Local Renormalisation Group Equations in Quantum Field Theory

H. OSBORN

DAMTP, University of Cambridge, Silver St.,
Cambridge CB3 9EW, England.

ABSTRACT

A local renormalisation group equation is formulated for renormalisable theories which describes the effect of local Weyl rescalings of the metric. In two dimensions the resulting equations are shown to correspond to Zamolodchikov's c-theorem and in four dimensions to give results which may have a similar significance.

The conventional renormalisation group describes a flow in the space of theories, as parameterised by a set of couplings, induced by a change in the overall mass scale. In quantum field theory or statistical mechanical models, there is either a natural or imposed cut off Λ at short distances but the theory becomes physically relevant, and also the amplitudes and other observables are largely independent of the initial cut off field theory when the correlation length, expressed in the units of $1/\Lambda$ diverges. In such a limit the dependence on Λ can be eliminated in favour of an arbitrary finite mass scale μ which corresponds to a point on the renormalisation flow trajectory in the neighbourhood of an infra red stable fixed point. In this neighbourhood the theory is characterised by a set of renormalisable couplings g^i. By introducing suitable explicit factors of μ g^i may be assumed to be dimensionless. In such continuum field theories the flow is generated by a differential operator of the form

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta^i\frac{\partial}{\partial g^i} \quad , \tag{1}$$

where $\beta^i(g)$, the β function, is a vector field on the space of couplings. In general $\beta^i(g) = k_i g^i + \mathrm{O}(g^2)$ where k_i results from the explicit μ dependence introduced in the basic action S and depends on the dimension of the operator $\mathcal{O}_i$ corresponding to g^i in S. For a strictly renormalisable theory in dimension d $\mathcal{O}_i$ has dimension d and $k_i = 0$.

For a field theory the physical amplitudes and other observables may be expressed in terms of the vacuum energy W defined by a functional integral over fields ϕ

$$e^W = \int d[\phi]\, e^{-S_0} \quad , \tag{2}$$

with S_0 formed from the initial action S and depending on the cut off Λ in such a way that $W(g)$ is a finite scalar function of the couplings g^i as $\Lambda \to \infty$. With an appropriate renormalisation scheme W is invariant along the renormalisation flow so that it obeys a homogeneous renormalisation group or Callan-Symanzik equation

$$\mathcal{D}W(g) = 0 \quad . \tag{3}$$

If W is also regarded as a function of a constant spatial metric $\gamma_{\mu\nu}$, so that $(\gamma_{\mu\nu}x^\mu x^\nu)^{\frac{1}{2}}$ denotes the length of x, then by dimensional analysis W is invariant under rescalings of length with a corresponding change in μ. This may also be expressed as a differential equation

$$\Big(\mu\frac{\partial}{\partial\mu} + 2\,\gamma^{\mu\nu}\frac{\partial}{\partial\gamma^{\mu\nu}}\Big)W = 0 \quad . \tag{4}$$

One approach to defining a local renormalisation group equation would be to allow the cut off and hence μ to be x dependent although the interpretation is not clear[1,2]. Here we consider the extension of the usual renormalisation group to describe arbitrary local rescalings of the length[3], as realised by Weyl transformations $\gamma_{\mu\nu} \to e^{-2\sigma}\gamma_{\mu\nu}$, with $\sigma(x)$ arbitrary, when the field theory is defined to a curved space background with general metric $\gamma_{\mu\nu}(x)$. Such Weyl transformations form an infinite dimensional abelian group extending the one dimensional group of constant rescalings and which leads to the usual renormalisation group. The fixed points under Weyl rescalings correspond to conformal field theories which, in two dimensions, are the basic building blocks of string theories. The conditions for Weyl invariance represent string equations of motion.

