Mathematical Tripos Part III Lent Term 2020
Black Holes: Examples Sheet 2 H.S. Reall

1. A general static, spherically symmetric metric can be written

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2,$$

where $d\Omega^2$ is the metric on a unit 2-sphere. Assume that $A(r)$ and $B(r)$ are analytic functions of r such that both have a simple zero at $r = r_+ > 0$ and are positive for $r > r_+$.

(a) Show that radial null geodesics are given by $t \pm r^* = \text{constant}$, where

$$r^* \equiv \int_{r_0}^{r} \frac{dx}{\sqrt{A(x)B(x)}},$$

with $r_0 > r_+$ an arbitrary constant. Show that $r^* \to -\infty$ as $r \to r_+$.

(b) Obtain the metric in ingoing Eddington-Finkelstein coordinates. Explain why this metric can be analytically continued through $r = r_+$.

2. Consider a particle with 4-velocity U in a stationary, asymptotically flat, space-time with timelike Killing vector field k. $E = -k \cdot U$ has the interpretation of "energy per unit mass measured at infinity" if the particle moves on a geodesic. For non-geodesic motion, this equation is used to *define* "energy per unit mass measured at infinity."

(a) Consider a unit mass particle P following an orbit of k at radius $r = r_P > 2M$ in the Schwarzschild spacetime. Assume that the force making this particle accelerate comes from a radial massless inelastic string, whose other end is held by an observer Q at infinity. If Q pulls the string through proper distance δS then what is the change δr_P in r_P ?

(b) What is the change δE in the energy of P measured by Q? This must equal the work $F\delta S$ done by Q where F is the force that the string exerts on Q, i.e., the tension at Q. Calculate F. Show that $F \to 1/(4M)$ as $r_P \to 2M$. What is the force measured by P as $r_P \to 2M$?

3. Consider 2d Minkowski spacetime with coordinates (t, x). Delete the points $(\pm 1, 0)$. Let Σ be a surface of constant t. Sketch $D(\Sigma)$, distinguishing the cases $t > 1$, $-1 < t < 1$ and $t < -1$.

4. Prove that the extrinsic curvature of a spacelike hypersurface with unit normal vector n^a satisfies $K_{ab} = (1/2)\mathcal{L}_n h_{ab}$. Deduce that a surface of constant t in a static spacetime has $K_{ab} = 0$.

5. Use isotropic coordinates to prove that a surface of constant t in the Schwarzschild spacetime is an asymptotically flat end (with $K_{ab} = 0$).

6. Consider a metric written in the $3 + 1$ form

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right).$$

Let $T = \partial/\partial t$, $X = N^i \partial/\partial x^i$ and $n = N^{-1}(T - X)$. (a) Verify that $n_a = -N(dt)_a$ and hence that n^a is the future-directed unit normal to surfaces of constant t. (b) By writing n_a in terms of T_a and X_a and using $K_{ab} = h_a^c h_b^d \nabla_c n_d$, show that $K_{ij} = (2N)^{-1}\left(\dot{h}_{ij} - D_i N_j - D_j N_i\right)$ where a dot denotes a derivative w.r.t. t, D_i is the Levi-Civita connection of h_{ij}, and $N_i = h_{ij}N^j$.

7. A perfect fluid has stress tensor $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, where ρ is the energy density, p the pressure, and u^a the 4-velocity of the fluid. Show that

   (a) the dominant energy condition is obeyed if, and only if, $\rho \geq |p|$;

   (b) the weak energy condition is obeyed if, and only if, $\rho \geq 0$ and $\rho + p \geq 0$;

   (c) the null energy condition is obeyed if, and only if, $\rho + p \geq 0$;

   (d) the strong energy condition is obeyed if, and only if, $\rho + 3p \geq 0$ and $\rho + p \geq 0$.

   A cosmological constant has $p = -\rho$. Which energy conditions does it violate? (Consider both signs for ρ.)

