Path Integrals and Feynman Graphs

1. Let

$$K(q,t;q_0,t_0) = \left(\frac{m}{2\pi i(t-t_0)}\right)^{\frac{1}{2}} e^{im\frac{(q-q_0)^2}{2(t-t_0)}}\,, \qquad t>t_0\,.$$

be the amplitude for a particle to propagate from q_0 at time t_0 to q at time t. Show that

$$\int \mathrm{d}q'\, K(q,t;q',t')\,K(q',t';q_0,t_0) = K(q,t;q_0,t_0)\,, \qquad t>t'>t_0\,.$$

2. Show that the expression given for K in the last example satisfies

$$-\frac{1}{2m}\frac{\partial^2}{\partial x^2}K(x,t;0,0) = i\frac{\partial}{\partial t}K(x,t;0,0) \qquad \text{and} \qquad \lim_{t\to 0} K(x,t;0,0) = \delta(x)$$

Use these facts to express the solution of the Schrödinger equation for a free particle $\Psi(x,t)$ in terms of initial data $\Psi(x,0)$ and $K(x,t;0,0)$. Check this result for $\Psi(x,0)=e^{ikx}$ with k constant.

3. For a harmonic oscillator with angular frequency ω

$$K(q,t;q_0,0) = \left(\frac{m\omega}{2\pi i\sin\omega t}\right)^{\frac{1}{2}} \exp\left(im\omega\,\frac{(q^2+q_0{}^2)\cos\omega t - 2qq_0}{2\sin\omega t}\right).$$

Verify

$$\mathrm{tr}\big(e^{-\beta H}\big) = \int \mathrm{d}q\, K(q,-i\beta;q,0)\,, \qquad \mathrm{tr}\big(Pe^{-\beta H}\big) = \int \mathrm{d}q\, K(q,-i\beta;-q,0)\,,$$

where H is the usual harmonic oscillator Hamiltonian and P is the parity operator. The operator trace is defined by $\mathrm{tr}(O)=\sum_n \langle n|O|n\rangle$ where $\{|n\rangle\}$ is a complete orthonormal set of states. Note that we must require $Re\beta>0$.

4. Consider the Lagrangian $L=\frac{1}{2}\dot q^2 - \frac{1}{2}c(t)\,q^2$. Show that the quantum amplitude K in this case can be approximated

$$K(q,t;q_0,t_0) \approx \frac{1}{(2\pi i\epsilon)^{\frac{1}{2}(N+1)}}\int \prod_{r=1}^{N}\mathrm{d}q_r\, e^{iS[q]}\,, \qquad S[q]=\sum_{r=0}^{N}\left(\frac{1}{2\epsilon}(q_{r+1}-q_r)^2 - \tfrac{1}{2}\epsilon c_r q_r^2\right),$$

where we take $q_{N+1}=q$, $t-t_0=T=(N+1)\epsilon$ and $c_r=c(t_r)$ for $t_r=t_0+r\epsilon$. Suppose Q_r, with $Q_0=q_0$, $Q_{N+1}=q$, is the solution of $\frac{\partial}{\partial q_r}S[q]=0$ for $r=1,\dots,N$. Let $q=Q+f$ where $f_0=f_{N+1}=0$. Show that $S[q]=S[Q]+S[f]$ where $S[f]=\frac{1}{2\epsilon}\sum_{r,s} f_r A_{N,rs} f_s$ and we define $j\times j$ matrices $\underline{A}_j$ by

$$\underline{A}_j = \begin{pmatrix} 2-\epsilon^2 c_1 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2-\epsilon^2 c_2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2-\epsilon^2 c_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2-\epsilon^2 c_{j-1} & -1 \\ 0 & 0 & 0 & \dots & -1 & 2-\epsilon^2 c_j \end{pmatrix}$$

Hence obtain

$$K(q,t;q_0,t_0) \approx \frac{1}{(2\pi i\epsilon \det \underline{A}_N)^{\frac{1}{2}}}\, e^{iS[Q]}\,.$$

By expanding along the j'th row or column show that $\det \underline{A}_j = (2-\epsilon^2 c_j)\det \underline{A}_{j-1} - \det \underline{A}_{j-2}$. For $c_r = 0$ show that $\det \underline{A}_N = N+1$. If $D(t_j) = \epsilon \det \underline{A}_j$ show that as $N \to \infty$ we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} D(t) = -c(t)D(t)\,, \qquad D(t_0) = 0\,, \quad D'(t_0) = 1\,.$$