If W is a finite functional of arbitrary local metrics $\gamma_{\mu\nu}(x)$ then it is straightforward to define the expectation value of the energy momentum tensor $T_{\mu\nu}(x)$ by functional differentiation

$$\big\langle T_{\mu\nu}(x)\big\rangle = 2\ell\,\frac{\delta}{\delta\gamma^{\mu\nu}(x)}W \quad , \tag{5}$$

where ℓ is some constant depending on conventions. Furthermore connected correlation functions at non coincident points are similarly obtained.

$$\big\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)\ldots\big\rangle = 2\ell\,\frac{\delta}{\delta\gamma^{\mu\nu}(x)}\,2\ell\,\frac{\delta}{\delta\gamma^{\sigma\rho}(y)}\ldots W \quad , \qquad x \neq y \quad . \tag{6}$$

Since we need to obtained well-defined local operator equations it is also necessary to be able to define general finite local operators $\mathcal{O}_i(x)$ corresponding to the set of renormalisable couplings g^i and which form a closed set under operator mixing. A very convenient formalism[4,5,6] which ensures this is to allow all couplings g^i to be arbitrary functions of x so that they act as sources for the associated local operators $\mathcal{O}_i$. Although translational invariance is lost the essential renormalisability of the theory should be maintained, although new counterterms depending on $\partial_\mu g^i$ will be necessary. Such terms play a vital role in ensuring that it is possible to obtain finite local composite operators by functional differentiation. We may therefore define

$$\big\langle \mathcal{O}_i(x) \big\rangle = \ell \, \frac{\delta}{\delta g^i(x)} W \quad , \tag{7}$$

with $\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \ldots \rangle$ also defined similarly to (5).

In this framework it is feasible to formulate finite equations representing the effect of infinitesimal local Weyl transformations $\delta\gamma^{\mu\nu} = 2\sigma\gamma^{\mu\nu}$. The generator of such Weyl transformations is

$$\Delta^W_\sigma = 2 \int dv \, \sigma \, \gamma^{\mu\nu} \frac{\delta}{\delta \gamma^{\mu\nu}} \quad , \qquad dv = d^d x \sqrt{\gamma} \quad , \tag{8}$$

while the associated corresponding variation of the couplings, as determined by the β function, is given by

$$\Delta^\beta_\sigma = \int dv \, \sigma \, \beta^i \frac{\delta}{\delta g^i} \quad , \tag{9}$$

The essential local renormalisation group equation then takes the form

$$\Delta^W_\sigma W = \Delta^\beta_\sigma W + \text{terms involving derivatives of } \gamma_{\mu\nu}, g^i, \sigma \quad . \tag{10}$$

When the derivative terms are absent, for a flat background and g^i, σ constant, this reduces to the usual Callan-Symanzik equation given by (1), (3) and (4). It is crucial that the additional terms in (10) should also be expressible as analogous local functional derivatives of W. Assuming the cut off theory preserves invariance under diffeomorphisms these derivative terms must also be invariant, and so derivatives of the metric appear also in the form of the curvature tensor. (10) is equivalent to the local equation

$$\gamma^{\mu\nu} T_{\mu\nu} = \beta^i \mathcal{O}_i + \nabla_\mu \mathcal{Z}^\mu + \text{curvature, } \partial_\mu g \text{ terms} \quad , \tag{11}$$

where $\mathcal{Z}^\mu$ is a vector operator and the other terms are local scalars. The additional pieces in (10) or (11) may be calculated directly in the loop expansion for renormalised perturbation theory, much effort has ensured knowledge of the curvature dependent terms in the trace of the energy momentum tensor at one and higher loops[7] and the $\partial_\mu g$ terms in particular renormalisable theories have also been recently calculated using dimensional regularisation[4,6]. These contributions are strongly constrained by power counting.

Two Dimensional Theories

The analysis of the additional terms in (10) is simplest for two dimensional field theories where we consider only terms proportional to the identity, or pure c-numbers. We assume therefore that the dimensions of all other operators are strictly positive and in particular neglect any vector operators that may be present. In this case (10) becomes explicitly

$$\Delta^W_\sigma W = \Delta^\beta_\sigma W + \frac{1}{\ell}\int dv\, \sigma\big(\tfrac{1}{2}\,\beta^\Phi R - \tfrac{1}{2}\,\chi_{ij}\partial_\mu g^i \partial^\mu g^j\big) - \frac{1}{\ell}\int dv\, \partial_\mu \sigma\, w_i \partial^\mu g^i \quad . \tag{12}$$

R is the scalar curvature while $\beta^\Phi(g)$ in a string context is the dilaton β function. $\chi_{ij}(g)$ is a symmetric tensor while $w_i(g)$ is a vector on the space of couplings. In renormalised perturbation theory only couplings g^i corresponding to strictly renormalisable interactions, so that $\mathcal{O}_i$ has dimension 2, appear.

