Optimal probes for withdrawal of uncontaminated fluid samples

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Withdrawal of fluid by a composite probe pushed against the face \( z=0 \) of a porous half-space \( z>0 \) is modeled assuming incompressible Darcy flow. The probe is circular, of radius \( a \), with an inner sampling section of radius \( aa \) and a concentric outer guard probe \( aa<r<a \). The porous rock in \( 0\leq z\leq \beta a \) is saturated with fluid 1, and the region \( z>\beta a \) is saturated with fluid 2; the two fluids have the same viscosity. It is assumed that the interface between the two fluids is sharp and remains so as it moves through the rock. The pressure in the probe is lower than that of the pore fluid in the rock, so that the fluid interface is convected with the fluids towards the probe. This idealized axisymmetric problem is solved numerically, and it is shown that an analysis based on far-field spherical flow towards a point sink is a good approximation when the nondimensional depth of fluid 1 is large, i.e., \( \beta \gg 1 \). The inner sampling probe eventually produces pure fluid 2, and this technique has been proposed for sampling pore fluids in rock surrounding an oil well [A. Hrametz, C. Gardner, M. Wais, and M. Proett, U.S. Patent No. 6,301,959 B1 (16 October 2001)]. Fluid 1 is drilling fluid filtrate, which has displaced the original pore fluid (fluid 2), a pure sample of which is required. The time required to collect an uncontaminated sample of original pore fluid can be minimized by a suitable choice of the probe geometry \( \alpha \) [J. Sherwood, J. Fitzgerald and B. Hill, U.S. Patent No. 6,719,049 B2 (13 April 2004)]. It is shown that the optimal choice of \( \alpha \) depends on the depth of filtrate invasion \( \beta \) and the volume of sample required. © 2005 American Institute of Physics.

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I. INTRODUCTION

Samples of hydrocarbon pore fluid from a newly drilled petroleum reservoir provide information of great value to oil companies. Such samples can be collected by a device lowered down the wellbore on a wireline.1 The sampling device clamps a probe against the wall of the wellbore, as shown schematically in Fig. 1. Fluid is then pumped from the rock, through the probe, and into a sampling chamber and is either analyzed downhole or is brought to the surface for later analysis.

During its construction, the well is filled with drilling fluid. This drilling fluid is usually a suspension of solid particles in liquid, which the drilling fluid engineer makes sufficiently dense so that the pressure in the wellbore is higher than the pore pressure in the surrounding rock. This prevents undesirable ingress of liquid or gas from the rock pores into the wellbore, but has the unfortunate consequence that the liquid phase of the drilling fluid flows from the wellbore into the rock. Solids within the drilling fluid are too large to penetrate far into pores: they collect at the rock surface and form a filtercake.2 The pores in the rock close to the wellbore therefore contain drilling fluid filtrate rather than original pore fluid. Hence, when a sample of pore fluid is withdrawn from the rock, the sample is at first highly contaminated with filtrate (Fig. 1). It is easy to distinguish the original hydrocarbons from the filtrate if the drilling fluid is water-based, but more difficult if the drilling fluid is oil-based. It is therefore important to minimize contamination caused by the filtrate from oil-based mud, in which case the filtrate is miscible with the original hydrocarbon. The level of contamination drops after a large volume of fluid has been pumped from the rock, but this takes a long time,3 thereby increasing the risk that the tool or wireline becomes stuck within the filtercake.4

A recent patent5 proposes that a guarded probe can be used to reduce the level of sample contamination. We show here that the flow in a model axisymmetric problem can be studied analytically. This enables us to estimate the optimal configuration for the sampling probe and the surrounding guard probe6 such that the time required to collect an uncontaminated sample is minimized.

The sampling probe has radius \( aa \), and is surrounded by a concentric annular guard probe which occupies the region \( aa<r<a \) (Fig. 2). The sampling and guard probes are assumed to be at the same pressure, in order to prevent cross flow from one probe at higher pressure to the other at lower pressure. The composite probe is pushed against the cylindrical rock surface of the wellbore. We consider the case in which the probe dimension \( a \) and depth of filtrate invasion \( \beta a \) are small compared to the wellbore radius \( r_w \), so that we can neglect the curvature of the wellbore rock surface. This enables us to consider a much simpler problem in which the composite probe is pushed against a plane rock surface \( z=0 \). The half-space \( z>0 \) is occupied by rock with porosity \( \phi \) and isotropic hydraulic permeability \( k \). Fluid can flow through the rock surface \( z=0 \) into the probes through the circle with \( 0<r<a \) from which drilling fluid filtercake has been removed, but the rest of the rock surface \( r>a \) is cov-
The interface between the two fluids at \( z = \beta a \) is assumed to be initially sharp, and to remain so as it is displaced through the porous medium. In practice dispersion will occur, both as the filtrate invades the porous medium and during subsequent withdrawal of fluid by the sampling device. This is discussed in Sec. VI.

