6}R - Y_{\rm gh}\Big)^2 - \frac{n_F}{2}\,R^2\bigg)\ .\end{aligned} \tag{5.4}$$

In terms of (2.5) this gives

$$\begin{aligned}\lambda^{(1)}\cdot\mathcal{R} = \frac{1}{16\pi^2\varepsilon}\Big(&-\frac{1}{20}(2n_V + n_F)F + \frac{1}{360}(62n_V + 11n_F)G\\ &+ n_V\big(2G^{\mu\nu}v_\mu v_\nu + Hv^2 - (\nabla^\mu v_\mu)^2 - v^2v^2\big)\Big)\ .\end{aligned} \tag{5.5}$$

At two loops the amplitudes depend on the cubic and quartic pieces in the expansion of $\mathcal{L}^q$, in particular

$$\mathcal{L}_3^q = g\ a_\mu \times a_\nu \cdot D'^c_\mu a_\nu - g\ D'^c_\mu\bar c\cdot a^\mu \times c\ .$$

One loop subdivergences are cancelled by taking

$$
\begin{aligned}
\Delta^{(1)}_{\mathrm{a}\mu\nu} &= \frac{g^2}{16\pi^2\varepsilon}\bigg(\frac{2}{3}(5C-4R)\big(-D^{c2}\gamma_{\mu\nu} + D^c_\mu D^c_\nu - 2F^{c\mu\nu}\times + v^2\gamma_{\mu\nu} - v_\mu v_\nu\big) \\
&\qquad\qquad + 4C\big(\nabla^\sigma v_\sigma\,\gamma_{\mu\nu} + 2v_\nu D^c_\mu + 2v_\mu v_\nu\big)\bigg) , \qquad (5.6)\\
\Delta^{(1)}_{\mathrm{gh}} &= D'^{c\,T}_{\mu} D'^{\mu(1)}_{\mathrm{o}} , \qquad D'^{\mu(1)}_{\mathrm{o}} = \frac{g^2}{16\pi^2\varepsilon}\,(D^c_\mu - v_\mu) ,
\end{aligned}
$$

which are in accord with (4.8) assuming the usual one loop results

$$
Z^{(1)}_g = \frac{2}{3}(11C-4R)\frac{g^2}{16\pi^2\varepsilon} , \quad Z^{(1)}_\beta = -\tfrac{1}{2}(3+\xi)\frac{Cg^2}{16\pi^2\varepsilon} , \quad Z^{(1)}_\gamma = -\xi\,\frac{Cg^2}{16\pi^2\varepsilon} , \qquad (5.7)
$$

for $\xi = 1$ and where $f_{acd}f_{bcd} = C\delta_{ab}$, $\mathrm{tr}(t_a t_b) = -R\delta_{ab}$. After subtraction of subdivergences and taking into account the various d-dependent factors in (2.6) we obtain

$$
\begin{aligned}
\lambda^{(2)}\cdot\mathcal{R} = \frac{n_V\,g^2}{(16\pi^2\varepsilon)^2}\bigg(&\frac{\varepsilon}{9}\Big(C-\frac{7}{8}R\Big)F \\
&- \frac{2}{3}(11C-4R)\big(2G^{\mu\nu}v_\mu v_\nu + Hv^2 - (\nabla^\sigma v_\sigma)^2 - 2v^2\nabla^\sigma v_\sigma - 2v^2v^2\big) \\
&+ \frac{\varepsilon}{6}(51C-20R)\big(2G^{\mu\nu}v_\mu v_\nu - (\nabla^\sigma v_\sigma)^2\big) + \frac{\varepsilon}{6}(29C-12R)Hv^2 \\
&- \frac{\varepsilon}{3}(7C-4R)v^2\nabla^\sigma v_\sigma - \frac{5\varepsilon}{9}(23C-4R)v^2v^2\bigg) . \qquad (5.8)
\end{aligned}
$$

An important consistency check is that the double poles are in accord with the renormalization group equation (2.8).

In addition we have calculated the M dependent terms in (4.23). At one loop

$$
\lambda^{(1)}\cdot\mathcal{M} = \frac{1}{16\pi^2\varepsilon}\Big(H\,\mathrm{tr}(M^2) + 2\,\mathrm{tr}(D^\psi_\mu M D^{\psi\mu} M)\Big) , \qquad (5.9)
$$

while at two loops

$$
\begin{aligned}
\lambda^{(2)}\cdot\mathcal{M} = \frac{1}{(16\pi^2\varepsilon)^2}\Big(&\tfrac{1}{2}(12-11\varepsilon)g^2H\,\mathrm{tr}(t^2M^2) + (12-5\varepsilon)g^2\,\mathrm{tr}(t^2D^\psi_\mu M D^{\psi\mu}M) \\
&+ (12-11\varepsilon)2g\partial^\mu g\,\mathrm{tr}(t^2 M D^\psi_\mu M) - 24\varepsilon\,\partial^\mu g\partial_\mu g\,\mathrm{tr}(t^2 2M^2)\Big) . \qquad (5.10)
\end{aligned}
$$

Again the double poles are in accord with the renormalization group equation (4.26) since

$$
\gamma^{(1)}_m = -6t^2\,\frac{g^2}{16\pi^2} . \qquad (5.11)
$$

From (5.5) and (5.8) we obtain, apart from (2.9), to two-loop order

$$
\begin{aligned}
\beta_a &= \frac{1}{16\pi^2}\Big(-\frac{1}{20}(2n_V+n_F)+\frac{2}{9}n_V\Big(C-\frac{7}{8}R\Big)h\Big)\ , \qquad h=\frac{g^2}{16\pi^2}\ ,\\
\chi^g = -2\chi^a &= \frac{n_V}{16\pi^2 g^2}\Big(4+\frac{4}{3}(51C-20R)h\Big)\ ,\\
\chi^f &= \frac{n_V}{16\pi^2 g^2}\Big(2+\frac{2}{3}(29C-16R)h\Big)\ , \qquad \chi^e=0\ ,\\
\chi^b &= \frac{n_V}{16\pi^2 g^3}\Big(4+\frac{16}{3}(11C-4R)h\Big)\ ,\\
\chi^c &= -\frac{n_V}{16\pi^2 g^4}\Big(8+\frac{4}{9}(341C-76R)h\Big)\ ,
\end{aligned}
\tag{5.12}
$$

while $16\pi^2\beta_K=-\frac{4}{3}+4t^2h$. We have checked that these expressions are consistent with the various conditions flowing from the finiteness of (3.9a...h). In particular from (3.15) and (3.20) to this order

$$
gW = 4Y = \frac{n_V}{16\pi^2}\Big(2+\frac{1}{3}(51C-20R)h\Big)\ . \tag{5.13}
$$

Using (3.19) and (3.22) along with

$$
\frac{1}{g}\beta^g(g) = -\beta_0\, h-\beta_1\, h^2-\beta_2\, h^2\ , \quad \beta_0=\frac{1}{3}(11C-4R)\ , \tag{5.14}
$$

gives

$$
128\pi^2\beta_c = \frac{2}{3}\beta_0\beta_1 n_V h^3+\Big(\beta_0\beta_2+\beta_0\beta_1\Big(\frac{17}{4}C-\frac{5}{3}R\Big)+\frac{2}{3}{\beta_1}^2\Big)n_V h^4, \tag{5.15a}
$$

$$
16\pi^2\beta_b = 16\pi^2\beta_b^0+\frac{1}{8}\beta_1 n_V h^2+\frac{1}{6}\Big(\beta_1\Big(\frac{17}{4}C-\frac{5}{3}R\Big)+\beta_2\Big)n_V h^3\ . \tag{5.15b}
$$

Of course beyond the lowest-order contribution, to order $\mathrm{O}(h^3)$ and $\mathrm{O}(h^2)$ in (5.15a) and (5.15b) respectively, the results are sensitive to the choice of renormalisation scheme. The lowest order terms obtained here agree, for the appropriate choice of β_0,β_1, with previous results for QED and pure gauge theories [12].

In addition from (5.9) and (5.10) we obtain

$$
\begin{aligned}
16\pi^2\,\beta_h &= 2-22t^2h\ ,\\
16\pi^2\kappa_r &= 4-20t^2h\ ,\\
16\pi^2\, g\,\kappa_s &= -\,44t^2h\ , \qquad 16\pi^2\, g^2\kappa_t=-96t^2h\ .
\end{aligned}
\tag{5.16}
$$

Since S,T are first nonzero at two loops it is straightforward to check that these results satisfy (4.35a, b).

As was mentioned in the introduction it may be possible to obtain a version of Zamolodchikov's c-theorem in four-dimensional field theories and to the extent that χ^g is positive definite this has been verified by our perturbative treatment as in (3.17b) with C defined by (3.18). For fermions coupled to gauge fields with a simple gauge group so that there is a single coupling g there is the usual UV-fixed point as $g \to 0$ so long as $\beta_0 > 0$ and $C \to C_0$ where for an $SU(N)$ gauge theory with f fermions in the fundamental representation C_0 is given by (1.2).