Of course W is arbitrary up to the addition of a local functional representing the usual arbitrariness in the definition of renormalised couplings. In this case we may consider

$$\delta W = -\frac{1}{\ell}\int dv\,\big(\tfrac{1}{2} bR - \tfrac{1}{2} c_{ij}\partial_\mu g^i \partial^\mu g^j\big) \quad , \tag{13}$$

It is easy to show that, using $\delta_\sigma R = 2\sigma R + 2\nabla^2\sigma$, this gives

$$\begin{aligned} &\delta\beta^\Phi = \beta^i\partial_i b \quad , \qquad \delta\chi_{ij} = \mathcal{L}_\beta c_{ij} = \beta^k\partial_k c_{ij} + \partial_i\beta^k c_{kj} + \partial_j\beta^k c_{ik} \quad , \\ &\delta w_i = \, -\partial_i b + c_{ij}\beta^j \quad . \end{aligned} \tag{14}$$

where $\mathcal{L}_\beta$ denotes the Lie derivative. In general it should be noted that β^Φ, χ_{ij}, w_i cannot be transformed to zero, except if $w_i = \partial_i X$ it is possible to take $w_i \to 0$ by a choice of b.

The extra terms in (12) may be regarded as an anomaly for local scale transformations and as usual they satisfy a consistency condition

$$\begin{aligned} 0 = \big[\Delta^W_\sigma - \Delta^\beta_\sigma, \Delta^W_{\sigma'} - \Delta^\beta_{\sigma'}\big] W = \frac{1}{\ell}\int dv\,(\sigma\partial_\mu\sigma' - \sigma'\partial_\mu\sigma)V^\mu \quad , \\ V_\mu = \partial_\mu\beta^\Phi - \chi_{ij}\partial_\mu g^i\beta^j + \beta^j\frac{\partial}{\partial g^j}\big(w_i\partial_\mu g^i\big) \quad . \end{aligned} \tag{15}$$

Since $g^i(x)$ is arbitrary the condition $V_\mu = 0$ becomes

$$\partial_i \beta^\Phi = \chi_{ij}\beta^j - \mathcal{L}_\beta w_i \quad , \qquad \mathcal{L}_\beta w_i = \beta^j \partial_j w_i + \partial_i \beta^j w_j \quad . \tag{16}$$

An alternative form is

$$\partial_i \tilde{\beta}^\Phi = \chi_{ij}\beta^j + (\partial_i w_j - \partial_j w_i)\beta^j \quad , \qquad \tilde{\beta}^\Phi = \beta^\Phi + w_i \beta^i \quad , \tag{17}$$

so that $\tilde{\beta}^\Phi$ is stationary when $\beta^i = 0$ and also

$$\beta^i \partial_i \tilde{\beta}^\Phi = \chi_{ij}\beta^i\beta^j \quad . \tag{18}$$

Of course these equations (16), (17) and (18) are invariant under (14) when also $\delta\tilde{\beta}^\Phi = c_{ij}\beta^i\beta^j$.

The result (18) is equivalent to Zamolodchikov's c-theorem[8] if χ_{ij} were positive and (17) indicates that $\tilde{\beta}^\Phi$ is effectively an action for the equations $\beta^i = 0$. (17) is sharper than (18), since the r.h.s. is constrained by integrability constraints. This equation, for $w_i = 0$, was also obtained by Zamolodchikov[8] but only in lowest order for perturbations around a conformal field theory.