We assume that both fluids have the same viscosity \( \mu \), though in practice this is unlikely and the displacement of more viscous fluid by less viscous fluid is unstable. If the filtrate viscosity is smaller than that of the original pore fluid, viscous fingering will have occurred when the filtrate invaded the rock. If the filtrate viscosity is larger than that of the pore fluid, fingering will occur as fluid is withdrawn by the probe. The Saffman-Taylor instability has been studied mainly in two-dimensional flows, but axisymmetric problems in porous media have also been investigated.\(^8\)

The filtrate density will, in general, differ from that of the original pore fluid, and in order to estimate the resulting buoyancy-driven velocities we assume a density difference \( \Delta \rho = 100 \text{ kg m}^{-3} \), a fluid viscosity \( \mu = 10^{-3} \text{ Pa s} \), and a rock permeability \( k = 100 \text{ mDarcy} = 10^{-13} \text{ m}^2 \). The acceleration due to gravity is \( g = 9.8 \text{ m s}^{-2} \), so that buoyancy-induced Darcy velocities are of the order \( k g \Delta \rho / \mu = 10^{-7} \text{ m s}^{-1} \). A sampling tool typically withdraws fluid at a rate\(^1\) \( Q_f = 0.6 \text{ gal min}^{-1} \approx 4 \times 10^{-5} \text{ m}^3 \text{ s}^{-1} \), so that radial Darcy velocities towards the probe equal the buoyancy-induced velocities at a distance of 8 m, well outside the radius of interest, which for a pumping time of 30 min in a rock of porosity \( \phi = 0.1 \) is less than 1 m. Gravity-driven motion of filtrate around a vertical well is discussed by Dussan V and Auzerais.\(^9\) The filtrate, if denser than the original pore fluid, flows downwards to the next impermeable rock layer, where it accumulates and where the interface between the filtrate and original pore fluid is no longer parallel to the axis of the well. Sampling would not intentionally be performed at such a point where the filtrate has accumulated. Gravitational effects are important in other axisymmetric Darcy flows, especially the problem of water coning,\(^10-14\), in which an oil/water interface or a saltwater/freshwater interface moves towards a well.

The problem discussed here is depicted schematically in Fig. 2. The simplified axisymmetric geometry, with a plane rock surface, permits a more detailed analysis than would be possible for a cylindrical wellbore. Analytic results for the motion of the interface between the two fluids are presented in Secs. II–III and are shown to agree with numerical computations in Sec. IV. In Sec. V we use these results to determine the optimal probe geometry \( a \) that minimizes the time required to collect an uncontaminated sample of the original pore fluid. The effect of dispersion is considered in Sec. VI, and in Sec. VII we discuss an approximate analysis of the cylindrical wellbore geometry.

**II. GOVERNING EQUATIONS**

We assume that the fluid velocity \( \mathbf{u} \) within the permeable rock is given by Darcy's law:

\[
\mathbf{u} = -

\]
\[ \mathbf{u} = -\frac{k}{\mu} \nabla p, \]  

where \( p \) is the pressure within the fluid. The fluids are taken to be incompressible, so that \( \nabla \cdot \mathbf{u} = 0 \) and the pore pressure \( p \) satisfies the Laplace equation

\[ \nabla^2 p = 0. \tag{2} \]

We measure all pressures relative to the pore pressure at infinity, and we nondimensionalize all lengths by the guard probe outer diameter \( a \) so that the boundary conditions for the Laplace Eq. (2) are

\[
p = p_0, \quad z = 0, \quad r < 1 \tag{3a}
\]

\[
\frac{\partial p}{\partial z} = 0, \quad z = 0, \quad r > 1 \tag{3b}
\]

\[
p \to 0, \quad z \to \infty. \tag{3c}
\]

