Functional Methods and Grassmann Integrals

1. Let, with $\underline{x}, \underline{b}, \underline{a} \in \mathbb{R}^n$,

$$e^{W(\underline{b})} = \int \mathrm{d}^n x \, e^{-S + \underline{b}\cdot\underline{x}} \,, \quad S = \tfrac{1}{2}\, \underline{x} \cdot \underline{A}\, \underline{x} + V(\underline{x}) \,, \quad \frac{\partial}{\partial \underline{b}} W(\underline{b}) = \underline{a} \,, \qquad \Gamma(\underline{a}) + W(\underline{b}) = \underline{b} \cdot \underline{a} \,.$$

Show that

$$\frac{\partial}{\partial \underline{a}} \Gamma(\underline{a}) = \underline{b} \,, \qquad \underline{W}'' \underline{\Gamma}'' = \underline{1} \,, \qquad \frac{\partial}{\partial \underline{b}} = \underline{W}'' \frac{\partial}{\partial \underline{a}} \,,$$

where $\underline{W}'', \underline{\Gamma}''$ are matrices formed by $\frac{\partial^2}{\partial b_i \partial b_j} W$, $\frac{\partial^2}{\partial a_i \partial a_j} \Gamma$. Find the first few terms in the perturbative expansion of $W(\underline{b})$ and correspondingly for $\Gamma(\underline{a})$.

2. Starting from $Z[J]$ determine $W[J]$ and $\Gamma[\varphi]$ for the free field theory for a scalar field $\phi(x)$ of mass m. What are

$$\frac{\delta^2}{\delta J(x) \delta J(y)} W[J] \Big|_{J=0} \,, \qquad \frac{\delta^2}{\delta \varphi(x) \delta \varphi(y)} \Gamma[\varphi] \Big|_{\varphi=0} \,,$$

in this case? Verify that they satisfy the expected inverse relations.

3. Starting from the definition of the effective action $\Gamma[\varphi]$ in terms of the generating functional for connected graphs $W[J]$,

$$W[J] + \Gamma[\varphi] = \int \mathrm{d}^d x \, J\varphi \,, \qquad \frac{\delta}{\delta J(x)} W[J] = \varphi(x) \,,$$

obtain

$$- \int \mathrm{d}^d z \, G_2(x,z) \, \Gamma_2(z,y) = \delta^d(x-y) \,, \qquad -i \frac{\delta}{\delta J(x)} = \int \mathrm{d}^d y \, G_2(x,y) \frac{\delta}{\delta \varphi(y)} \,,$$

where

$$G_n(x_1, \dots, x_n) = (-i)^{n-1} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W[J] \,, \quad \Gamma_n(x_1, \dots, x_n) = -i \frac{\delta}{\delta \varphi(x_1)} \cdots \frac{\delta}{\delta \varphi(x_n)} \Gamma[\varphi] \,.$$

Show that

$$G_3(x_1, x_2, x_3) = \int \mathrm{d}^d y_1 \mathrm{d}^d y_2 \mathrm{d}^d y_3 \, G_2(x_1, y_1) G_2(x_2, y_2) G_2(x_3, y_3) \, \Gamma_3(y_1, y_2, y_3) \,.$$

Also obtain relations in diagrammatic form for G_4, G_5 in terms of G_2, Γ_3, Γ_4 and Γ_5.

4. Evaluate, for $\theta_i, \bar{\theta}_i$ Grassmann variables

$$Z = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \mathrm{d}^n x \prod_{i=1}^n \mathrm{d}\bar{\theta}_i \, \mathrm{d}\theta_i \; e^{-S(x,\theta,\bar{\theta})} \,, \quad S(x,\theta,\bar{\theta}) = \tfrac{1}{2} w_i(x) w_i(x) + \bar{\theta}_i w_{i,j}(x) \, \theta_j \,,$$

where $w_{i,j}(x) = \frac{\partial}{\partial x_j} w_i(x)$ and assuming $\det[w_{i,j}(x)] > 0$.

If $w_i(x) = x_i + \frac{1}{2} g_{ijk} x_j x_k$, $g_{ijk} = g_{ikj}$, what are the Feynman rules for a perturbative expansion? Work out the two loop contributions to $Z(g)$ (this is much simpler if g_{ijk} is completely symmetric).

If $w_i(x) \to \lambda w_i(x)$ in S show that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} e^{-S(x,\theta,\bar{\theta})} = - \left(\theta_j \frac{\partial}{\partial x_j} + \lambda \, w_j(x) \frac{\partial}{\partial \bar{\theta}_j} \right) \left(\bar{\theta}_i w_i(x) \, e^{-S(x,\theta,\bar{\theta})} \right) .$$

By letting $x_i \to \lambda x_i$, $g_{ijk} \to g_{ijk}/\lambda$ what does this imply for $Z(g)$?