Solve this for $c(t) = \omega^2$. For any N show that the eigenvalues of $\underline{A}_N$ are $\lambda_n = 2(1-\cos\frac{n\pi}{N+1}) - \epsilon^2\omega^2$ for $n = 1,\dots N$ and that $\lambda_n \approx \epsilon^2(\frac{n^2\pi^2}{T^2} - \omega^2)$ as $N \to \infty$. Use $\prod_{n=1}^{N}(2\cos\theta - 2\cos\frac{n\pi}{N+1}) = \sin(N+1)\theta/\sin\theta$ to work out $\det \underline{A}_N$ for large N in this case. How does this compare with using the large N approximation for λ_n?

5. Prove the identity

$$G(\partial/\partial b)F(b) = F(\partial/\partial u)G(u)e^{ub}|_{u=0}$$

by assuming the functions F and G are expandable as power series (by linearity, it then suffices to consider $F(x) = x^n$ and $G(x) = x^m$). Extend this to the case of many variables.

6. Determine all connected one loop graphs, with their appropriate symmetry factors, which contribute to $\langle\phi(x_1)\phi(x_2)\rangle$, $\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle$ and $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$, for an interaction given by $V(\phi) = \frac{1}{6}g\phi^3 + \frac{1}{24}\lambda\phi^4$.

7. Using the expansion of

$$Z = \exp\Big(\frac{1}{2}\frac{\partial}{\partial \underline{x}}\cdot\underline{A}^{-1}\frac{\partial}{\partial \underline{x}}\Big)\exp\big(-V(\underline{x})\big)\Big|_{\underline{x}=\underline{0}}\,,$$

where $V(\underline{0}) = 0$ and if $V_{i_1\dots i_n} = \partial_{i_1}\dots\partial_{i_n}V(\underline{x})\big|_{\underline{x}=\underline{0}}$ with $V_i = V_{ij} = 0$, determine possible three loop connected vacuum diagrams which are one particle irreducible (cannot be made disconnected by cutting one line). If $V^{(n)}$ denotes the n'th derivative of V at $\underline{x} = \underline{0}$ and if for a given three loop diagram the contribution is $\mathrm{O}(\underline{A}^{-r}\big(V^{(3)})^{s_3}(V^{(4)})^{s_4}(V^{(5)})^{s_5}(V^{(6)})^{s_6}\big)$ show that we must have $r - \sum_n s_n = 2$ and $2r = \sum_n n s_n$. Hence verify that there are two diagrams involving just $V^{(4)}$ and one just $V^{(6)}$ and two just $V^{(3)}$ and 8 in all. What are the symmetry factors for each diagram?

8. A zero dimensional model for the functional integral in quantum field theory is obtained by considering

$$Z(\lambda) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int \mathrm{d}x\; e^{-\frac{1}{2}x^2 - \frac{1}{24}\lambda x^4}\,, \qquad \lambda > 0\,.$$

Obtain the N'th order perturbation expansion $Z_N(\lambda) = \sum_{n=0}^{N}(-1)^n\big(\frac{\lambda}{24}\big)^n\frac{(4n)!}{2^{2n}(2n)!n!}$ and hence $W(\lambda) = \log Z(\lambda) = -\frac{1}{8}\lambda + \frac{1}{12}\lambda^2 - \frac{11}{96}\lambda^3 + \mathrm{O}(\lambda^4)$. Show how the coefficients in the expansion of $W(\lambda)$ are the sums of the symmetry factors of the relevant connected vacuum diagrams (there is one at two loops, two at three loops and four at four loops).

Why is $Z(\lambda) < 1$? If you have access to Mathematica or equivalent plot $Z_N(\lambda)$ for $\lambda = 0.1$ against N and show that there is a region of N where the result appears to converge to a precise answer before blowing up for larger N. By expanding $e^{-\frac{1}{2}x^2}$ obtain a convergent strong coupling expansion in terms of powers $\lambda^{-\frac{1}{4}(2n+1)}$, $n = 0, 1, \dots$. How many terms does one need for $\lambda = 0.1$ to get the previous result?