The problem is analogous to that of an electrified disc at potential \( p_0 \) in unbounded space, and the solution of the Laplace Eq. (2) with boundary conditions (3) is

\[
p = \frac{2p_0}{\pi} \int_0^\infty s^{-1} J_0(sr) \sin s e^{-sz} ds, \tag{4}\]

where \( J_n \) is a Bessel function (of the first kind) of order \( n \). When \( z = 0 \) the integral (4) is a special case of the Weber-Schafheitlin discontinuous integral,\(^\text{15}\) with

\[
p = \frac{2p_0}{\pi} \sin^{-1}(r^{-1}), \quad r > 1, \quad z = 0 \tag{5a}
\]

\[
p = p_0, \quad r < 1, \quad z = 0. \tag{5b}
\]

In the far field the flow approximates to a spherically symmetric radial flow towards a point sink, and we see from (5) that \( p \sim 2p_0/\pi r \), on the plane \( z = 0 \). The flow rate into the inner sampling probe of (non-dimensional) radius \( a \) is

\[
Q_s(a) = -2\pi a^2 \int_0^a rudr = -\frac{8\pi ap_0 k}{\mu} \int_0^a r dr \int_{z=0}^\infty \sin s J_0(sr) ds
\]

\[
= -\frac{4\pi a^2 p_0}{\mu} \frac{a^2}{1 + (1 - a^2)^{1/2}}, \tag{6}
\]

where we have used the relation \( \int_0^\infty J_0(t) dt = \frac{\pi}{2} J_1(z) \). Note that \( p_0 < 0 \), so that \( Q_s > 0 \). The total volumetric flow rate of fluid into the composite probe is

\[
Q_s = Q_s(a = 1) = -4\pi a^2 p_0 k/\mu. \tag{7}
\]

The fraction of the total flow which passes through the sample probe is therefore

\[
\frac{Q_s}{Q_t} = \frac{a^2}{1 + (1 - a^2)^{1/2}} \approx 1 - (1 - a^2)^{1/2}, \tag{8}
\]

shown in Fig. 3.

\[\text{FIG. 3. The sample probe volumetric flow rate } Q_s \text{, expressed as a fraction of the total flow } Q_t \text{ into the combined sample and guard probes, as given by (8).}\]

III. DARCY FLOW TOWARDS THE PROBE

A. Motion of fluid particles

The Darcy velocity (1) within the rock is

\[
u_r = \frac{2p_0 k}{\pi a \mu} \int_0^\infty e^{-sz} \sin s J_1(sr) ds, \tag{9a}\]

\[
u_z = \frac{2p_0 k}{\pi a \mu} \int_0^\infty e^{-sz} \sin s J_0(sr) ds, \tag{9b}\]

and in a rock of porosity \( \phi \) the velocity of a fluid particle is higher than the Darcy velocity by a factor \( \phi^{-1} \). We may use (9) to compute trajectories of fluid particles as they travel towards the probe, and similarly, we can determine the time that any fluid particle takes to reach the probe.

Our main interest lies in the motion of the interface between the filtrate and original pore fluid. This interface is initially at \( z = \beta \), and it is clear that the first original pore fluid to reach the probe will be that which is initially on the axis of symmetry at \( (r,z) = (0,\beta) \). If the probe were to be approximated by a point sink of constant strength \( Q_s > 0 \), the Darcy velocity \( u_z(z) \) along the axis \( r=0 \) would be

\[
u_z = -\frac{Q_s}{2\pi a^2 z}, \tag{10}\]

and the time taken for the original pore fluid to first reach the probe would be

\[
T = \frac{2\pi \beta^3 a^3 \phi}{3Q_t}. \tag{11}\]

The timescale \( T \) is the time taken to pump a volume of fluid equal to the pore volume of a hemisphere of radius \( \beta a \). Darcy flow depends linearly on the pressure gradient. In the absence of any effects due to gravity or interfacial tension, the motion of fluid particles and of the filtrate/pore-fluid interface depends only on the total volume withdrawn by the probe. Nevertheless, it is convenient to work with velocities and pumping time, rather than with displacements and pumped volumes.
When the probe is a disc of (dimensional) radius $a$, the velocity along the axis $r=0$ is, by (9b),

$$u_z = \frac{2 \rho_0 \phi}{\pi a \mu (z^2 + 1)} = -\frac{Q_t}{2 \pi a \mu (z^2 + 1)},$$

from which we find that the time taken for the original pore fluid to first reach the probe is

$$T_d = T(1 + 3 \beta^{-2}),$$

with $T_d > T$ since the fluid velocity close to the disc is smaller than that close to a point sink.