An infrared stable fixed point may be realised perturbatively if $0 < \beta_0 \ll -\beta_1$ in (5.14), when,

$$h_* = \frac{g_*^2}{16\pi^2} \approx \frac{\beta_0}{|\beta_1|}\Big(1 + \frac{\beta_0\beta_2}{{\beta_1}^2}\Big) \ .$$

In the large-N limit this situation may be achieved with h_*N expressible as a power series in $1/N$, as was realised some time ago [15], if there are sufficiently many fermions. Choosing $f = \frac{11}{2}N - k$ with $k \ll N$ then, since $R = \frac{1}{2}f$,

$$\beta_0 = \frac{2}{3}\,k \ , \quad -\beta_1 \approx \frac{25}{2}\,N^2 - \frac{13}{3}\,Nk \ , \quad -\beta_2 \approx \frac{701}{12}\,N^3 \ , \tag{5.17}$$

using the results of Tarasov et al. [20] for β_2 which they obtained in a $\overline{MS}$ scheme consistent with our calculations. In this case

$$h_*N \approx \frac{4}{75}\,\frac{k}{N}\Big(1 + \frac{548}{75^2}\,\frac{k}{N}\Big) \ . \tag{5.18}$$

Although in the mass-independent dimensional regularisation scheme used here this fixed point is independent of fermion mass terms it is only relevant to the long distance behaviour when all physical masses are zero so we suppose that $M = 0$. From (5.15b) we find, despite the unpleasant numerical coefficient in β_2 in (5.17), the relatively simple form

$$C_* - C_0 \approx -\frac{8}{5}\,k^2 - \frac{1}{5}\Big(\frac{28}{25}\Big)^2\frac{k^3}{N} \ . \tag{5.19}$$

The arbitrariness in the choice of regularisation procedure disappears at the fixed point where also $\beta_c = 0$. At this fixed point the anomalous dimension of the scalar operator $\frac{1}{4}[F^{\mu\nu} \cdot F_{\mu\nu}]$ becomes $\beta^{g\prime} \approx 2\beta_0^2/|\beta_1| \approx (4/15)^2 k^2/N^2$ and also

$$\gamma_m(g) \approx 3\,Nh + \frac{31}{4}\,(Nh)^2 \ , \quad \gamma_m(g_*) \approx \frac{4}{25}\,\frac{k}{N} + \frac{3}{25}\,\Big(\frac{14}{25}\Big)^2\Big(\frac{k}{N}\Big)^2 \ . \tag{5.20}$$

The interpretation of these results is nevertheless unclear but will perhaps be of relevance when the scale-invariant theory defined in this fashion is better understood.

6. Scalar Field Theories

The gauge theories treated in sections 4 and 5 depend only on a single coupling g. This restriction may be relaxed by considering renormalisable field theories for a multi-component scalar field ϕ_i when the basic coupling becomes a symmetric tensor $g_{ijk\ell}$. Neglecting any other fields the initial Lagrangian has the form

$$\begin{aligned} \mathcal{L} &= K(A,\phi) + V(\phi) \ , \\ K(A,\phi) &= \tfrac{1}{2}(D^\mu\phi)^T D_\mu\phi \ , \qquad D_\mu\phi = \partial_\mu\phi + A_\mu\phi \ , \quad A_\mu^T = -A_\mu \ , \end{aligned} \tag{6.1}$$

for $V(\phi)$ a general quartic polynomial in ϕ, $\partial_i\partial_j\partial_k\partial_\ell V = g_{ijk\ell}$ and A_μ a background gauge field.

Conventionally in order to ensure a finite quantum field theory it is sufficient to take

$$\begin{aligned} \tilde{\mathcal{L}}_{\mathrm{o}} &= \mathcal{L}_{\mathrm{o}} - \mu^{-\varepsilon}\tfrac{1}{4}\,\mathrm{tr}\big(\mathcal{K}F_{\mu\nu}F^{\mu\nu}\big) \ , \\ \mathcal{L}_{\mathrm{o}} &= K(A,\phi_{\mathrm{o}}) + V_{\mathrm{o}}(\phi_{\mathrm{o}}) \ , \qquad \phi_{\mathrm{o}i} = \mu^{-\frac{1}{2}\varepsilon} Z_{ij}\phi_j \ , \\ V_{\mathrm{o}}(\phi_{\mathrm{o}}) &= \mu^{-\varepsilon}\big(V(\phi) + L^V(\phi)\big) \ , \quad L^V = \mathcal{O}\big(V''^2, V'''^4, V''V'''^2\big) \ , \end{aligned} \tag{6.2}$$

assuming $V(\phi)$ is gauge invariant under gauge transformations on A_μ, ϕ, so that $\partial_\mu V(\phi) = V'(\phi)^T D_\mu\phi$. The corresponding β functions are defined as usual as in (2.4), in particular

$$\hat{\beta}^V(\phi) = \mu\frac{d}{d\mu}V(\phi)\Big|_{V_{\mathrm{o}},\,\phi_{\mathrm{o}}} \ , \qquad \partial_i\partial_j\partial_k\partial_\ell\hat{\beta}^V = \hat{\beta}^g_{ijk\ell} = -\varepsilon\, g_{ijk\ell} + \beta^g_{ijk\ell} \ . \tag{6.3}$$

To three loop order, with dimensional regularisation and minimal subtraction so that in (6.2) L^V and $Z-1$ contain just poles in ε,

$$\begin{aligned} \hat{\beta}^V &= \varepsilon V + V_i\hat{\gamma}_{ij}\phi_j \\ &\quad + \frac{1}{16\pi^2}\tfrac{1}{2}\,V_{ij}V_{ij} - \frac{1}{(16\pi^2)^2}\tfrac{1}{2}\,V_{ij}V_{ik\ell}V_{jk\ell} \\ &\quad + \frac{1}{(16\pi^2)^3}\Big(\frac{1}{4}\,g_{ijmn}\,g_{k\ell mn}V_{ik}V_{j\ell} - \frac{3}{16}\,g_{ik\ell m}\,g_{jk\ell m}V_{in}V_{jn} + 2\,g_{ik\ell m}V_{ij}V_{k\ell n}V_{jmn} \\ &\quad - \frac{1}{4}\,g_{k\ell mn}V_{ij}V_{ik\ell}V_{jmn} - \frac{1}{8}\,V_{ik\ell}V_{jk\ell}V_{imn}V_{jmn} + \frac{1}{2}\zeta(3)\,V_{ijk}V_{i\ell m}V_{\ell jn}V_{nmk}\Big) \ , \\ \hat{\gamma}_{ij} &= -\tfrac{1}{2}\varepsilon\,\delta_{ij} + \frac{1}{(16\pi^2)^2}\frac{1}{12}\,g_{ik\ell m}\,g_{jk\ell m} - \frac{1}{(16\pi^2)^3}\frac{1}{16}\,g_{ik\ell m}\,g_{jknp}\,g_{\ell mnp} \ , \end{aligned} \tag{6.4}$$

where $\partial_i V = V_i, \dots, V_{ijk\ell} = g_{ijk\ell}$. The result for $\hat{\beta}^V$ was partially contained in our previous background field calculations [17] and was completed by using old results for the ε expansion of individual three loop graphs [21]. In general Z is arbitrary up to $Z \to OZ$ for $O^TO = 1$ but conventionally this is resolved by requiring that Z and hence γ is symmetric.

If the scalar field theory described by $\mathcal{L}$ in (6.1) is extended to curved space then it is convenient to take

$$K(A,\phi) = \tfrac{1}{2}(D^\mu\phi)^T D_\mu\phi + \tfrac{1}{8}(d-2)\phi^T\phi\, H \ , \qquad H = \frac{R}{d-1} \ , \tag{6.5}$$

since this gives a conformally invariant contribution to the action for general d when $\delta\gamma^{\mu\nu} = 2\sigma\,\gamma^{\mu\nu}$ and $\delta\phi_i = \frac{1}{2}(d-2)\sigma\phi_i$. Furthermore if the couplings in V are allowed to be x dependent and A_μ is also regarded as an arbitrary anti-symmetric gauge field then, in order to define a finite quantum field theory, it is sufficient, by discarding total derivatives, to include additional counterterms restricted to the form

$$\tilde{\mathcal{L}}_{\rm o} = \mathcal{L}_{\rm o} + \mathcal{Q}(\phi_{\rm o}) - \mu^{-\varepsilon}\tilde\lambda\cdot\mathcal{R} \ , \tag{6.6a}$$

$$\mathcal{L}_{\rm o} = K(A_{\rm o},\phi_{\rm o}) + V_{\rm o}(\phi_{\rm o}) \ , \qquad A_{{\rm o}\mu} = A_\mu + N_I(D_\mu g)_I \ , \ \ N_I^T = -N_I \ , \tag{6.6b}$$

$$\tilde\lambda\cdot\mathcal{R} = \lambda\cdot\mathcal{R} + \tfrac{1}{4}\,{\rm tr}(\mathcal{K}F_{\mu\nu}F^{\mu\nu}) + \tfrac{1}{2}\,{\rm tr}(\mathcal{P}_{IJ}F^{\mu\nu})(D_\mu g)_I(D_\nu g)_J \ , \tag{6.6c}$$

$$\mathcal{Q} = \eta\, H + \delta_I\,(D^2g)_I + \tfrac{1}{2}\epsilon_{IJ}\,(D_\mu g)_I(D^\mu g)_J \ , \tag{6.6d}$$

