Supplementary notes

Tensor components

Consider a basis $\{e_\mu\}$ for $T_p(M)$ and let $\{f^\mu\}$ be the dual basis for $T_p^\star(M)$. In the abstract index notation we denote the μth basis vector as e_μ^a. Here μ simply labels which basis vector we are talking about. Similarly the μth dual basis covector as f_a^μ. Let a vector X^a have components X^μ, i.e., $X^a = X^\mu e_\mu^a$. We then have $X^\mu = f_a^\mu X^a$. Similarly a covector η_a has components $\eta_\mu = e_\mu^a \eta_a$. And for a tensor we have

$$T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = f_{a_1}^{\mu_1} \dots f_{a_r}^{\mu_r} e_{\nu_1}^{b_1} \dots e_{\nu_s}^{b_s} T^{a_1 \dots a_r}{}_{b_1 \dots b_s}$$

Covariant derivative

A covariant derivative of a vector field Y^a is tensor field of type $(1,1)$. In the abstract index notation we should really write it as $(\nabla Y)^a{}_b$ but it is often written instead as $\nabla_b Y^a$. To avoid confusion, remember that $\nabla_b Y^a \equiv (\nabla Y)^a{}_b$. In a basis the components are

$$\nabla_\mu Y^\nu \equiv (\nabla Y)^\nu{}_\mu = e_\mu^b f_a^\nu (\nabla Y)^a{}_b = f_a^\nu (\nabla_\mu Y)^a$$

(recall ∇_μ means ∇_{e_μ}). Now each component Y^ν can be regarded as a function (possibly defined only on some region of the manifold). So we can also calculate the covariant derivative of this function, which is the same as the gradient of this function, i.e., a covector field $(\nabla(Y^\nu))_a$. The μ component of this covector field is

$$\nabla_\mu(Y^\nu) = e_\mu^a (\nabla(Y^\nu))_a = \nabla_\mu(Y^\nu) = \nabla_\mu(f_a^\nu Y^a) \neq \nabla_\mu Y^\nu$$

Exterior derivative

We defined the exterior derivative in a coordinate basis as

$$(dX)_{\mu_1 \dots \mu_{p+1}} = (p+1)\partial_{[\mu_1} X_{\mu_2 \dots \mu_{p+1}]}$$

As an exercise I asked you to prove that these components transform in the right way under a change of coordinate basis, so that this does indeed define a $(p+1)$-form. Here's the solution to this exercise. Use the above expression to define the components of dX in a chart with coordinates x^μ and introduce another chart with coordinates x'^μ. The components of X in the two charts are related by

$$X_{\mu_1\ldots\mu_p} = \frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\nu_p}}{\partial x^{\mu_p}} X'_{\nu_1\ldots\nu_p} \tag{1}$$

hence

$$\begin{aligned}
(dX)_{\mu_1\ldots\mu_{p+1}} &= (p+1)\frac{\partial}{\partial x^{[\mu_1}}\left(\frac{\partial x'^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x'^{\nu_{p+1}}}{\partial x^{\mu_{p+1}]}} X'_{\nu_2\ldots\nu_{p+1}}\right) \\
&= \left(\frac{\partial}{\partial x^{[\mu_1}} X'_{\nu_2\ldots\nu_{p+1}}\right)\frac{\partial x'^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x'^{\nu_{p+1}}}{\partial x^{\mu_{p+1}]}} \\
&= \left(\frac{\partial}{\partial x'^{\nu_1}} X'_{\nu_2\ldots\nu_{p+1}}\right)\frac{\partial x'^{\nu_1}}{\partial x^{[\mu_1}}\frac{\partial x'^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x'^{\nu_{p+1}}}{\partial x^{\mu_{p+1}]}} \\
&= \left(\frac{\partial}{\partial x'^{\nu_1}} X'_{\nu_2\ldots\nu_{p+1}}\right)\frac{\partial x'^{[\nu_1}}{\partial x^{\mu_1}}\frac{\partial x'^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x'^{\nu_{p+1}]}}{\partial x^{\mu_{p+1}}} \\
&= \left(\frac{\partial}{\partial x'^{[\nu_1}} X'_{\nu_2\ldots\nu_{p+1}]}\right)\frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}}\frac{\partial x'^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x'^{\nu_{p+1}}}{\partial x^{\mu_{p+1}}} \\
&= (dX)'_{\nu_1\ldots\nu_{p+1}}\frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}}\frac{\partial x'^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x'^{\nu_{p+1}}}{\partial x^{\mu_{p+1}}}
\end{aligned}$$

which is the tensor transformation law. The second equality uses the fact that second partial derivatives are symmetric and hence drop out when we antisymmetrize. The fourth equality uses the result that, for any $t^\mu{}_\nu$ we have

$$t^{\nu_1}{}_{[\mu_1}\ldots t^{\nu_r}{}_{\mu_r]} = t^{[\nu_1}{}_{\mu_1}\ldots t^{\nu_r]}{}_{\mu_r}$$

(This same result is needed to verify that the RHS of (1) is antisymmetric, or more generally that antisymmetrization is independent of the choice of basis.) The proof of this result is (σ denotes a permutation of $\{1, 2, \ldots, r\}$)

$$\begin{aligned}
t^{\nu_1}{}_{[\mu_1}\ldots t^{\nu_r}{}_{\mu_r]} &= \frac{1}{r!}\sum_\sigma \text{sign}(\sigma) t^{\nu_1}{}_{\mu_{\sigma(1)}}\ldots t^{\nu_r}{}_{\mu_{\sigma(r)}} \\
&= \frac{1}{r!}\sum_\sigma \text{sign}(\sigma) t^{\nu_{\sigma^{-1}(1)}}{}_{\mu_1}\ldots t^{\nu_{\sigma^{-1}(r)}}{}_{\mu_r} \\
&= t^{[\nu_1}{}_{\mu_1}\ldots t^{\nu_r]}{}_{\mu_r}
\end{aligned}$$

where the second equality just rearranges the order of the factors in the sum and the final equality uses the fact that σ and σ^{-1} have the same sign, and summing over σ is equivalent to summing over σ^{-1}.