Suppose that a fluid particle, initially on the interface at $(r_\beta, \beta)$, arrives at the probe at $(r_0, 0)$ at time $t_0$. Streamlines which arrive at the probe within the disc $0 \leq r \leq r_0$ bring original pore fluid at time $t_0$, whereas those arriving over $r_0 < r < 1$ bring filtrate, as sketched in Fig. 4. In consequence, the fraction of the total fluid arriving at the probe which is original pore fluid at time $t_0$ is

$$f_i = 1 - (1 - r_0^2)^{1/2},$$

and the fraction that is filtrate is

$$F_i = 1 - f_i = (1 - r_0^2)^{1/2}.$$  

If $r_0 < \alpha$, and if we restrict ourselves to the fluid $Q_t$ arriving at the inner sampling probe, the fraction of $Q_t$ which is original pore fluid is

$$f_s = \frac{1 - (1 - r_0^2)^{1/2}}{1 - (1 - \alpha^2)^{1/2}}, \quad r_0 < \alpha$$

and the fraction that is filtrate is

$$F_s = 1 - f_s = \frac{(1 - r_0^2)^{1/2} - (1 - \alpha^2)^{1/2}}{1 - (1 - \alpha^2)^{1/2}}, \quad r_0 < \alpha.$$  

The original pore fluid fraction $f_s$ of fluid collected by the sampling probe is zero until time $T_d$, when $r_0=0$. As $r_0$ increases, $f_s$ increases until $f_s=1$ when $r_0=\alpha$. We now investigate the time at which this occurs.

### B. Streamlines close to the axis $r=0$

If $r \ll z$, we may expand $J_1(sr)$ and $J_0(sr)$ in the expressions (9) for $u$ to obtain

$$u_r = -\frac{Q_t}{2 \pi a^2} \frac{r z}{(z^2 + 1)^2} + \cdots,$$

$$u_z = -\frac{Q_t}{2 \pi a^2} \left[ \frac{1}{z^2 + 1} - \frac{r^2}{2 (z^2 + 1)} \left( \frac{3z^2 - 1}{2 (z^2 + 1)} \right) \right] + \cdots.$$  

The expressions (18) are not valid in a region close to the probe, where $z \ll r$, but we assume that $r \ll 1$, so that this region is small and errors may be neglected. The equation for streamlines may be obtained by integrating

$$\frac{dr}{dz} = \frac{dz}{dr},$$

and the fluid particle reaches the probe at $(r_0, 0)$, where

$$r_0 = \frac{r_\beta}{(\beta^2 + 1)^{1/2}}.$$  

We can now obtain the $O(r^2)$ correction (18b) to the axial velocity $u_z$ along the streamline that starts from $(r_\beta, \beta)$:

$$u_z = \frac{d z}{d t} = -\frac{Q_t}{2 \pi a^3 \phi} \left[ \frac{1}{z^2 + 1} - \frac{r_\beta^2}{2 (\beta^2 + 1)} \left( \frac{3z^2 - 1}{2 (z^2 + 1)^2} + \cdots \right) \right].$$

After integrating (22), we find that the time $t_0$ at which the fluid particle reaches $(r_0, 0)$ is

$$t_0 = T \left[ 1 + \frac{3}{\beta^2} + \frac{3r_\beta^2}{2 (1 - \beta^2)} + \cdots \right].$$

### C. Far-field spherical flow

If we are sufficiently far from the probe, the flow will differ little from that due to a point sink at the origin. If we neglect deviations from radial flow close to the probe, the time taken for a fluid particle initially at $(r_\beta, \beta)$ to reach the probe is predicted to be

$$t_0 = \frac{2 \pi a^3 \phi}{3 Q_t} (\beta^2 + r_\beta^2)^{1/2} = T \left( 1 + \frac{3r_\beta^2}{2 \beta^2} + \cdots \right).$$

We may use (21) to express $r_\beta$ in terms of $r_0$, and we then observe that (24) agrees with (23) to $O(r_0^2)$ if $\beta \gg 1$.