The crucial equation (12) contains in concise form relations for correlation functions which may be obtained by functional differentiation and then restricting the flat space and also g^i constant. In two dimensions the essential content is found by considering just two point correlation functions. Thus by taking the derivative of (12) with respect to $\gamma^{\mu\nu}$ or g^i we obtain

$$\big\langle T_{\rho\rho}(x)\, T_{\mu\nu}(0)\big\rangle - \big\langle \Theta(x)\, T_{\mu\nu}(0)\big\rangle = \ell\, \beta^\Phi \big(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu\big)\delta^2(x) \quad , \tag{19a}$$

$$\big\langle T_{\mu\mu}(x)\, \mathcal{O}_i(0)\big\rangle - \big\langle \Theta(x)\, \mathcal{O}_i(0)\big\rangle = \ell\, w_i \partial^2\delta^2(x) \quad , \qquad \Theta = \beta^i \mathcal{O}_i \quad , \tag{19b}$$

Furthermore (12) contains the renormalisation group equations, with $\mathcal{D}$ as in (1),

$$\mathcal{D}\big\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\big\rangle = 0 \quad , \tag{20a}$$

$$\mathcal{D}\big\langle \mathcal{O}_i(x)\, T_{\mu\nu}(0)\big\rangle + \partial_i\beta^j\big\langle \mathcal{O}_j(x)\, T_{\mu\nu}(0)\big\rangle = -\,\ell\, \partial_i\beta^\Phi\big(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu\big)\delta^2(x) \quad , \tag{20b}$$

$$\mathcal{D}\big\langle \mathcal{O}_i(x)\, \mathcal{O}_j(0)\big\rangle + \partial_i\beta^k\big\langle \mathcal{O}_k(x)\, \mathcal{O}_j(0)\big\rangle + \partial_j\beta^k\big\langle \mathcal{O}_i(x)\, \mathcal{O}_k(0)\big\rangle = -\,\ell\chi_{ij}\partial^2\delta^2(x) \,. \tag{20c}$$

The differentiation automatically generates the anomalous dimension matrix $\partial_i\beta^j$ for the operators $\mathcal{O}_j$. This r.h.s. of (19a,b) and (20c) may be regarded as alternative definitions of β^Φ, w_i and χ_{ij}, χ_{ij} arises as a result of the additional divergences present in the two point correlation functions of operators $\mathcal{O}_i$ not removed by conventional renormalisations which ensure that $\mathcal{O}_i$ is a finite local operator. The r.h.s.

of (20b) is dictated by compatibility with (19a) and it is not difficult to derive the essential result (16) by consistency.

To see in more detail the connection with Zamolodchikov's c-theorem we write

$$\begin{aligned} \big\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\big\rangle &= \big(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu\big)\big(\partial^2\delta_{\sigma\rho}-\partial_\sigma\partial_\rho\big)\Omega(t) \quad , \\ \big\langle T_{\mu\nu}(x)\, \mathcal{O}_i(0)\big\rangle &= \big(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu\big)\partial^2\Omega_i(t) \quad , \\ \big\langle \mathcal{O}_i(x)\, \mathcal{O}_j(0)\big\rangle &= \partial^2\partial^2\Omega_{ij}(t) \quad , \qquad t=\tfrac{1}{2}\ln\mu^2x^2 \quad . \end{aligned} \tag{21}$$

With these definitions, choosing $\ell=-4\pi$, we may obtain

$$\begin{aligned} &\Omega'(t)-\beta^i\Omega_i'(t)=-2\beta^\Phi \quad , \qquad \Omega_i'(t)-\beta^j\Omega_{ij}'(t)=-2w_i \quad , \\ &\Omega''+\mathcal{L}_\beta\Omega'=0 \quad , \qquad \Omega_i''+\mathcal{L}_\beta\Omega_i'=2\partial_i\beta^\Phi \quad , \qquad \Omega_{ij}''+\mathcal{L}_\beta\Omega_{ij}'=2\chi_{ij} \quad , \end{aligned} \tag{22}$$

where $\Omega'(t)=\partial_t\Omega(t)$ (clearly any constants in Ω, Ω_i or Ω_{ij} are irrelevant). The Zamolodchikov metric is defined by