where $g_I \equiv g_{ijk\ell}$. In these results we have assumed manifest invariance, as is automatic in the background field formalism using dimensional regularisation, under simultaneous gauge transformations on ϕ, A_μ, $\delta\phi = -\omega\phi$, $\delta A_\mu = \partial_\mu\omega + [A_\mu,\,\omega]$, $\omega^T = -\omega$, and also on the couplings in V, $\delta V(\phi) = V'(\phi)^T\omega\phi$, so that $(D_\mu g)_I$ is defined as the appropriate covariant derivative. Hence $\lambda\cdot\mathcal{R}$ is just as in (2.5), depending on the dimensionless coupling g_I, but with $\partial_\mu \to D_\mu$. In general in (6.6a) $\mathcal{Q}$ involves operators of dimension two or less formed by a quadratic polynomial in ϕ, with no derivatives, proportional to H or two covariant derivatives of the couplings in V so that the overall dimension of $\mathcal{Q}$ is four. However in (6.6d), and henceforth, we have assumed for simplicity that $V(\phi)$ contains no terms cubic in ϕ which implies that we can also assume the structure for $\mathcal{Q}$, and hence also for η, δ_I and ϵ_{IJ},

$$\mathcal{Q}(\phi_{\rm o}) = \mu^{-\varepsilon}L^{\mathcal{Q}}(\phi) \ , \qquad L^{\mathcal{Q}}(\phi) = \mathcal{O}\big(\phi^2, V''(\phi)\big) \ , \tag{6.7}$$

where $L^{\mathcal{Q}}$ contains only poles in ε. In order to obtain renormalisation group equations as in (2.7) additional β functions are necessary, in particular

$$\begin{gathered}\hat\beta^V + \beta^{\mathcal{Q}} - \beta_{\tilde\lambda}\cdot\mathcal{R} = \mu\frac{d}{d\mu}V\Big|_{V_{\rm o}+\mathcal{Q}-\mu^{-\varepsilon}\tilde\lambda\cdot\mathcal{R},\,\phi_{\rm o}} \ , \quad \beta^A_\mu = \mu\frac{d}{d\mu}A_\mu\Big|_{A_{\rm o},\,g_{\rm o}} = \rho_I(D_\mu g)_I \ , \\ \beta^{\mathcal{Q}} = \beta^\eta\, H + \beta^\delta_I\,(D^2g)_I + \tfrac{1}{2}\beta^\epsilon_{IJ}\,(D_\mu g)_I(D^\mu g)_J \ , \\ \beta_{\tilde\lambda}\cdot\mathcal{R} = \beta_\lambda\cdot\mathcal{R} + \tfrac{1}{4}\,{\rm tr}(\beta_{\mathcal{K}}F_{\mu\nu}F^{\mu\nu}) + \tfrac{1}{2}\,{\rm tr}(\beta_{IJ}F^{\mu\nu})(D_\mu g)_I(D_\nu g)_J \ .\end{gathered} \tag{6.8}$$

As before we obtain*

$$\Big(\big(\hat\beta^V + \beta^{\mathcal{Q}}\big)\cdot\frac{\partial}{\partial V} + \beta^A_\mu\cdot\frac{\partial}{\partial A_\mu} - (\hat\gamma\phi)\cdot\frac{\partial}{\partial\phi} - \varepsilon\Big)\tilde{\mathcal{L}}_{\rm o} = \mu^{-\varepsilon}\beta_{\tilde\lambda}\cdot\mathcal{R} \ , \tag{6.9}$$

* If $V(\phi) = \sum_n v_n P_n(\phi)$, for P_n a complete set of functions of x and monomials in ϕ of degree up to 4, and also $B(\phi) = \sum_n b_n P_n(\phi)$ then $B\cdot\dfrac{\partial}{\partial V} = \sum_n b_n\dfrac{\partial}{\partial v_n}$.

which implies

$$\beta^{\mathcal{Q}} = \Big(\varepsilon - \hat\beta^V\cdot\frac{\partial}{\partial V} - \beta^A_\mu\cdot\frac{\partial}{\partial A_\mu} + (\hat\gamma\phi)\cdot\frac{\partial}{\partial\phi}\Big)L^{\mathcal{Q}} - \beta^{\mathcal{Q}}\cdot\frac{\partial}{\partial V}L^V \ , \tag{6.10a}$$

$$\beta_{\tilde\lambda}\cdot\mathcal{R} = \Big(\varepsilon - \hat\beta^g_I\frac{\partial}{\partial g_I} - \beta^A_\mu\cdot\frac{\partial}{\partial A_\mu}\Big)\tilde\lambda\cdot\mathcal{R} + \beta^{\mathcal{Q}}\cdot\frac{\partial}{\partial V}L^{\mathcal{Q}} \ , \tag{6.10b}$$

so that $\beta^{\mathcal{Q}}$, $\beta_{\tilde\lambda}$ are determined as usual just from the simple ε poles in $L^{\mathcal{Q}}$, $\tilde\lambda$.

Since $\tilde{\mathcal{L}}_{\rm o}$ has been constructed for arbitrary x dependent couplings finite local composite operators may be defined by functional differentiation,

$$\big[B(\phi)\big] = B\cdot\frac{\delta}{\delta V}\tilde S_{\rm o} \ , \qquad \big[(D^\mu\phi)^T\omega\phi\big] = \omega\cdot\frac{\delta}{\delta A_\mu}\tilde S_{\rm o} \ , \quad \omega^T = -\omega \ . \tag{6.11}$$

Using gauge invariance of $\tilde S_{\rm o}$ it is easy to derive the equation

$$\begin{aligned}\nabla_\mu\big[(D^\mu\phi)^T\omega\phi\big] - \big[(D^\mu\phi)^T D_\mu\omega\,\phi\big] = \big[V'(\phi)^T\omega\phi\big] - (\omega\phi)\cdot\frac{\delta}{\delta\phi}\tilde S_{\rm o} \ ,\\ D_\mu\omega = \partial_\mu\omega + [A_\mu,\omega] \ ,\end{aligned} \tag{6.12}$$

while it is also useful to require the relation

$$\big[(D_\mu\phi)^T\sigma\phi\big] = \tfrac{1}{2}\big(\partial_\mu\big[\phi^T\sigma\phi\big] - \big[\phi^T D_\mu\sigma\,\phi\big]\big) \ , \qquad \sigma^T = \sigma \ , \tag{6.13}$$

by taking it as a definition of the l.h.s.

Within this framework it is straightforward to adapt the treatment of sections 2 and 3 to derive a finite expression for the trace of the energy momentum tensor. Assuming in (3.1) $\Delta\phi_i = \phi_i$, so that $\delta K(A,\phi) = 4\sigma\, K(A,\phi) + \frac{1}{2}\nabla_\mu(\phi^T\phi\partial^\mu\sigma) - \frac{1}{4}\varepsilon\,\phi^T\phi\nabla^2\sigma$, then using (3.6) (3.2) is replaced by

$$\begin{aligned}\gamma^{\mu\nu}T_{\mu\nu} &= \varepsilon\tilde{\mathcal{L}}_{\rm o} - \Delta V_{\rm o} - \Delta\mathcal{Q} + \nabla_\mu\big(I^{\mathcal{Q}} + \tilde I\big)^\mu - \phi\cdot\frac{\delta}{\delta\phi}\tilde S_{\rm o} \ ,\\ \Delta V_{\rm o} + \Delta\mathcal{Q} &= \Big(4 - \phi\cdot\frac{\partial}{\partial\phi}\Big)V_{\rm o} + \Big(2 - \phi\cdot\frac{\partial}{\partial\phi}\Big)\mathcal{Q} = (\Delta V)\cdot\frac{\partial}{\partial V}\tilde{\mathcal{L}}_{\rm o} \ ,\\ \Delta V(\phi) &= \Big(4 - \phi\cdot\frac{\partial}{\partial\phi}\Big)V(\phi) \ , \qquad I^{\mathcal{Q}}_\mu = -\tfrac{1}{4}\varepsilon\,\partial_\mu(\phi_{\rm o}^T\phi_{\rm o}) + 2\,\partial_\mu\eta + (2-\varepsilon)\delta_I(D_\mu g)_I \ ,\end{aligned} \tag{6.14}$$

where $\tilde I_\mu$ comes from just the $\tilde\lambda\cdot\mathcal{R}$ term in $\tilde{\mathcal{L}}_{\rm o}$. In ΔV the couplings in V are just multiplied by their canonical dimension so that $-\Delta V$ is the classical expression for $\gamma^{\mu\nu}T_{\mu\nu}$ after using the equation of motion. (6.14) may be simplified by using (6.9) and (6.11), noting that $\tilde{\mathcal{L}}_{\rm o}$ contains no derivatives of the couplings in ΔV, to give

$$\begin{aligned}\gamma^{\mu\nu}T_{\mu\nu} = {}& \big[\hat\beta^V + \beta^{\mathcal{Q}} - \Delta V\big] + \big[(D^\mu\phi)^T\beta^A_\mu\phi\big] - \mu^{-\varepsilon}\beta_{\tilde\lambda}\cdot\mathcal{R}\\ & - \big((1+\hat\gamma)\phi\big)\cdot\frac{\delta}{\delta\phi}\tilde S_{\rm o} + \nabla_\mu\big(J^\mu + \mu^{-\varepsilon}\tilde Z^\mu(\hat\beta^g,\beta^A)\big) \ .\end{aligned} \tag{6.15}$$