8. Consider two Lorentzian metrics on a manifold M related by a conformal transformation $\bar{g} = \Omega^2 g$ where Ω is a positive function on M.

   (a) Show that g and $\bar{g}$ have the same null geodesics.

   (b) Show that the Ricci tensor of g is related to the Ricci tensor of $\bar{g}$ by

$$R_{ab} = \bar{R}_{ab} + 2\Omega^{-1}\bar{\nabla}_a\bar{\nabla}_b\Omega + \bar{g}_{ab}\bar{g}^{cd}\left(\Omega^{-1}\bar{\nabla}_c\bar{\nabla}_d\Omega - 3\Omega^{-2}\partial_c\Omega\partial_d\Omega\right)$$

   where $\bar{\nabla}$ is the Levi-Civita connection associated with $\bar{g}$.

   (c) Let ψ be a solution of the equation $g^{ab}\nabla_a\nabla_b\psi + \xi R\psi = 0$. We say that the equation is *conformally invariant* if there exists a constant p such that $\bar{\psi} \equiv \Omega^p\psi$ is a solution of the equation in a spacetime with metric $\bar{g} = \Omega^2 g$ whenever ψ solves the equation in a spacetime with metric g. Determine the value of ξ for which this equation is conformally covariant.

9. The Robinson-Bertotti metric is

$$ds^2 = -\lambda^2 dt^2 + M^2\left(\frac{d\lambda}{\lambda}\right)^2 + M^2 d\Omega^2$$

   This is the product $AdS_2 \times S^2$ where AdS_2 denotes 2d anti-de Sitter spacetime. By replacing the time coordinate t by one of the radial null coordinates $u = t + M/\lambda$, $v = t - M/\lambda$ show that the singularity at $\lambda = 0$ is merely a coordinate singularity. By introducing the new coordinates (U, V), defined by $u = \tan(U/2)$, $v = -\cot(V/2)$, obtain the maximal analytic extension of the RB metric and deduce its Penrose diagram (more precisely: deduce the Penrose diagram of the AdS_2 part of the RB metric). Is this spacetime globally hyperbolic?

10. Let Σ be a spacelike hypersurface with future directed timelike unit normal n^a, induced metric $h_{ab} = g_{ab} + n_a n_b$ and extrinsic curvature $K_{ab} = h_a^c h_b^d \nabla_c n_d$. Let S be a compact orientable 2d surface within Σ with unit normal m_a. On S, let $U_{\pm}^a = (n^a \pm m^a)/\sqrt{2}$. (a) Show that $U_{\pm}^a$ are future-directed null vectors orthogonal to S and $U_+ \cdot U_- = -1$. (b) Consider a null geodesic congruence containing the geodesics orthogonal to S with tangent $U_{\pm}^a$ there. On S we can choose (in the notation of lectures) $U^a = U_{\pm}^a$ and $N^a = U_{\mp}^a$. Show that the projection operator P_b^a can be written as $P_b^a = h_b^a - m^a m_b$. (c) On S, the expansion of the geodesics orthogonal to S is $\theta_{\pm} = P^{ab}\nabla_a U_b$. Since P^{ab} is a projection onto directions tangential to S, this expression involves only derivatives tangential to S so we can replace U_b by its value on S, i.e., $U_{\pm b}$. Show that this gives

$$\theta_{\pm} = (h^{ab} - m^a m^b)K_{ab} \pm k$$

    where k is the trace of the extrinsic curvature of S viewed as a surface in Σ. (d) Let Σ be a *time-symmetric* hypersurface, i.e., $K_{ab} = 0$. Can S be trapped? Show that S is marginally trapped if, and only if, $k = 0$. (This is the condition for S to be a *minimal surface* in Σ.) (e) Let $K_{ab} = J_{(a}M_{b)}$ where J^a and M^a are tangential to Σ and orthogonal to each other. Assume that M_a is tangent to S. Show that the results in (d) extend to this case. (A surface of constant t in the Kerr geometry has K_{ab} of this form.)