Mathematical Tripos Part III **Lent Term 2012**
Advanced Quantum Field Theory: Examples 3 **H. Osborn**

Feynman Graph and RG Calculations

1. Consider a theory in 4 dimensions which contains both scalar fields ϕ and spinor fields ψ. The interaction lagrangian contains vertices of different types, labelled by r, which have the general structure $\lambda_r\, \phi^{b_r} \bar{\psi}^{f_r} \psi^{f_r}$. Show that the superficial degree of divergence of a diagram with E_B external boson lines, E_F external fermion lines, and n_r vertices of type r is

$$D = 4 - E_B - \tfrac{3}{2} E_F + \textstyle\sum_r n_r \delta_r \qquad \text{with} \qquad \delta_r = b_r + 3 f_r - 4\ .$$

Relate δ_r to the dimension of the coupling constant λ_r. A fermion propagator behaves like $1/p$ for large momentum p.

2. What renormalisable, polynomial interactions are allowed for a scalar field in $d = 2$ spacetime dimensions? Draw all divergent connected one particle irreducible diagrams which arise for ϕ^4 theory in $d = 2$. Are the examples consistent with the statement that this theory can be rendered finite in the operator approach by normal ordering? [In terms of diagrams, normal ordering is equivalent to the restriction that propagators cannot begin and end on the same vertex.] What renormalisable interactions are allowed for a spinor (fermion) field in $d = 2$?

3. With $c, c' > 0$ let

$$I_1(c) = \int_0^\infty \!\mathrm{d}x\, \frac{x^{1-\varepsilon}}{x+c}\ ,\ I_2(c,c') = \int_0^\infty \!\mathrm{d}x\, \frac{x^{1-\varepsilon}}{(x+c)(x+c')}\ ,\ I_3(c) = \int_0^\infty \!\mathrm{d}x \int_0^\infty \!\mathrm{d}y\, \frac{x^{1-\varepsilon}\, y^{1-\varepsilon}}{(x+c)(y+c)(x+y+c)}\ .$$

For what ε are these integrals convergent? Using $x^{-\lambda}\Gamma(\lambda) = \int_0^\infty \mathrm{d}\alpha\, \alpha^{\lambda-1} e^{-\alpha x}$ for $x > 0$ and also the identity $\Gamma(1-\varepsilon)\Gamma(\varepsilon) = \pi/\sin \pi\varepsilon$ evaluate the integrals to obtain

$$I_1(c) = -\frac{\pi}{\sin \pi\varepsilon}\, c^{1-\varepsilon}\ ,\ I_2(c,c') = \frac{\pi}{\sin\pi\varepsilon}\, \frac{c^{1-\varepsilon} - c'^{1-\varepsilon}}{c - c'}\ ,\ I_3(c) = \Gamma(1-\varepsilon)^2 \big(\Gamma(2\varepsilon - 1) - \Gamma(\varepsilon)^2\big)\, c^{1-2\varepsilon}\ .$$

Determine the divergent parts of I_1, I_2, I_3 as given by poles in ε. Note that for $\varepsilon = 0$ I_3 has subdivergences when $x \to \infty$ or $y \to \infty$. Explain why the subdivergences may be subtracted by considering

$$I_3(c) - \frac{2}{\varepsilon}\, I_1(c) \sim \frac{1}{\varepsilon^2}(1 - \tfrac{1}{2}\varepsilon)\, c\,,$$

which does not have any $\ln c$ divergent terms.

4. Using $A^{-1} = \int_0^\infty \mathrm{d}\alpha\, e^{-\alpha A}$ prove the general Feynman parameter identity

$$\frac{1}{A_1 A_2 \dots A_n} \ = \ (n{-}1)! \int_0^1 \!\mathrm{d}x_1 \int_0^1 \!\mathrm{d}x_2 \dots \int_0^1 \!\mathrm{d}x_n\, \frac{\delta(1 - x_1 - x_2 - \dots - x_n)}{(x_1 A_1 + x_2 A_2 + \dots + x_n A_n)^n}\ .$$

5. Using $(x^2)^{-\lambda}\Gamma(\lambda) = \int_0^\infty \mathrm{d}\alpha\ \alpha^{\lambda-1}\, e^{-\alpha x^2}$ evaluate $\int \mathrm{d}^d x\ (x^2)^{-\lambda}\, e^{i\, p\cdot x}$. Check that the result is consistent with the standard Fourier inversion formula. Let $G_0(x)$ be the Green function whose Fourier transform is $1/p^2$. Calculate $\int \mathrm{d}^d x\ G_0(x)^n\, e^{i\, p\cdot x}$ for $n = 2, 3$. Show that the poles in $\varepsilon = 4 - d$ are in exact agreement with one and two loop momentum space calculations. What happens for $d \approx 3$? Why should we consider $n = 3, 4, 5$ in this case?