$\tilde{Z}_\mu$ is the sum of the obvious extension of (3.8) to this case and

$$\begin{aligned}&-\mathcal{A}_{IJ}\,(D^2 g)_I(\beta^A_\mu g)_J - \tfrac{1}{2}\mathcal{B}_{IJK}\,(D_\mu g)_I(D^\mu g)_J(\beta^A_\mu g)_K\\ &-\mathrm{tr}\big(\mathcal{P}_{IJ}\beta^{A\nu}\big)(D_\mu g)_I(D_\nu g)_J - \tfrac{1}{2}\mathrm{tr}\big(\{\mathcal{K},\,\beta^{A\nu}\}F_{\mu\nu}\big) - \mathrm{tr}\big(\mathcal{P}_{IJ}F_{\mu\nu}\big)\hat\beta^g_I(D^\nu g)_J\ ,\end{aligned}\tag{6.16}$$

resulting from the presence of A_μ in the covariant derivatives in $\lambda\cdot\mathcal{R}$ and also the extra terms in (6.6c). From (6.6d) and (6.14), using

$$\Big(\hat\beta^g\cdot\frac{\partial}{\partial g} - (\hat\gamma\phi)\cdot\frac{\partial}{\partial\phi}\Big)K(A,\phi_{\rm o}) = \Big(\hat\beta^g\cdot\frac{\delta}{\delta g} - (\hat\gamma\phi)\cdot\frac{\delta}{\delta\phi}\Big)\int K(A,\phi_{\rm o}) + \tfrac{1}{4}\varepsilon\,\nabla^2\big(\phi_{\rm o}^T\phi_{\rm o}\big)\ ,$$

we obtain

$$\begin{aligned}J_\mu ={}& (D_{{\rm o}\mu}\phi_{\rm o})^T N_I\hat\beta^g_I\,\phi_{\rm o} + \tfrac{1}{4}\varepsilon\,\partial_\mu\big(\phi_{\rm o}^T\phi_{\rm o}\big)\\ &+ I^{\mathcal{Q}}_\mu + \epsilon_{IJ}\,(D_\mu g)_I\hat\beta^g_J - \partial_\mu(\delta_I\hat\beta^g_I) + 2\,\delta_I(D_\mu\hat\beta^g)_I + \delta_I(\beta^A_\mu g)_I\ .\end{aligned}\tag{6.17}$$

From (6.14) the $\varepsilon\,\partial_\mu(\phi_{\rm o}^T\phi_{\rm o})$ terms cancel in J_μ. This is a direct consequence of the particular d-dependent factors in (6.5), other choices lead to non minimal terms in subsequent equations which complicates the analysis. Since all other terms in (6.15) are finite operators the sum of the last two terms, which are a total divergence, must also be finite. Hence we write

$$J_\mu = \big[(D_\mu\phi)^T S\phi\big] + J_{\Theta\mu} + \mu^{-\varepsilon}S\cdot\frac{\delta}{\delta A^\mu}\int\tilde\lambda\cdot\mathcal{R}\ ,\qquad S^T = -S\ ,\tag{6.18a}$$

$$J_{\Theta\mu} = \partial_\mu\big(2\,\eta - \delta_I\hat B^g_I\big) + \delta_I\big((2-\varepsilon)D_\mu g + 2D_\mu\hat B^g + B^A_\mu g\big)_I + \epsilon_{IJ}\,(D_\mu g)_I\hat B^g_J\ ,\tag{6.18b}$$

$$\hat B^g_I = \hat\beta^g_I - (Sg)_I\ ,\qquad B^A_\mu = \beta^A_\mu + D_\mu S = P_I(D_\mu g)_I\ ,\quad P_I = \rho_I + \partial_I S\ .\tag{6.18c}$$

In order for (6.18a), given (6.17), to be possible it is necessary that

$$N_I\hat\beta^g_I = S + N_I(Sg)_I \quad\text{or}\quad N_I\hat B^g_I = S \quad\Rightarrow\quad S = -N^1_I g_I\ ,\tag{6.19}$$

so that S is determined by the simple poles in N_I. From (6.18a) we may then write

$$\nabla_\mu\big(J^\mu + \mu^{-\varepsilon}\tilde Z^\mu(\hat\beta^g,\beta^A)\big) = \nabla_\mu\big(\big[(D^\mu\phi)^T S\phi\big] + J^\mu_\Theta + \mu^{-\varepsilon}\tilde Z^\mu(\hat B^g, B^A)\big)\ ,\tag{6.20}$$

and since the operators appearing in J^μ_Θ are independent of $\tilde Z^\mu$ they must be separately finite, at least up to contributions with a vanishing divergence, and hence

$$J_{\Theta\mu} = \partial_\mu[T] + [U_I](D_\mu g)_I\ .\tag{6.21}$$

The finiteness of $\tilde Z^\mu$ leads to similar conditions to (3.9a,...h) involving $\hat B^g$ and B^A, apart from the extra terms in (6.17). Since the couplings for the operators appearing in J^μ_Θ are present only in $V_{\rm o}$ without any derivatives then from (6.18b) and (6.21) we can write

$$2\,\eta - \delta_I\hat B^g_I = T\cdot\frac{\partial}{\partial V}V_{\rm o}\ ,\tag{6.22a}$$

$$(2-\varepsilon)\delta_I + \delta_J\big(2\,\partial_I\hat B^g + P_I g\big)_J + \epsilon_{IJ}\hat B^g_J = U_I\cdot\frac{\partial}{\partial V}V_{\rm o}\ ,\tag{6.22b}$$

$$T(\phi) = L^{\delta,1}_I(\phi)g_I\ ,\qquad U_I(\phi) = -\,3\,L^{\delta,1}_I(\phi) - L^{\epsilon,1}_{IJ}(\phi)g_J\ ,\tag{6.22c}$$

where $\delta_I(\phi_{\rm o}) = \mu^{-\varepsilon}\sum_n L_I^{\delta,n}(\phi)\,\varepsilon^{-n}$ with a similar expression for ϵ_{IJ}.

If (6.12) is used then (6.15) becomes

$$
\begin{aligned}
\gamma^{\mu\nu}T_{\mu\nu} = {}& \big[\hat{B}^V + \beta^{\mathcal{Q}} - \Delta V\big] + \big[(D^\mu\phi)^T B^A_\mu\phi\big] - \mu^{-\varepsilon}\beta_{\tilde{\lambda}}\cdot\mathcal{R} \\
& - \big((1+\hat{\gamma}+S)\phi\big)\cdot\frac{\delta}{\delta\phi}\tilde{S}_{\rm o} + \nabla_\mu\big(J^\mu_\Theta + \mu^{-\varepsilon}\tilde{Z}^\mu(\hat{B}^g, B^A)\big)\ .
\end{aligned} \tag{6.23}
$$

where

$$
\hat{B}^V(\phi) = \hat{\beta}^V(\phi) + V'(\phi)^T S\phi\ . \tag{6.24}
$$

The definition for $\hat{B}^g$ in (6.18c) is consistent with that for $\hat{B}^V$ in (6.24). The replacement of the β functions β^V, β^A_μ by B^V, B^A_μ in the final expression for the trace of the energy momentum tensor in (6.23) is a reflection of the underlying gauge invariance since this leads to an ambiguity in the definition of the β functions which is compensated by the additional S dependent piece in (6.18c), (6.24). An entirely equivalent situation arises in relation to diffeomorphisms in non linear σ models [22], in both cases the energy momentum tensor is independent of such arbitrariness.

It is important to check that the consistency conditions (6.19) and (6.22a,b) are in accord with renormalisation group equations, such as (6.9), determining the higher order poles. Applying $\hat{\beta}^g_I\partial_I$ to (6.19) and using (6.8) and $(Sg)_I\partial_I\,\hat{\beta}^g_J = (S\hat{\beta}^g)_J$ shows that this is equivalent to

$$
\hat{\beta}^g_I\frac{\partial}{\partial g_I}S = -\rho_I\hat{B}^g_I \quad \text{or} \quad P_I\hat{B}^g_I = 0\ . \tag{6.25}
$$

Similarly applying $\varepsilon - \hat{\beta}^V\cdot\dfrac{\partial}{\partial V} + (\hat{\gamma}\phi)\cdot\dfrac{\partial}{\partial\phi}$ to both sides of (6.22a), along with (6.10a) and (6.25), results in*

$$
2\,\beta^\eta - \beta^\delta_I\hat{B}^g_I = -\hat{\beta}^V\cdot\frac{\partial}{\partial V}T + T\cdot\frac{\partial}{\partial V}\hat{\beta}^V\ , \tag{6.26}
$$

while (6.22b) corresponds to

$$
\begin{aligned}
(2-\varepsilon)\beta^\delta_I &+ \beta^\delta_J\big(2\,\partial_I\hat{B}^g + P_I g\big)_J + \beta^\epsilon_{IJ}\,\hat{B}^g_J \\
&= -\hat{B}^V\cdot\frac{\partial}{\partial V}U_I - U_J\big(\partial_I\hat{B}^g + P_I g\big)_J + U_I\cdot\frac{\partial}{\partial V}\hat{B}^V\ .
\end{aligned} \tag{6.27}
$$

Since T is a scalar we may also let $\hat{\beta}^V \to \hat{B}^V$ on the r.h.s. of (6.26). The $\mathcal{O}(\varepsilon)$ parts of (6.25), (6.26) and (6.27) are equivalent to the definitions of S, T and U_I in (6.19) and (6.22c).