$$G_{ij}(t)=\tfrac{1}{8}(x^2)^2\big\langle \mathcal{O}_i(x)\, \mathcal{O}_j(0)\big\rangle \quad , \tag{23}$$

and is positive for unitary theories. Using (21) and (22) we find

$$G_{ij}=\tfrac{1}{2}\Omega_{ij}''-\tfrac{1}{2}\Omega_{ij}'''+\tfrac{1}{8}\Omega_{ij}''''=\chi_{ij}+\mathcal{L}_\beta c_{ij} \quad , \qquad c_{ij}=-\tfrac{1}{2}\Omega_{ij}'+\tfrac{1}{2}\Omega_{ij}''-\tfrac{1}{8}\Omega_{ij}''' \quad . \tag{24}$$

Clearly therefore χ_{ij} is equivalent to the positive G_{ij} under the arbitrariness expressed by (16). At the same time we therefore define

$$\begin{aligned} W_i=w_i+c_{ij}\beta^j &= -\tfrac{1}{2}\Omega_i'+\tfrac{1}{2}\Omega_i''-\tfrac{1}{8}\Omega_i''' \quad , \\ \tfrac{1}{3}C=\beta^\Phi+W_i\beta^i &= -\tfrac{1}{2}\Omega'+\tfrac{1}{2}\Omega''-\tfrac{1}{8}\Omega''' \quad , \end{aligned} \tag{25}$$

where C is equivalent to Zamolodchikov's definition, if $F=\frac{1}{8}z^4\langle T_{zz}(x)T_{zz}(0)\rangle$, $G=\frac{1}{8}z^2x^2\langle T_{zz}(x)T_{\mu\mu}(0)\rangle$, $H=\frac{1}{8}(x^2)^2\langle T_{\mu\mu}(x)T_{\nu\nu}(0)\rangle$ then $C=4F-2G-\frac{3}{4}H$, $H=G_{ij}\beta^i\beta^j$. Since (16,17) are invariant under (14)

$$\tfrac{1}{3}\partial_i C=G_{ij}\beta^j+(\partial_iW_j-\partial_jW_i)\beta^j \quad , \tag{26a}$$

$$\tfrac{1}{3}C' = -\,\beta^i\partial_iC=-G_{ij}\beta^i\beta^j\leq 0 \quad , \tag{26b}$$

where (26b) is the essential result of Zamolodchikov's c-theorem. There is no simple definition for W_i in terms of linear combinations of correlation functions for $x\neq$

0, similar to G_{ij} in (23) or C, which remains valid in the limit of a conformal field theory. For $\beta^i = 0 \quad W_i = w_i$ is determined by a contact term $\propto \partial_z{}^2\delta^2(x)$ in the operator product of $T_{zz}(x)$ and $\mathcal{O}_i(0)$ which is presumably only present for marginal operators. Given the definition (25) for W_i it is straightforward to see directly that $\frac{1}{8}(x^2)^2\langle T_{\mu\mu}(x)\,\mathcal{O}_i(0)\rangle = \frac{1}{2}\Omega_i'' - \frac{1}{2}\Omega_i''' + \frac{1}{8}\Omega_i'''' = \mathcal{L}_\beta W_i + \partial_i\beta^\Phi = \partial_i\frac{1}{3}C - (\partial_i W_j - \partial_j W_i)\beta^j = G_{ij}\beta^j$. When $\beta^i = 0$ $\langle T_{\mu\mu}(x)\,T_{\nu\nu}(0)\rangle \propto \partial^2\delta^2(x)$ but this can be removed by a counterterm breaking two dimensional reparameterisation invariance. In such conformal field theories $\langle T_{zz}(x)\,T_{zz}(0)\rangle = 2C/z^4$ so that C becomes the central charge in the Virasoro algebra. Many authors[9] have calculated the renormalisation flow of C in conformal field theories perturbed by just relevant operators, i.e. operators with dimension slightly less than 2, and found consistency with the framework of the c-theorem.

