Mathematical Tripos Part III H.S. Reall

GENERAL RELATIVITY: Examples 3 Michaelmas 2017

1. Let $\phi : M \to N$ be a diffeomorphism. Let ∇ be a covariant derivative on M. The push-forward of ∇ is a covariant derivative $\tilde{\nabla}$ on N defined by

$$\tilde{\nabla}_X T = \phi_* \left(\nabla_{\phi^*(X)} \left(\phi^*(T) \right) \right)$$

where X is a vector field and T a tensor field on N. (In words: pull-back X and T to M, evaluate the covariant derivative there and then push-forward the result to N.)

(a) Check that this satisfies the properties of a covariant derivative. (b) Show that the Riemann tensor of $\tilde{\nabla}$ is the push-forward of the Riemann tensor of ∇. (c) Let ∇ be the Levi-Civita connection defined by a metric g on M. Show that $\tilde{\nabla}$ is the Levi-Civita connection defined by the metric $\phi_*(g)$ on N.

2. (a) Use the Leibniz rule to derive the formula for the Lie derivative of a covector ω valid in any coordinate basis:

$$(\mathcal{L}_X \omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu$$

(Hint: consider $(\mathcal{L}_X \omega)(Y)$ for a vector field Y.)

(b) Use normal coordinates to argue that one can replace partial derivatives with covariant derivatives to obtain the basis-independent result (where ∇ is the Levi-Civita connection)

$$(\mathcal{L}_X \omega)_a = X^b \nabla_b \omega_a + \omega_b \nabla_a X^b$$

(c) Show that the Lie derivative of a metric tensor is given in a coordinate basis by

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu X^\rho + g_{\rho\nu} \partial_\mu X^\rho$$

(d) Show that this can be written in the basis-independent form

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a$$

3. (a) Let X and Y be two vector fields. Show that

$$\mathcal{L}_X(\mathcal{L}_Y Q) - \mathcal{L}_Y(\mathcal{L}_X Q) = \mathcal{L}_{[X,Y]} Q,$$

when Q is either a function or a vector field. Deduce that the result holds if Q is a tensor field.

(b) Demonstrate that if a Riemannian or Lorentzian manifold has two "independent" isometries then it has a third, and define what is meant by independent here.

(c) Consider the unit sphere with metric

$$ds^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2.$$

Show that

$$\frac{\partial}{\partial \phi} \qquad \text{and} \qquad \sin\phi \, \frac{\partial}{\partial \theta} + \cot\theta \cos\phi \, \frac{\partial}{\partial \phi}$$

are Killing vectors. What is the third? Are there any more?

4. Let K^a be a Killing vector field and T_{ab} the energy momentum tensor. Let $J^a = T^a{}_b K^b$. Show that $\nabla_a J^a = 0$. J^a is a *conserved current.*

5. (a) Show that a Killing vector field K^a satisfies the equation

$$\nabla_a \nabla_b K^c = R^c{}_{bad} K^d.$$

[*Hint:* use the identity $R^a{}_{[bcd]} = 0$.]

(b) Deduce that in Minkowski spacetime the components of Killing covectors are linear functions of the coordinates.

6. Consider Minkowski spacetime in an inertial frame, so the metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

(a) Let K^a be a Killing vector field. Write down Killing's equation in the inertial frame coordinates.

(b) Using the result of the previous problem, show that the general solution can be written in terms of a constant antisymmetric matrix $a_{\mu\nu}$ and a constant covector b_μ.

(c) Identify the isometries corresponding to Killing fields with (i) $a_{\mu\nu} = 0$ (ii) $a_{0i} = 0$, $b_\mu = 0$, (iii) $a_{ij} = 0$, $b_\mu = 0$ (where i, j take values from 1 to 3).

(d) Identify the conserved quantities along a timelike geodesic corresponding to cases (i) to (iii).

7. Consider the energy-momentum tensor describing a point mass at the origin: in "almost inertial" coordinates it is $T_{00}(t, \mathbf{x}) = M\delta^3(\mathbf{x})$, $T_{0i} = T_{ij} = 0$. Determine the linearized gravitational field produced by this energy momentum tensor, assuming it to be independent of t. For what values of $R = |\mathbf{x}|$ is the linear approximation valid?

8. In "almost inertial" coordinates the energy momentum tensor of a straight *cosmic string* aligned along the z-axis is

$$T_{\mu\nu} = \mu\delta(x)\delta(y)\text{diag}(1, 0, 0, -1),$$

where μ is a small positive constant. Terms of order μ^2 are to be ignored. Look for a time-independent solution of the linearized Einstein equation, finding $h_{11} = h_{22} = -\lambda$ as the only non-zero components of the perturbed metric tensor, where $\lambda \equiv 8\mu \log(r/r_0)$, $r = \sqrt{x^2 + y^2}$, and r_0 is an arbitrary length.