---

* For a theory with a single component scalar field ϕ when $V(\phi) = \frac{1}{4!}g\phi^4 + \frac{1}{2}m^2\phi^2 + \ldots$ and $\beta^V(\phi) = \frac{1}{4!}\beta^g(g)\phi^4 - \frac{1}{2}\gamma_m(g)m^2\phi^2 + \ldots$ then if $T(\phi) = -d(g)\phi^2 + \ldots$ we have $\beta^{\mathcal{Q}}(\phi) = \beta_\eta(g)\frac{1}{2}\phi^2 H - d'(g)\nabla^2 g\,\phi^2 + \ldots$ and (6.26) gives $\beta_\eta = \gamma_m d$ as in refs. [11,12]

The requirement for the additional counterterms in (6.6a,b) is first seen at two loops. Using background field methods and extending our previous calculations for a curved space background [17] then as part of the necessary two loop counterterms we find,

$$-\frac{1}{(16\pi^2)^2}\,\frac{1}{\varepsilon}\,\frac{1}{24}\Big((D_\mu V)_{ijk}\,(D^\mu V)_{ijk}+\frac{1}{6}R\,V_{ijk}V_{ijk}\Big)\,. \tag{6.28}$$

Using $(D_\mu V)_{ijk} = g_{ijk\ell}(D_\mu\phi)_\ell + (D_\mu g)_{ijk\ell}\,\phi_\ell$ this can be cast in the form (6.6a,b) so that, apart from the two loop contribution to $\hat\gamma_{ij}$ in (6.4),

$$\begin{aligned}\big(\rho_I^{(2)}h_I\big)_{ij} &= -\,\frac{1}{(16\pi^2)^2}\,\frac{1}{6}\,g_{k\ell m[i}\,h_{j]k\ell m}\ , \qquad S_{ij}^{(2)}=0\ ,\\ \beta^{\mathcal{Q}}(\phi)^{(2)} &= -\,\frac{1}{(16\pi^2)^2}\,\frac{1}{12}\,(D_\mu g)_{ik\ell m}\,(D^\mu g)_{jk\ell m}\,\phi_i\phi_j\ ,\\ U_I(\phi)^{(2)}h_I &= \frac{1}{(16\pi^2)^2}\,\frac{1}{12}\,g_{ik\ell m}\,h_{jk\ell m}\,\phi_i\phi_j\ ,\end{aligned} \tag{6.29}$$

for arbitrary $h_{ijk\ell}$. At three loops we obtain using the methods of [17], after carefully taking into account the various d-dependent factors in (2.6) and (6.5)

$$\lambda^{(3)}\!\cdot\!\mathcal{R} = \frac{1}{(16\pi^2)^3}\,\frac{1}{\varepsilon}\,\frac{1}{864}\Big(\frac{1}{3}F g_I g_I + 2G^{\mu\nu}(D_\mu g)_I(D_\nu g)_I + H(D_\mu g)_I(D^\mu g)_I - (D^2 g)_I(D^2 g)_I\Big), \tag{6.30}$$

and similarly to (6.28) there is an additional counterterm*

$$\begin{aligned}&\frac{1}{(16\pi^2)^3}\,\frac{1}{\varepsilon^2}\,\frac{1}{24}\Big(2(1-\tfrac{1}{4}\varepsilon)g_{ijk\ell}\,V_{ijm}\,(D^2V)_{k\ell m} - \varepsilon\,(D^2 g)_{ijk\ell}\,V_{ijm}\,V_{k\ell m}\Big)\\ &+\frac{1}{(16\pi^2)^3}\,\frac{1}{\varepsilon^2}\,\frac{1}{18}\Big(-(1-\tfrac{7}{8}\varepsilon)(D_\mu g)_{ijk\ell}\,(D^\mu g)_{ijkm}\,V_{\ell m} + \tfrac{1}{2}\,\varepsilon\,(D^2 g)_{ijk\ell}\,g_{ijkm}\,V_{\ell m}\Big)\,.\end{aligned} \tag{6.31}$$

By discarding total derivatives this can be expressed in the form (6.6a,b), determining $\mathcal{Q}^{(3)}$ and $N_I^{(3)}$. We have checked that the results satisfy the pole equations such as (6.10a,b) and give

$$\begin{aligned}\big(\rho_I^{(3)}h_I\big)_{ij} &= \frac{1}{(16\pi^2)^3}\,\frac{1}{8}\,g_{k\ell mn}\,g_{k\ell p[i}\,h_{j]pmn}\ , \qquad S_{ij}^{(3)}=0\ ,\\ \beta^{\mathcal{Q}}(\phi)^{(3)} &= \frac{1}{(16\pi^2)^2}\,\frac{1}{8}\Big(g_{k\ell m(n}\,(D_\mu g)_{i)k\ell p}\,(D^\mu g)_{jmnp}\,\phi_i\phi_j - (D^2 g)_{k\ell mn}\,g_{ik\ell p}\,g_{jmnp}\,\phi_i\phi_j\\ &\qquad + \frac{7}{6}(D_\mu g)_{ik\ell m}\,(D^\mu g)_{jk\ell m}\,V_{ij}(\phi) + \frac{2}{3}(D^2 g)_{ik\ell m}\,g_{jk\ell m}\,V_{ij}(\phi)\Big)\,.\end{aligned} \tag{6.32}$$

* Apart from results contained in [17] we have used, in terms of notation defined there for flat space $G_0(x,z)^2_{\rm R}\,G_0(y,z)^2_{\rm R}\,G_0(x,y) \sim -\frac{1}{(16\pi^2)^3}\,\frac{1}{3}\Big\{\Big(\frac{1}{\varepsilon^2}-\frac{1}{4\varepsilon}\Big)(\partial_x^2+\partial_y^2) - \frac{1}{\varepsilon}\,\partial_z^2\Big\}\delta^d(x,z)\delta^d(y,z)$ where $G_0(x,y)^2_{\rm R} = G_0(x,y)^2 - \frac{1}{8\pi^2\varepsilon}\delta^d(x,y)$.

The result for $\beta_I^{\delta(3)}$ is in accord with (6.27) given U_I and ϵ_{IJ} from (6.29). In addition

$$T(\phi)^{(3)} = \frac{1}{(16\pi^2)^3}\,\frac{1}{12}\Big(-\tfrac{1}{2}\,g_{k\ell mn}\,g_{ik\ell p}\,g_{jmnp}\,\phi_i\phi_j + \tfrac{1}{3}\,g_{ik\ell m}\,g_{jk\ell m}\,V_{ij}(\phi)\Big)\ .$$

From (6.26) this allows the calculation of the lowest order, four loop, contribution to β^η,

$$\begin{aligned}2\,\beta^\eta(\phi) = \frac{1}{(16\pi^2)^4}\,\frac{1}{6}\Big(-\frac{1}{6}\,N_{k\ell}\,g_{kmni}\,g_{\ell mnj} - g_{pqmn}\,g_{pqk\ell}\,g_{kmri}\,g_{\ell nrj} \\ + g_{rpq\ell}\,g_{rmnk}\,g_{\ell mni}\,g_{kpqj}\Big)\phi_i\phi_j\ , \quad N_{ij} = g_{ik\ell m}\,g_{jk\ell m}\ .\end{aligned} \tag{6.33}$$

This agrees with previous results [11,12] for a single component field.

From (6.30) we can also determine the lowest contribution to the metric on the space of scalar couplings, as in (3.17a,b),

$$\chi_{IJ}^{g(3)} = \frac{1}{(16\pi^2)^3}\,\frac{1}{72}\,\delta_{IJ}\ , \qquad W_I^{(3)} = \frac{1}{(16\pi^2)^3}\,\frac{1}{216}\,g_I\ . \tag{6.34}$$

Since $W_I \propto \partial_I(g_J g_J)$ (3.17a) becomes $8\,\partial_I\tilde\beta_b = \chi^g_{IJ}\beta^g_J$ to this order. At four loops then $8\tilde\beta_b = W_I\beta^g_I$ so β_b remains zero. To the next order we expect corrections to χ^g_{IJ} and from an analysis of the relevant diagrams it is clear that there is a unique possibility $\chi^g_{IJ}h_Ih_J \propto g_{ijk\ell}\,h_{ijmn}\,h_{k\ell mn}$ and that therefore $W_I = \frac{1}{12}\partial_I(\chi^g_{JK}g_Jg_K)$. In this case these terms do not contribute, in association with the one loop β^g, to β_b to five loop order and we then obtain,

$$8\,\beta_b = \frac{1}{(16\pi^2)^5}\,\frac{1}{144}\Big(g_{ijk\ell}\,g_{ijmn}\,g_{pqkm}\,g_{pq\ell n} - \frac{1}{18}N_{ij}\,N_{ij}\Big)\ . \tag{6.35}$$

Also from (6.30) and (3.20) we obtain

$$\chi_{IJ}^{a(3)} = -\frac{1}{(16\pi^2)^3}\,\frac{1}{144}\,\delta_{IJ}\ , \qquad Y^{(3)} = \frac{1}{(16\pi^2)^3}\,\frac{1}{432}\,g_Ig_I\ . \tag{6.36}$$

Hence (3.21a,b) give

$$\begin{aligned}8\,\beta_c = -\frac{1}{(16\pi^2)^6}\,\frac{1}{2160}\Big(6\,g_{ijk\ell}\,g_{k\ell mn}\,g_{mnpq}\,g_{iprs}\,g_{jqrs} \\ +12\,g_{ijk\ell}\,g_{k\ell mn}\,g_{mrpq}\,g_{jspq}\,g_{inrs} - N_{ij}\,g_{imk\ell}\,g_{jmpq}\,g_{k\ell pq}\Big)\ .\end{aligned} \tag{6.37}$$

At higher orders the integrability of (3.16a,b), or (3.17a), provide non trivial constraints on the form of β^g_I and the metric χ^g_{IJ}. These arise initially at three loops for β^g and five loops for χ^g when there are seven potential independent terms, although W_I is still a total derivative $\frac{1}{20}\partial_I(\chi^g_{JK}g_Jg_K)$, and there are five linear relations on the coefficients for integrability. The compatibility of the three loop expression for β^g with a gradient flow was shown some time ago by Wallace and Zia [23].