Four Dimensional Field Theories

The previous discussion may be extended to four dimensional field theories, although with significant additional complications since many more terms are possible[3,6]. While in two dimension the essential operator equation on a curved background but for constant couplings is

$$\gamma^{\mu\nu}T_{\mu\nu} = \beta^i\mathcal{O}_i + \tfrac{1}{2}\beta^\Phi R \quad , \tag{27}$$

in four dimensions

$$\begin{aligned}\gamma^{\mu\nu}T_{\mu\nu} &= \beta^i\mathcal{O}_i - \beta_a\,F - \beta_b\,G - \beta_c\,\tfrac{1}{9}R^2 + d\,\tfrac{1}{3}\nabla^2 R \quad , \\ F &= R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{3}R^2 \quad , \\ G &= R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2 \quad , \end{aligned} \tag{28}$$

neglecting any curvature terms involving lower dimension operators. Each of the coefficients $\beta_a, \beta_b, \beta_c$ has been considered as possible candidate for a generalisation of the c-theorem[10,11]. Neglecting lower dimension operators the local renormalisation group equation becomes (taking $\ell = -1$ now)

$$\begin{aligned}\Delta_\sigma^W W &= \Delta_\sigma^\beta W + \int dv\,\sigma\,\mathcal{B} + \int dv\,\partial_\mu\sigma\mathcal{Z}^\mu - \int dv\,\nabla^2\sigma\,\mathcal{A} \quad , \\ \mathcal{B} &= \beta_a\,F + \beta_b\,G + \tfrac{1}{9}\beta_c\,R^2 \\ &\quad + \tfrac{1}{3}\chi_i^e\partial_\mu g^i\,\partial^\mu R + \tfrac{1}{6}\chi_{ij}^f\partial_\mu g^i\partial^\mu g^j\,R + \tfrac{1}{2}\chi_{ij}^g\partial_\mu g^i\partial_\nu g^j\,G^{\mu\nu}\end{aligned}$$

$$+ \tfrac{1}{2}\chi^a_{ij}\, \nabla^2 g^i \nabla^2 g^j + \tfrac{1}{2}\chi^b_{ijk}\, \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k + \tfrac{1}{4}\chi^c_{ijk\ell}\, \partial_\mu g^i \partial^\mu g^j \partial_\nu g^k \partial^\nu g^\ell \quad ,$$
$$\mathcal{Z}_\mu = G_{\mu\nu}\, w_i \partial^\nu g^i + \tfrac{1}{3} R\, Y_i \partial_\mu g^i + S_{ij}\, \partial_\mu g^i \nabla^2 g^j + \tfrac{1}{2} T_{ijk}\, \partial_\nu g^i \partial^\nu g^j \partial_\mu g^k \quad ,$$
$$\mathcal{A} = \tfrac{1}{3} d\, R + U_i \nabla^2 g^i + \tfrac{1}{2} V_{ij} \partial_\mu g^i \partial^\mu g^j \quad , \tag{29}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta} R$ is the Einstein tensor. This is equivalent to (28) when $\partial_\mu g^i = 0$. Despite complications it is nevertheless possible to once more impose the consistency condition

$$\big[\Delta^W_\sigma - \Delta^\beta_\sigma, \Delta^W_{\sigma'} - \Delta^\beta_{\sigma'}\big] W = 0 \quad . \tag{30}$$

It is still important to recognise that the parameters $\beta_a, \beta_b, \ldots$ are arbitrary as a consequence of possible local variations δW involving an integral over terms of the same form as appearing in $\mathcal{B}$ in (29). In this case there are several independent consistency relations but amongst these we may obtain

$$8\partial_i \beta_b = \chi^g_{ij}\, \beta^j - \mathcal{L}_\beta w_i \quad , \tag{31}$$