5. Show that, for $\theta_i, \bar\theta_i, \eta_i, \bar\eta_i$, $i = 1, 2, \ldots, n$ independent Grassmann variables and $\underline{B}$ an invertible $n \times n$ matrix,

$$\int \prod_{i=1}^{n} \mathrm{d}\bar\theta_i \mathrm{d}\theta_i \, \exp\big(- \bar\theta_i B_{ij} \theta_j + \bar\eta_i \theta_i + \bar\theta_i \eta_i \big) = \det \underline{B} \, \exp\big(\bar\eta_i (\underline{B}^{-1})_{ij} \eta_j \big) \, ,$$

and

$$\langle \theta_i \bar\theta_j \rangle = - \frac{\partial}{\partial \bar\eta_i} \frac{\partial}{\partial \eta_j} \exp\big(\bar\eta_i (\underline{B}^{-1})_{ij} \eta_j \big) \Big|_{\eta = \bar\eta = 0} = (\underline{B}^{-1})_{ij} \, .$$

6. Show that

$$\int \mathrm{d}^{2n}\theta \, \exp\big(\tfrac{1}{2} \theta_i A_{ij} \theta_j + \eta_i \theta_i \big) = \mathrm{Pf}(\underline{A}) \, \exp\big(\tfrac{1}{2} \eta_i (\underline{A}^{-1})_{ij} \eta_j \big) \, , \quad \mathrm{d}^{2n}\theta = \mathrm{d}\theta_{2n} \ldots \mathrm{d}\theta_1 \, ,$$

where θ_i and η_i are Grassmann variables, $\underline{A}$ is an invertible antisymmetric matrix and the Pfaffian $\mathrm{Pf}(\underline{A}) = \frac{1}{2^n n!} \varepsilon_{i_1 \ldots i_{2n}} A_{i_1 i_2} \ldots A_{i_{2n-1} i_{2n}}$. Why does $\mathrm{Pf}(\underline{A})^2 = \det \underline{A}$? Hence obtain

$$\langle \theta_i \theta_j \rangle = \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j} \exp\big(\tfrac{1}{2} \eta_i (\underline{A}^{-1})_{ij} \eta_j \big) \Big|_{\eta = 0} = -(\underline{A}^{-1})_{ij} \, .$$

Show that $\delta(\theta) = \theta_1 \ldots \theta_{2n}$ plays the role of a δ-function for Grassmann integrals over $\theta_1, \ldots, \theta_{2n}$.

7. For the simple fermionic oscillator with energy eigenstates $|0\rangle, |1\rangle$ and $H = b^\dagger b \omega$ let $|\theta\rangle = |0\rangle + \theta |1\rangle$, $\langle \bar\theta | = \langle 0 | + \bar\theta \langle 1 |$ where $\theta, \bar\theta$ are Grassmann variables. Show that

$$\mathrm{tr}\big(e^{-\beta H} \big) = \int \mathrm{d}\theta \mathrm{d}\bar\theta \, e^{\bar\theta \theta} \, \langle \bar\theta | e^{-\beta H} | \theta \rangle \, , \qquad \int \mathrm{d}\bar\theta \mathrm{d}\theta \, e^{\theta \bar\theta} \, |\theta\rangle \langle \bar\theta | = 1 \, .$$

Let $\beta = (N+1)\epsilon$ and show that this can be expressed for large N as a path integral over products of $\langle \bar\theta_{i+1} | e^{-\epsilon H} | \theta_i \rangle \approx \exp\big((1 - \epsilon\omega) \bar\theta_{i+1} \theta_i \big)$ so that

$$\mathrm{tr}\big(e^{-\beta H} \big) \approx \int \prod_{i=1}^{N+1} \mathrm{d}\bar\theta_i \mathrm{d}\theta_i \, \exp\big(- \bar\theta_i B_{ij} \theta_j \big) = \det(\underline{B}) \, ,$$

where we take $\bar\theta_{N+1} = \bar\theta$, $\theta_{N+1} = -\theta$ and $\underline{B}$ is a $(N+1) \times (N+1)$ matrix of the form

$$\underline{B} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 1 - \epsilon\omega \\ -1 + \epsilon\omega & 1 & \ldots & 0 & 0 \\ 0 & -1 + \epsilon\omega & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -1 + \epsilon\omega & 1 \end{pmatrix} .$$

Show that

$$\det \underline{B} = 1 + (1 - \epsilon\omega)^{N+1} \to 1 + e^{-\beta\omega} \quad \text{as} \quad N \to \infty \, .$$

8. Let $S = \int_0^\beta \mathrm{d}\tau \big(\tfrac{1}{2} \dot{x}^2 + \bar\theta \, \dot\theta + \tfrac{1}{2} \lambda^2 w'(x)^2 + \lambda w''(x) \, \bar\theta \, \theta \big)$. Assume $x(\tau)$ and the Grassmannian $\theta(\tau), \bar\theta(\tau)$ are all periodic with period β. Show that

$$\mathcal{Q}\Big(\bar\theta \cdot w' \, e^{-S} \Big) = - \frac{\partial}{\partial \lambda} e^{-S} \, , \text{ for } \mathcal{Q} = \int_0^\beta \mathrm{d}\tau \left(\theta \, \frac{\delta}{\delta x} + \big(\lambda \, w'(x) + \dot{x} \big) \frac{\delta}{\delta \bar\theta} \right) , \quad \bar\theta \cdot w' = \int_0^\beta \mathrm{d}\tau \, \bar\theta w'(x) \, .$$

Hence show that

$$Z = \int \mathrm{d}[x] \, \mathrm{d}[\bar\theta] \, \mathrm{d}[\theta] \, e^{-S} \, .$$

is independent of λ if $\lambda \neq 0$. Calculate Z for $\lambda w'(x) = \omega \, x$ if $Z = 1$ for $\omega = 0$.