7. Yukawa Couplings

Apart from gauge theories and scalar fields with quartic interaction terms the remaining renormalisable field theories are those involving Yukawa couplings of scalar fields with Dirac fermions. The basic interaction is described by

$$\mathcal{L}_\psi = \bar\psi\big(\gamma^\mu \overleftrightarrow{\nabla}_\mu + M(\phi)\big)\psi \ , \qquad \partial_i M = \Gamma_i = \Gamma_i{}^\dagger \ . \tag{7.1}$$

In general the couplings Γ_i may involve γ_5 but to avoid any potential difficulties with dimensional regularisation we exclude this possibility here. The hermitian matrix Γ_i then defines a new coupling with a corresponding β function which to two loop order [24] for no gauge interactions becomes

$$\begin{aligned}
\hat\beta_i^\Gamma &= \Gamma_j\hat\gamma_{ji} + \frac{1}{16\pi^2}\Big(2\,\Gamma_j\Gamma_i\Gamma_j + \tfrac12\{\Gamma^2,\Gamma_i\}\Big)\\
&\quad + \frac{1}{(16\pi^2)^2}\Big(2\,[\Gamma_j,\Gamma_k]\Gamma_i\Gamma_j\Gamma_k - \Gamma_j\{\Gamma^2,\Gamma_i\}\Gamma_j - \tfrac18\{\Gamma_j\Gamma^2\Gamma_j,\Gamma_i\}\\
&\qquad\qquad - 2\,g_{ijk\ell}\Gamma_j\Gamma_k\Gamma_\ell - 4\,\mathrm{tr}(\Gamma_j\Gamma_k)\big(\Gamma_j\Gamma_i\Gamma_k + \tfrac38\{\Gamma_j\Gamma_k,\Gamma_i\}\big)\Big) \ , \qquad (7.2)\\
\hat\gamma_{ij} &= -\tfrac12\varepsilon\,\delta_{ij} + \frac{1}{16\pi^2}\,2\,\mathrm{tr}(\Gamma_i\Gamma_j)\\
&\quad + \frac{1}{(16\pi^2)^2}\Big(\frac{1}{12}\,g_{ik\ell m}\,g_{jk\ell m} - 3\,\mathrm{tr}(\Gamma^2\Gamma_{(i}\Gamma_{j)}) - 2\,\mathrm{tr}(\Gamma_k\Gamma_i\Gamma_k\Gamma_j)\Big) \ .
\end{aligned}$$

When Γ_i is allowed to be x-dependent it is necessary for a consistent renormalisation procedure to introduce an arbitrary external gauge field $A_\mu^\psi = -A_\mu^{\psi\dagger}$ in (7.1) so that $\nabla_\mu \to D_\mu^\psi = \nabla_\mu + A_\mu^\psi$. The counterterms depending on the fermion field ψ may then be absorbed by $\psi \to \psi_{\rm o} = \mu^{-\varepsilon}Z_\psi\psi$ and also $A_\mu^\psi \to A_{{\rm o}\mu}^\psi = A_\mu^\psi + N\cdot D_\mu\Gamma$ as well as $M(\phi)\to M_{\rm o}(\phi_{\rm o})$, $\Gamma_{{\rm o}i} = \mu^{\frac12\varepsilon}(\Gamma_i + L_i^\Gamma)$, where N and L_i^Γ contain just poles in ε. As in (6.8) we may define an appropriate β function $\beta_\mu^{A^\psi}$. When the discusion of the previous section is extended to this case these additional terms give rise to modifications akin to (6.24) arising from the essential arbitrariness under $\Gamma_i \to U^\dagger\Gamma_i U$ for $U^\dagger U = 1$.

At one loop as part of the necessary counterterms we find a contribution

$$-\frac{1}{16\pi^2\varepsilon}\Big(2\,\mathrm{tr}(D_\mu M D^\mu M) + \frac13 R\,\mathrm{tr}(M^2) + \bar\psi\Gamma_i\gamma^\mu \overleftrightarrow{D}_\mu^\psi\Gamma_i\psi\Big) \ , \tag{7.3}$$

for $D_\mu M = \partial_\mu M + [A_\mu^\psi, M]$. Using $D_\mu M \to \Gamma_i D_\mu\phi_i + D_\mu\Gamma_i\,\phi_i$, $D_\mu\Gamma_i = \partial_\mu\Gamma_i + [A_\mu^\psi,\Gamma_i] + A_{\mu\,ij}\Gamma_j$, this can be cast into the required renormalised form leading to the one loop contribution to γ_{ij} in (7.2) and

$$\begin{aligned}
\beta_{\mu\,ij}^A &= -\frac{4}{16\pi^2}\,\mathrm{tr}(\Gamma_{[i}\,D_\mu\Gamma_{j]}) \ , \qquad \beta_\mu^{A^\psi} = -\frac{1}{16\pi^2}\,\tfrac12[\Gamma_i, D_\mu\Gamma_i] \ ,\\
\beta^{\mathcal{Q}}(\phi) &= -\frac{2}{16\pi^2}\,\mathrm{tr}(D_\mu\Gamma_i\,D^\mu\Gamma_j)\phi_i\phi_j \ .
\end{aligned} \tag{7.4}$$

At two loops the essential counterterms, for the purposes of this paper, are given by

$$\begin{aligned}\lambda^{(2)}\cdot\mathcal{R} = \frac{1}{(16\pi^2)^2}\,\frac{1}{\varepsilon}\,\frac{1}{6}\Big(\frac{1}{8}\,F\,\mathrm{tr}(\Gamma_i\Gamma_i) + 2\,G^{\mu\nu}\mathrm{tr}(D_\mu\Gamma_i D_\nu\Gamma_i)\\ + H\,\mathrm{tr}(D_\mu\Gamma_i D^\mu\Gamma_i) - \mathrm{tr}(D^2\Gamma_i D^2\Gamma_i)\Big)\ .\end{aligned} \tag{7.5}$$

This expression, apart from the F term, is similar in form to (6.30), both are determined by the finiteness requirements of (3.9e,f) neglecting higher order $\mathcal{O}(\beta)$ terms. From (7.5) the lowest order contribution to the metric for the Yukawa couplings is

$$h{\cdot}\chi^{g(2)}{\cdot}h = \frac{1}{(16\pi^2)^2}\,\frac{4}{3}\,\mathrm{tr}(h_i h_i)\ , \qquad W^{(2)}{\cdot}h = h{\cdot}\frac{\partial}{\partial\Gamma}\,\frac{1}{(16\pi^2)^2}\,\frac{1}{6}\,\mathrm{tr}(\Gamma^2)\ , \tag{7.6}$$

for any hermitian matrix h_i of the same dimension as Γ_i. It is easy to see that (3.17a) now becomes

$$h{\cdot}\frac{\partial}{\partial\Gamma}\,8\tilde\beta_b = h{\cdot}\chi^g{\cdot}\beta^\Gamma\ , \tag{7.7}$$

which is solved, to lowest order with the one loop β^Γ in (7.2), by

$$8\tilde\beta_b = W{\cdot}\beta^\Gamma\ , \quad \text{or} \quad \beta_b = 0. \tag{7.8}$$

At the next order the corrections to (7.6) have not been calculated but should have the general form,

$$\begin{aligned}(16\pi^2)^3\,h{\cdot}\chi^{g(3)}{\cdot}h ={}& x\,\mathrm{tr}(h^2\Gamma^2) + y\,\mathrm{tr}(\widehat{h_i\Gamma_i}\widehat{h_j\Gamma_j}) + z\,\mathrm{tr}(h_i\Gamma_j h_i\Gamma_j) + u\,\mathrm{tr}(h_i h_j\Gamma_i\Gamma_j)\\ &+ v\,\mathrm{tr}(h_i h_j\Gamma_j\Gamma_i) + a\,\mathrm{tr}(h^2)\mathrm{tr}(\Gamma^2) + b\,\mathrm{tr}(h_i\Gamma_i)\mathrm{tr}(h_j\Gamma_j)\\ &+ c\,\mathrm{tr}(h_i h_j)\mathrm{tr}(\Gamma_i\Gamma_j) + d\,\mathrm{tr}(h_i\Gamma_j)\mathrm{tr}(h_i\Gamma_j) + e\,\mathrm{tr}(h_i\Gamma_j)\mathrm{tr}(h_j\Gamma_i)\ ,\\ (16\pi^2)^3\,W^{(3)}{\cdot}h ={}& h{\cdot}\frac{\partial}{\partial\Gamma}\,\frac{1}{24}\Big((x+y+v)\mathrm{tr}(\Gamma^2\Gamma^2) + (z+u)\mathrm{tr}(\Gamma_i\Gamma_j\Gamma_i\Gamma_j)\\ &+ (a+b)\mathrm{tr}(\Gamma^2)\mathrm{tr}(\Gamma^2) + (c+d+e)\mathrm{tr}(\Gamma_i\Gamma_j)\mathrm{tr}(\Gamma_i\Gamma_j)\Big)\ ,\end{aligned} \tag{7.9}$$

where $\widehat{h_i\Gamma_i} = \frac{1}{2}(h_i\Gamma_i + \Gamma_i h_i)$. The requirement that (7.7) is integrable, along with (7.6) and the two loop β function in (7.2), then imposes conditions on this expression for χ^g which entail

$$\tfrac{1}{2}(d+e) = y+v = 2x+\tfrac{2}{3} = \tfrac{1}{2}u-\tfrac{2}{3} = z-2 = c+\tfrac{10}{3}\ , \quad a=2b\ .$$