Show that the perturbed metric can be written in cylindrical polar coordinates as

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + (1 - \lambda)(\mathrm{d}r^2 + r^2\mathrm{d}\phi^2).$$

Make a change of radial coordinate given by $(1 - \lambda)r^2 = (1 - 8\mu)r'^2$ to obtain

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r'^2 + (1 - 8\mu)r'^2\mathrm{d}\phi^2,$$

and change the angular coordinate to obtain

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r'^2 + r'^2\mathrm{d}\phi'^2.$$

Is this Minkowski spacetime? Show intuitively how a distant object may give rise to double images.

9. Consider a large thin spherical shell of mass M and radius R which rotates slowly about the z-axis (in "almost inertial" coordinates) with angular velocity Ω, so that terms of $\mathcal{O}(R^2\Omega^2)$ can be neglected. Introduce a shell density $\rho = M\delta(r-R)/(4\pi R^2)$ where $r^2 = x^2+y^2+z^2$, and a 4-velocity $u^\mu = (1, -\Omega y, \Omega x, 0)$. The energy-momentum tensor has components $T^{\mu\nu} = \rho u^\mu u^\nu$. We can regard this source as a superposition of two sources, one for which only T_{00} is nonzero, and one for which only T_{0i} is nonzero.

   (a) Solve the linearized Einstein equations sourced by T_{00}. Show that the result agrees with Newtonian theory.

   (b) Consider the perturbation sourced by T_{0i}. Argue that the only non-vanishing components are h_{0i}, which satisfy $\nabla^2 h_{0i} = -16\pi T_{0i}$. Consider the combination $h_{01} + ih_{02}$ and work in spherical polar coordinates. You should find that the RHS of the linearized Einstein equation is proportional to $\sin\theta e^{i\phi}$, i.e., to a spherical harmonic with $l = m = 1$. Since a general solution to the Laplace equation can be expanded as a sum over spherical harmonics, this implies that the solution must be of the form $h_{01} + ih_{02} = f(r)\sin\theta e^{i\phi}$. Hence obtain the solution

$$h_{0i} = \begin{cases} \omega(y, -x, 0) & r < R \\ \omega\frac{R^3}{r^3}(y, -x, 0) & r > R \end{cases}$$

   where $\omega = 4M\Omega/(3R)$. Note that this decays as $1/r^2$ at large r. This is a general result: rotation of the source affects $h_{\mu\nu}$ at $\mathcal{O}(1/r^2)$, subleading compared to the $\mathcal{O}(1/r)$ contribution arising from the total mass.

   (c) Consider a free particle moving non-relativistically inside the shell. Working to first order in the perturbation, show that the geodesic equation implies $\ddot{\mathbf{x}} = 2\mathbf{w} \wedge \dot{\mathbf{x}}$ where $\mathbf{w} = (0, 0, \omega)$ and a dot denotes a derivative with respect to t. This is the same as the equation of a slowly moving free particle in Minkowski spacetime, using a reference frame rotating with angular velocity $-\mathbf{w}$ with respect to an inertial frame (the RHS is the Coriolis force). Hence, inside the shell, the background Minkowski frame is rotating with angular velocity $-\mathbf{w}$ with respect to a local inertial frame, i.e., the local inertial frames are rotating with angular velocity $+\mathbf{w}$ with respect to the background Minkowski frame. The local inertial frames are *dragged around* by the rotating shell. This is the *Lense-Thirring effect* (1918).

10. (Optional question.) Show that the second order terms in the expansion of the Ricci tensor around Minkowski spacetime are

$$\begin{aligned} R^{(2)}_{\mu\nu}[h] &= \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} + \partial^\sigma h^\rho{}_\nu\partial_{[\sigma}h_{\rho]\mu} \\ &+ \frac{1}{2}\partial_\sigma\left(h^{\sigma\rho}\partial_\rho h_{\mu\nu}\right) - \frac{1}{4}\partial^\rho h\partial_\rho h_{\mu\nu} - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu}h_{\nu)\rho} \end{aligned}$$

11. (a) Use the linearized Einstein equation to show that, in vacuum,

$$\langle \eta^{\mu\nu} R^{(2)}_{\mu\nu}[h]\rangle = 0$$

   (b) Show that

$$\langle t_{\mu\nu}\rangle = \frac{1}{32\pi}\langle \partial_\mu \bar{h}_{\rho\sigma}\partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2}\partial_\mu\bar{h}\partial_\nu\bar{h} - 2\partial_\sigma\bar{h}^{\rho\sigma}\partial_{(\mu}\bar{h}_{\nu)\rho}\rangle$$

   (c) Show that $\langle t_{\mu\nu}\rangle$ is gauge invariant.