6. For the scalar field theory with $V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{24} \mu^\varepsilon \lambda \phi^4$, with $d = 4 - \varepsilon$ and μ an arbitrary mass scale so that λ is dimensionless, the counterterms necessary for a finite four dimensional theory are written as

$$\mathcal{L}_{\text{c.t.}}(\phi) = -\tfrac{1}{2} A\, (\partial\phi)^2 - \tfrac{1}{2} B\, \phi^2 - \tfrac{1}{24} \mu^\varepsilon \lambda D\, \phi^4\ .$$

At one and two loops the non zero results of calculations are, for $\hat\lambda = \lambda/16\pi^2$,

$$A^{(2)} = -\frac{\hat\lambda^2}{12\varepsilon}\,, \quad B^{(1)} = \frac{\hat\lambda m^2}{\varepsilon}\,, \quad B^{(2)} = \hat\lambda^2 m^2\Big(\frac{2}{\varepsilon^2} - \frac{1}{2\varepsilon}\Big)\,, \quad D^{(1)} = \frac{3\hat\lambda}{\varepsilon}\,, \quad D^{(2)} = 3\hat\lambda^2\Big(\frac{3}{\varepsilon^2} - \frac{1}{\varepsilon}\Big)\,.$$

Find corresponding expressions for λ_0, m_0^2 and determine $\beta_\lambda(\lambda)$ and $\gamma_{m^2}(\lambda)$ to two loop order. Check that the ε^{-2} poles have the necessary coefficients.

7. Let $\hat\tau_n(p_1,\dots,p_n)$ be the one particle irreducible functions in four dimensions with n external lines with incoming momenta p_i after removal of a factor $i(2\pi)^4\delta(\sum p_i)$. In ϕ^4 theory, as calculated using dimensional regularisation in the $\overline{MS}$ scheme, show that to one loop we may obtain the finite results

$$\hat\tau_2(p,-p) = -p^2 - m^2 + \frac{\lambda m^2}{32\pi^2}\Big(1 - \ln\frac{m^2}{\mu^2}\Big)\,, \quad \hat\tau_4(p_1,p_2,p_3,p_4) = -\lambda - \frac{\lambda^2}{32\pi^2}\Big(f(s)+f(t)+f(u)\Big)\,,$$

where

$$f(s) = \int_0^1 \mathrm{d}\alpha\, \ln\frac{m^2 - \alpha(1-\alpha)s}{\mu^2}\,, \qquad s = -(p_1+p_2)^2\,,\ t = -(p_1+p_3)^2\,,\ u = -(p_1+p_4)^2\,.$$

Show that $\hat\tau_2(p,-p)$ and $\hat\tau_4(p_1,p_2,p_3,p_4)$ satisfy the RG equations

$$\Big(\mu\frac{\partial}{\partial\mu} + \frac{3\lambda^2}{16\pi^2}\frac{\partial}{\partial\lambda} + \frac{\lambda m^2}{16\pi^2}\frac{\partial}{\partial m^2}\Big)\hat\tau_n = \begin{cases} \mathrm{O}(\lambda^2), & n=2; \\ \mathrm{O}(\lambda^3), & n=4. \end{cases}$$

For $s=t=u$ and $-s \gg m^2$ in $\hat\tau_4$ verify that

$$\hat\tau_4(p_1,p_2,p_3,p_4) = -\Big(\frac{1}{\lambda} - \frac{3}{32\pi^2}\ln\Big(\frac{-s}{\mu^2}\Big)\Big)^{-1}\,,$$

satisfies the equation with zero right hand side. Suppose $\hat\tau_4(p,p,-p,-p) = -\lambda'$ for $p^2 = -m^2$ is an alternative definition of the coupling. Find λ' in terms of λ and express $\hat\tau_4$ in terms of λ' to $\mathrm{O}(\lambda'^2)$. Note that $\hat\tau_4$ is then independent of μ but that the limit $m^2 \to 0$ is singular.

8. Consider a scalar field ϕ where $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{6}\mu^{\frac{1}{2}\varepsilon} g\phi^3$ in dimension d and $\varepsilon = 6-d$. Here μ is an arbitrary mass scale so that g is dimensionless. Draw the one-loop one particle irreducible graph which contributes to the propagator at order g^2. Show that the divergent part of the corresponding integral using dimensional regularisation for the six dimensional theory is

$$-\frac{1}{\varepsilon}\frac{g^2}{(4\pi)^3}\left(m^2 + \tfrac{1}{6}p^2\right)\,,$$

where p is the external momentum. Using dimensional regularisation compute the divergence corresponding to the one particle irreducible one-loop graph giving a g^3 correction to three point function. Find also the one loop divergence for the one point function. Show that these divergences in six dimensions may be cancelled by introducing the counterterm Lagrangian

$$\mathcal{L}_{\text{c.t.}}(\phi) = \frac{1}{\varepsilon}\frac{1}{6(4\pi)^3}\left(\tfrac{1}{2}g^2(\partial\phi)^2 + \mu^{-\varepsilon}V''(\phi)^3\right)\,.$$

Check that $\mathcal{L}_{\text{c.t.}}$ has dimension d.

9. For the six dimensional ϕ^3 theory of the previous question express the bare g_0, m_0^2 in terms of the dimensionless coupling g and m^2 and also an arbitrary scale mass μ to lowest order. Determine the beta function $\beta_g(g)$ and also $\gamma_{m^2}(g)$ to lowest order and show that $\beta_g(g) < 0$ for g small.

10. For a theory with multiple couplings g^i the beta function $\beta^i(g) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g^i$ defines a vector field. Show that under a redefinition $g^i \to g'^i(g)$ we have

$$\beta'^i(g') = \frac{\partial g'^i}{\partial g^j}\,\beta^j(g)\,.$$

Show that for a single coupling with $\beta(g) = b_1 g^3 + b_2 g^5 + \mathrm{O}(g^7)$ and $g'(g) = g + \mathrm{O}(g^3)$ the first two terms in the beta function are invariant. Show also that it is possible to choose $g'(g)$ so that all terms other than the first two are zero.