Although this does not determine (7.9) entirely it does enable the calculation of the lowest order non zero contribution to β_b,

$$\begin{aligned}8\beta_b = \frac{1}{(16\pi^2)^4}\,\frac{2}{9}\Big(&\frac{1}{8}\,\mathrm{tr}(\Gamma_i\Gamma^2\Gamma_i\Gamma^2) + \mathrm{tr}(\Gamma_i\Gamma_j\Gamma_i\Gamma_j\Gamma^2) + \mathrm{tr}(\Gamma_i\Gamma_j\Gamma_i\Gamma_k\Gamma_j\Gamma_k)\\ &- \mathrm{tr}(\Gamma_i\Gamma_j\Gamma_k\Gamma_i\Gamma_j\Gamma_k) + 3\,\mathrm{tr}(\Gamma_i\Gamma_j)\,\mathrm{tr}(\Gamma_i\Gamma_j\Gamma^2 + \Gamma_i\Gamma_k\Gamma_j\Gamma_k)\\ &+ g_{ijk\ell}\,\mathrm{tr}(\Gamma_i\Gamma_j\Gamma_k\Gamma_\ell) - \frac{1}{24}\,N_{ij}\,\mathrm{tr}(\Gamma_i\Gamma_j)\Big)\ .\end{aligned} \tag{7.10}$$

Besides partially determining (7.9) the integrability of (7.7) also depends on the particular structure for the two loop β function in (7.2).* In addition the $\mathcal{O}(g)$ terms in (7.10), which arise from the appropriate terms in $\beta^{\Gamma(2)}$, are also entailed by the additional terms in the one loop expression for β^g,

$$h_I\,\Delta\beta^g_I = h_{ijk\ell}\,\frac{1}{16\pi^2}\Big(-48\,\mathrm{tr}(\Gamma_i\Gamma_j\Gamma_k\Gamma_\ell) + 8\,g_{ijkm}\,\mathrm{tr}(\Gamma_m\Gamma_\ell)\Big)\ , \tag{7.11}$$

in association with the lowest order contribution to the metric for the scalar couplings (6.34) when variations of $\tilde\beta_b$ with respect to g_I are considered.

For the counterterms corresponding to the bosonic part of (7.3) we find to the next order

$$\begin{aligned}\frac{1}{(16\pi^2\varepsilon)^2}\Big\{ & -4(1-\tfrac14\varepsilon)\Big(\mathrm{tr}\big(\Gamma_iD_\mu M\,\Gamma_iD^\mu M\big)+\frac16(1-\tfrac16\varepsilon)\,R\,\mathrm{tr}\big(\Gamma_iM\Gamma_iM\big)\Big)\\ & -2(1-\tfrac34\varepsilon)\Big(\mathrm{tr}\big(\Gamma^2D_\mu M\,D^\mu M\big)+\frac16(1-\tfrac16\varepsilon)\,R\,\mathrm{tr}\big(\Gamma^2M^2\big)\Big)\\ & -2(1-\tfrac14\varepsilon)\,\mathrm{tr}\big(\Gamma_iD_\mu\Gamma_i\,D^\mu M\,M + D_\mu\Gamma_i\,\Gamma_iMD^\mu M\big)\\ & +4(1-\tfrac12\varepsilon)\,\mathrm{tr}\big(D_\mu\Gamma_i\,MD^\mu\Gamma_i\,M\big)+4(1-\tfrac34\varepsilon)\,\mathrm{tr}\big(D^2\Gamma_i\,M\Gamma_iM\big)\\ & -\varepsilon\,\mathrm{tr}\big(D^2\Gamma_i\{\Gamma_i\,,M^2\}\big)\\ & -2(1-\tfrac14\varepsilon)\,\mathrm{tr}\big(D_\mu\Gamma_iD^\mu\Gamma_j\big)V_{ij}+\varepsilon\,\mathrm{tr}\big(D^2\Gamma_i\,\Gamma_j\big)V_{ij}\Big\}\ .\end{aligned} \tag{7.12}$$

To achieve this expression, as well as (7.5), it is essential to take account of the d-dependent factor in (6.5) which modifies the cancellation of subdivergences from straightforward minimal subtraction. The $D_\mu\phi D^\mu\phi$ and $R\,\phi^2$ terms in (7.12) are then in accord with the form of $K(A,\phi_{\rm o})$ in (6.5) since $\frac18(d-2)/(d-1)\approx\frac1{12}(1-\frac16\varepsilon)$. This result determines the appropriate Z_{ij} corresponding to the two loop Yukawa coupling contributions to γ_{ij} in (7.2). From (7.12) we may find $A_{{\rm o}\mu}$ and $\mathcal{Q}$ to two loop order since if $M\to\Gamma_i\phi_i$ and (7.12) is expressed in the form

$$\tfrac12(D^\mu\phi)^TZ^2D_\mu\phi+(D^\mu\phi)^TV_\mu\phi+\tfrac12\phi^TU\phi\ ,\qquad Z^T=Z\ ,$$

then

$$\begin{aligned}A_{{\rm o}\mu} &= A_\mu+Z^{-1}V_{\mu-}Z^{-1}-\tfrac12[D_\mu Z,Z^{-1}]\ ,\qquad V_{\mu\pm}=\tfrac12(V_\mu\pm V_\mu^T)\ ,\\ \mathcal{Q} &= \mu^{-\varepsilon}\tfrac12\,\phi^T\big(D^2Z^2-D_\mu V^\mu_+ +(V_{\mu-}-\tfrac12D_\mu Z^2)Z^{-2}(V^\mu_- +\tfrac12D^\mu Z^2)+U\big)\phi\ .\end{aligned}$$

A useful consistency check on these results is provided by the obvious extension of the pole equations such as (6.10a) to this case, when it is necessary to use all the one loop

* For a general $\beta_i^{\Gamma(2)}=P\,\Gamma_j\Gamma_k\Gamma_i\Gamma_j\Gamma_k+Q\,\Gamma_k\Gamma_j\Gamma_i\Gamma_j\Gamma_k+R\,\frac12\Gamma_j\{\Gamma^2,\Gamma_i\}\Gamma_j+S\,\frac12\{\Gamma_j\Gamma^2\Gamma_j,\Gamma_i\}+I\,\mathrm{tr}(\Gamma_j\Gamma_k)\Gamma_j\Gamma_i\Gamma_k+J\,\mathrm{tr}(\Gamma_j\Gamma_k)\frac12\{\Gamma_j\Gamma_k,\Gamma_i\}+K\,\mathrm{tr}(\Gamma^2\Gamma_{(i}\Gamma_{j)})\Gamma_j+L\,\mathrm{tr}(\Gamma_k\Gamma_i\Gamma_k\Gamma_j)\Gamma_j+\mathcal{O}(g)$ then it is necessary that $Q-2R+8S=0,\ Q+I-2J=0,\ 2R-2K+L=0$ which are of course compatible with (7.2).

results in (7.4). Furthermore we then obtain at two loops the extra pieces as compared to (6.29)

$$\Delta\beta^A_{\mu\, ij} = \frac{1}{(16\pi^2)^2}\Big(4\,\mathrm{tr}\big(\Gamma_k\Gamma_{[i}\Gamma_k D_\mu\Gamma_{j]}\big)+3\,\mathrm{tr}\big(\Gamma^2\{\Gamma_{[i}, D_\mu\Gamma_{j]}\}\big) \\ + \tfrac{1}{2}\,\mathrm{tr}\big([\Gamma_k, D_\mu\Gamma_k][\Gamma_i,\Gamma_j]\big)\Big)\ , \tag{7.13a}$$

$$\Delta\beta^{\mathcal{Q}}(\phi) = \frac{1}{(16\pi^2)^2}\Big\{\Big(2\,\mathrm{tr}\big(\Gamma_k D_\mu\Gamma_i\,\Gamma_k D^\mu\Gamma_j + D_\mu\Gamma_k\, D^\mu\Gamma_i\,\Gamma_k\Gamma_j + \Gamma_k D_\mu\Gamma_i\, D^\mu\Gamma_k\,\Gamma_j\big) \\ - 2\,\mathrm{tr}\big(D_\mu\Gamma_k\,\Gamma_i D^\mu\Gamma_k\,\Gamma_j\big)-4\,\mathrm{tr}\big(D^2\Gamma_k\,\Gamma_i\Gamma_k\Gamma_j\big) \\ + 3\,\mathrm{tr}\big(\Gamma^2 D_\mu\Gamma_i\, D^\mu\Gamma_j + D_\mu\Gamma^2\, D^\mu(\Gamma_i\Gamma_j)\big)+2\,\mathrm{tr}\big(D_\mu\Gamma_k\, D^\mu\Gamma_k\,\Gamma_i\Gamma_j\big) \\ - \tfrac{1}{2}\,\mathrm{tr}\big([\Gamma_k, D_\mu\Gamma_k][\Gamma_i, D_\mu\Gamma_j]\big)-\mathrm{tr}\big(\{\Gamma_k, D^2\Gamma_k\}\Gamma_i\Gamma_j\big)\Big)\phi_i\phi_j \\ + \mathrm{tr}\big(D_\mu\Gamma_i D^\mu\Gamma_j\big)V_{ij}(\phi) + 2\,\mathrm{tr}\big(D^2\Gamma_i\,\Gamma_j\big)V_{ij}(\phi)\Big\}\ . \tag{7.13b}$$