which is similar in form to (16). The other relations, which determine β_c and generalise earlier results obtained using dimensional regularisation[12], are sensitive to the presence of lower dimension operators[3] so that for instance the expression (28) for $\gamma^{\mu\nu} T_{\mu\nu}$ is modified by additional terms $\beta^a_\eta \mathcal{O}^m_a \frac{1}{3} R + \tau^a \nabla^2 \mathcal{O}^m_a$ for a basis of operators $\mathcal{O}^m_a$ of canonical dimension 2. None of the results involve the coefficient β_a in (28) since F is the square of the Weyl tensor. β_a may be connected with the spin 2 part of the correlation function $\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\rangle$ and analogous equation to (26b) derived but they contain an extra term spoiling the monotonic flow[11]. β_b is not related to two point functions but should be definable by 3 point correlation functions of $T_{\mu\nu}$ in the limit of a conformal field theory when $\beta^i = 0$. There is no connection to any central charge of the conformal group in this case although in both two and four dimensions it is the coefficient of the term in the trace of the energy momentum tensor that integrates to give the Euler number that is relevant in our discussion.

Although χ^g_{ij} has no presently known connection with a positive definite metric, as occurs in two dimensions, the result (31) is nevertheless non trivial. Writing (31) as

$$8\partial_i \tilde{\beta}_b = \chi^g_{ij} \beta^j + (\partial_i w_j - \partial_j w_i)\beta^j \quad , \qquad 8\tilde{\beta}_b = 8\beta_b + w_i \beta^i \quad , \tag{32}$$

there are consistency conditions necessary for the integrability of $\tilde{\beta}_b$. A general renormalisable field theory in four dimensions is characterised by the following dimensionless parameters

$$\begin{array}{lll} \text{gauge coupling } g & \text{interaction } g\bar{\psi}\gamma^\mu A^a_\mu t_a \psi \ , \ t_a{}^\dagger = -t_a & , \\ \text{Yukawa coupling } \Gamma_i & \bar{\psi}\Gamma_i \phi_i \psi \ , \ \Gamma_i^\dagger = \Gamma_i & , \\ \text{quartic scalar coupling } \lambda_{ijk\ell} & \tfrac{1}{24}\lambda_{ijk\ell}\phi_i\phi_j\phi_k\phi_\ell & , \end{array} \tag{33}$$

for a simple gauge group. In perturbation theory at low orders $w_i = \partial_i X$ and the effective metric χ^g_{ij} has been calculated for each of the renormalisable couplings in (33) to the first order in which there is a non zero result. If n_V is the dimension of the gauge group straightforward calculations[6] give

$$\begin{aligned} \chi^g_{ij} dg^i dg^j = {} & \frac{1}{16\pi^2}\frac{4n_V}{g^2}(dg)^2 && \text{1 loop} \\ & + \frac{1}{(16\pi^2)^2}\frac{4}{3}\,\mathrm{tr}(d\Gamma_i d\Gamma_i) && \text{2 loops} \\ & + \frac{1}{(16\pi^2)^3}\frac{1}{72}\, d\lambda_{ijk\ell}\, d\lambda_{ijk\ell} && \text{3 loops} \quad . \end{aligned} \tag{34}$$

This result is renormalisation scheme independent. At 2 loops there are corrections $\propto (dg)^2$ which have also been calculated and at 3 loops we expect further modifications to the first two terms of (34) as well as cross terms $\propto dg d\Gamma_i$. To show the non trivial nature of (32) we consider the one loop gauge coupling dependence of the Yukawa β function

$$\beta^\Gamma_i \sim \frac{g^2}{16\pi^2}\, 3\{t^2, \Gamma_i\} \quad . \tag{35}$$

Inserting this in (32), with $w_i \to 0$, leads to a prediction for the 2 loop Yukawa coupling contribution to the gauge coupling β function

$$\beta^g \sim \frac{2g^3}{(16\pi^2)^2}\frac{1}{n_V}\,\mathrm{tr}(t^2\Gamma^2) \quad , \tag{36}$$

which is in accord with direct calculation. There are other consistency checks involving the Yukawa β function at two loops and also the gauge coupling contribution to $\beta^\lambda_{ijk\ell}$ which may also be verified[6]. Such conditions on β functions were suggested some time ago by Wallace and Zia[13] for purely scalar field theories, but they found no constraints on $\beta^\lambda_{ijk\ell}$ due to the large number of possible terms in an arbitrary metric at $\mathrm{O}(\lambda^2)$. Here the relation between (35) and (36) arises since the lowest order non zero results in (34) occur at differing numbers of loops for the different couplings.