The general discussion of section 6 may be extended to include Yukawa couplings if we write for the additional contributions an analogous form to (6.8)

$$\Delta\beta^A_\mu = \rho{\cdot}D_\mu\Gamma\ , \qquad \Delta\beta^{\mathcal{Q}} = \beta^\delta{\cdot}D^2\Gamma + \tfrac{1}{2}\,D_\mu\Gamma{\cdot}\beta^\epsilon{\cdot}D^\mu\Gamma\ . \tag{7.14}$$

The corresponding version of (6.25) gives

$$\Big(g_I\frac{\partial}{\partial g_I} + \tfrac{1}{2}\,\Gamma{\cdot}\frac{\partial}{\partial\Gamma}\Big)S = -\rho_I g_I - \tfrac{1}{2}\,\rho{\cdot}\Gamma\ , \tag{7.15}$$

so that it is clear that S remains zero to two loop order. At one loop from (7.4)

$$(\rho{\cdot}h)_{ij} = -\frac{4}{16\pi^2}\,\mathrm{tr}(\Gamma_{[i}h_{j]})\ , \qquad h{\cdot}\beta^\epsilon(\phi){\cdot}h = -\frac{4}{16\pi^2}\,\mathrm{tr}(h_ih_j)\phi_i\phi_j\ . \tag{7.16}$$

To lowest order (6.27) becomes

$$2\,\beta^\delta{\cdot}h = -h{\cdot}\beta^\epsilon{\cdot}\beta^\Gamma - \beta^\Gamma{\cdot}\frac{\partial}{\partial\Gamma}U{\cdot}h - U{\cdot}\Big(h{\cdot}\frac{\partial}{\partial\Gamma}\beta^\Gamma + (\rho{\cdot}h)\Gamma\Big)+(U{\cdot}h){\cdot}\frac{\partial}{\partial V}\beta^V\ , \tag{7.17}$$

where

$$U(\phi){\cdot}h = \frac{2}{16\pi^2}\,\mathrm{tr}(h_i\Gamma_j)\phi_i\phi_j\ . \tag{7.18}$$

Using the results (7.16) and (7.18) in (7.17) with the one loop β^Γ from (7.2) and

$$\beta^V(\phi) = \frac{1}{16\pi^2}\Big(\tfrac{1}{2}\,V_{ij}(\phi)V_{ij}(\phi) + 2\,V_i(\phi)\mathrm{tr}(\Gamma_i\Gamma_j)\phi_j - 2\,\mathrm{tr}\big(M(\phi)^4\big)\Big) \tag{7.19}$$

gives

$$\beta^\delta(\phi){\cdot}h = -\frac{2}{(16\pi^2)^2}\Big\{\Big(\mathrm{tr}\big(\widehat{h_k\Gamma_k}\Gamma_i\Gamma_j\big)+2\,\mathrm{tr}\big(h_k\Gamma_i\Gamma_k\Gamma_j\big)\Big)\phi_i\phi_j - \mathrm{tr}\big(h_i\Gamma_j\big)V_{ij}(\phi)\Big\}\ , \tag{7.20}$$

which can also be read off directly from (7.13b). From (6.22c) we then find

$$T(\phi) = -\frac{1}{(16\pi^2)^2}\,\frac{1}{2}\,\Big\{\Big(\mathrm{tr}\big(\Gamma^2\Gamma_i\Gamma_j\big)+2\,\mathrm{tr}\big(\Gamma_k\Gamma_i\Gamma_k\Gamma_j\big)\Big)\phi_i\phi_j - \mathrm{tr}(\Gamma_i\Gamma_j)\,V_{ij}(\phi)\Big\}\ . \tag{7.21}$$

Using these results it is then possible to use (6.26) to calculate the lowest order three loop contribution to β^η,

$$\begin{aligned}2\,\beta^\eta(\phi) = \frac{1}{(16\pi^2)^3}\Big\{\Big(&-2\,\mathrm{tr}\big(\Gamma_k\Gamma_\ell\Gamma_k\Gamma_\ell\Gamma_i\Gamma_j\big)+\mathrm{tr}\big(\Gamma^2\{\Gamma_k\Gamma_i\Gamma_k,\Gamma_j\}\big)-4\,\mathrm{tr}\big(\Gamma_\ell\Gamma_k\Gamma_\ell\Gamma_i\Gamma_k\Gamma_j\big)\\ &+4\,\mathrm{tr}\big(\Gamma_k\Gamma_\ell\Gamma_i\Gamma_\ell\Gamma_k\Gamma_j\big)-\tfrac{1}{2}\,\mathrm{tr}\big(\Gamma_k\Gamma^2\Gamma_k\Gamma_i\Gamma_j\big)+\tfrac{1}{2}\,\mathrm{tr}\big(\Gamma^2\Gamma_i\Gamma^2\Gamma_j\big)\\ &+6\,\mathrm{tr}\big(\Gamma_k\Gamma_\ell\big)\mathrm{tr}\big(\Gamma_k\Gamma_\ell\Gamma_i\Gamma_j\big)\Big)\phi_i\phi_j - \tfrac{1}{2}\,\mathrm{tr}\big(\Gamma_i\Gamma_j\big)V_{k\ell i}(\phi)\,V_{k\ell j}(\phi)\Big\}\ .\end{aligned} \tag{7.22}$$

In principle it would also be possible to use (7.13b) to determine β^ϵ to two loops and then use an extension of (7.17) to find β^δ at three loops and hence the next order corrections to (7.22). The appearance of curvature dependent pieces in $\mathcal{Q}$ is significant since they are directly related to the so called improvement terms in the flat space energy momentum tensor for scalar theories.

8. Conclusion

The analysis described in this paper shows how curvature dependent renormalisation quantities may be obtained from those restricted to flat space but involving local composite operators and their products. However there is also an essential constraint on ordinary β functions, as shown in the previous section for the Yukawa coupling at two loops, arising from the integrability of the equations for the variation of the natural four dimensional analogue of the Virasoro central charge. A further illustration of the non trivial aspects of the integrability condition is provided by considering a scalar-fermion field theory with also a coupling to a quantum gauge field. At one loop, if t_a, t^ϕ_a are the gauge group generators acting on the fermion, scalar fields $V'(\phi)^T t^\phi_a\phi = 0,\ [t_a,\Gamma_i] = t^\phi_{a\,ij}\Gamma_j$, then there is an additional term in the Yukawa β function

$$\Delta\beta^\Gamma_i = \frac{g^2}{16\pi^2}\,3\,\big\{t^2,\Gamma_i\big\}\ . \tag{8.1}$$

Using the metric in (7.6) we find

$$8\,\Delta\tilde\beta_b = \frac{4\,g^2}{(16\pi^2)^3}\,\mathrm{tr}(t^2\Gamma^2)\ . \tag{8.2}$$

Hence considering now the variation with respect to g and using (5.12) this gives

$$\Delta\beta^g = \frac{2\,g^3}{(16\pi^2)^2}\,\frac{1}{n_V}\,\mathrm{tr}(t^2\Gamma^2)\ , \tag{8.3}$$

which is in accord with a direct two loop calculation [24]. Similarly there is also an extra contribution to the scalar coupling β function,

$$\Delta\beta^V(\phi) = \frac{3}{16\pi^2}\Big(g^2\, V'(\phi)^T t^{\phi 2}\phi + \tfrac{1}{2}\, g^4\, \phi^T t^\phi_a t^\phi_b \phi\, \phi^T t^\phi_a t^\phi_b \phi\Big) \,. \tag{8.4}$$

This then gives

$$8\,\Delta\tilde\beta_b = \frac{1}{(16\pi^2)^4}\,\frac{1}{12}\,\Big(g^2\,\mathrm{tr}(N t^{\phi 2}) + 6\, g^4\, g_{ijk\ell}\,(t^\phi_a t^\phi_b)_{ij}(t^\phi_a t^\phi_b)_{k\ell}\Big) \,, \tag{8.5}$$

for N_{ij} as in (6.33), which implies

$$\Delta\beta^g = \frac{1}{(16\pi^2)^3}\,\frac{1}{24}\,\frac{1}{n_V}\,\Big(g^3\,\mathrm{tr}(N t^{\phi 2}) + 12\, g^5\, g_{ijk\ell}\,(t^\phi_a t^\phi_b)_{ij}(t^\phi_a t^\phi_b)_{k\ell}\Big) \,. \tag{8.6}$$

We are not aware of a three loop calculation with which this can be compared.

In order to derive a genuine c-theorem it is necessary that the metric given by χ^g should be positive. This is true for the lowest order contributions calculated in (5.12), (6.34) and (7.6). In essence this is a consequence of the finiteness condition for (3.9e) since $\mathcal{A}_{ij}$ is the leading divergent part of the composite operator two point function, see (2.17), and in our conventions should be negative definite. (In two dimensions the positivity of the analogous metric in the perturbative treatment discussed here would be implied by a similar argument as that for $\mathcal{A}_{ij}$ which is clearly related to Zamolodchikov's discussion [3] of positivity in general.) It may be possible to bound the remaining $\mathcal{O}(\beta)$ terms in (3.9e) and hence show positivity in general. In any event at least at weak coupling the effective metric on the space of coupling constants is positive and the flow of the C function defined here by $\tilde\beta_b$ is monotonic.

Although many parts of this paper are perhaps rather technical in character it demonstrates that the analysis of the conditions for finiteness of correlation functions involving composite operators on a general curved space background lead to non trivial constraints on the expressions for β functions that are not revealed in conventional discussions of renormalisation. The essential results contained in sections 2 and 3 are relatively simple in character although as always detailed calculations in specific theories tend to be complicated. It would be interesting to derive similar expressions within a more general procedure.

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