To derive a c-theorem it is essential to go beyond perturbative considerations. At lowest order, or for a free theory,

$$\beta_b^0 = \frac{1}{90 \times 64\pi^2}\big(62n_V + 11n_F + n_S\big) \quad , \tag{37}$$

for n_V vectors, n_F Dirac fermions and n_S scalars. Cardy, who first proposed[14] the possibility of a four dimensional c-theorem based on β_b, has also suggested an interesting application of a possible 4 dimensional c-theorem to QCD when at short distances it is described by free gluons and quarks and at large distances, assuming confinement and no quark mass terms, by an effective chiral Lagrangian describing the massless Goldstone bosons arising from spontaneously broken chiral symmetry. Such scenarios are beyond the scope of the treatment here. However the results in four dimensions, with the positivity of the metric in (34), should rule out the possibility of limit cycles or chaotic behaviour[15] in the renormalisation flow described by perturbatively calculated β functions in the neighbourhood of the origin in the space of couplings, i.e. where perturbative calculations are valid.

References

1. G.M. Shore, *Nucl. Phys.* **B286** (1987) 349.
2. V. Periwal, *Comm. Math. Phys.* **120** (1988) 71.
3. H. Osborn, *Nucl. Phys.* **B363** (1991) 486.
4. H. Osborn, *Nucl. Phys.* **B294** (1987) 595.
5. H. Osborn, *Phys. Lett.* **B222** (1989) 97.
6. I. Jack and H. Osborn, *Nucl. Phys.* **B343** (1990) 647.
7. I. Jack and H. Osborn, *Nucl. Phys.* **B234** (1984) 331;
   I. Jack, *Nucl. Phys.* **B234** 365 (1984); **B253** (1985) 323.
8. A.B. Zamolodchikov, *JETP Lett.* **43** (1986) 43; *Sov. J. Nucl. Phys.* **46** (1988) 1090.
9. A.W.W. Ludwig and J.L. Cardy, *Nucl. Phys.* **B285** [FS19] (1987) 687;
   R.G. Pogosynan, *Sov. J. Nucl. Phys.* **48** (1987) 763;
   D.A. Kastor, E.J. Martinec and S.H. Shenker, *Nucl, Phys.* **B316** (1989) 590;
   A.N. Redlich, *Phys. Lett.* **216B** (1989) 129;

Y. Kitazawa et al. *Nucl. Phys.* **B306** (1988) 425;
C. Crnokovic, G. Sotkov and M. Stanishkov, *Phys. Lett.* **226B** (1989) 297;
M.T. Grisaru, A. Lerda, S. Penati and D. Zanon, *Phys. Lett.* **234B** (1990) 88; *Nucl, Phys.* **B342** (1990) 564;
A. Capelli and J.I. Latorre, *Nucl. Phys.* **B340** (1990) 659;
M. Lässig, *Nucl, Phys.* **B334** (1990) 652;
D.Z. Freedman. J.I. Latorre and X. Vilasís, *Mod. Phys. Lett.* **A6** (1991) 531.

10. A. Capelli, D. Friedan and J.I. Latorre, *Nucl. Phys.* **B352** (1991) 616.
11. G.M. Shore, *Phys. Lett.* **B253** (1991) 380; **256** (1991) 407.
12. L.S. Brown and J.C. Collins, *Ann. Phys.* **130** (1981) 215;
S.J. Hathrell, *Ann. Phys.* **139** (1982) 136; **142** (1982) 34;
M.D. Freeman, *Ann. Phys.* **153** (1984) 339.
13. D.J. Wallace and R.K.P. Zia, *Phys. Lett.* **48A** (1974) 327; *Ann. Phys.* **92** (1975) 142.
14. J.L. Cardy, *Phys. Lett.* **B215** (1988) 749.
15. P.H. Damgaard, CERN preprint TH-6073/91.