In the limit $t \to \infty$ we may estimate the fraction of filtrate in the fluid withdrawn by the combined guard and sample probes. We approximate the flow towards the probe by the...
flow towards a point sink of strength $Q_s$, and use geometrical arguments similar to those of Hammond. At time $t$ fluid will have been withdrawn from a hemispherical region of nondimensional radius $R$, where

$$\frac{2 \pi \alpha^3 R^3 \phi}{3} = Q_s t. \quad (25)$$

The area of the hemispherical surface is $2 \pi R^2$, and it lies entirely within the filtrate if $R < \beta$, i.e., if $t < T$, where $T$ is defined by (11). If $t > T$, that part of the hemisphere which lies within the filtrate-invaded zone has an area of $2 \pi R \beta$. The fraction of filtrate $F_i$ in the total flow is therefore

$$F_i = 1, \quad t < T \quad (26a)$$

$$F_i = \frac{\beta}{R} = \left(\frac{T}{t}\right)^{1/3}, \quad t > T. \quad (26b)$$

Hence for $t \gg T$, by comparing (26b) and (15) we expect the radial position $r_0$ at which the interface between the filtrate and original pore fluid enters the probe at time $t_0$ to be

$$r_0^2 \sim 1 - \left(\frac{T}{t_0}\right)^{2/3}, \quad t_0 \to \infty. \quad (27)$$

Combining this with (17), we obtain an explicit expression for the filtrate contamination $F_i(r_0)$ in the sampling probe. $F_i$ drops to zero when $r_0 = \alpha$, i.e., at the finite time $t_0 = T(1 - \alpha^3)^{-1/2}$, whereas the contamination (26) in the combined probe decays as $r^{-1/3}$, and takes an infinite time to become zero.

By (24) we may express $r_0$ in terms of $r_\beta$:

$$r_0^2 \sim 1 - \left(1 + \frac{r_\beta^2}{\beta^2}\right)^{-3/2}, \quad t_0 \to \infty. \quad (28)$$

Note that when $r_\beta \ll \beta$, (28) predicts $r_0 \approx (3/2)^{1/2} r_\beta / \beta$, which differs from (21) only by a factor $(3/2)^{1/2} \approx 1.22$ when $\beta \gg 1$.

The results of this section have been obtained by combining a simple far-field spherical flow approximation with an expression (8) for the flow into the probe obtained from a detailed knowledge of flow close to the probe. We shall see in Sec. IV that these results agree well with full numerical results when $\beta \gg 1$.

IV. NUMERICAL RESULTS

The fluid velocity (9) may be evaluated numerically using routines developed by Lucas and Stone and Lucas, and can then be integrated by means of the NAG routine D02CJF in order to follow fluid streamlines. We nondimensionalize time by $T$, so that pore fluid first arrives at the probe after a nondimensional time $t = 1 + 3 \beta^{-2}$. We consider by way of example the case $\beta = 10$.

Figure 5 shows the radial position $r_0$ at which a particle enters the probe as a function of the initial radial position $r_\beta$. Also shown is the prediction (21) valid for small $r$, and the prediction (28) valid for $t_0 \gg T$ when $\beta \gg 1$.

The corresponding time at which the original pore fluid arrives at $r_0$ is shown in Fig. 6, as is the analytic prediction (23b) for small $r_0$ and the asymptotic prediction (27) for $t_0 \gg T$.

Figure 7 shows the filtrate fraction $F_i$ in the total flow $Q_s$ to the combined probe, defined by (15), together with the asymptote (26) for large $t$. Also shown are the computed filtrate fractions $F_i$ (17) in sampling probes of diameter $\alpha = 0.2, 0.4, 0.6, 0.8$. Note that the sampling probe contamination levels drop to zero in finite time, as predicted in Sec. III C.

We see from Figs. 5–7 that when $\beta = 10$ there is good agreement between the full numerical results and the approximate analysis of Sec. III C. This allows us (when $\beta \gg 1$) to use the analysis of Sec. III C in order to determine the sampling probe radius $\alpha_{\text{opt}}$ which minimizes the total time required to acquire a sample. This is discussed in Sec. V.

V. OPTIMAL CHOICE FOR $\alpha$ IN THE AXISYMMETRIC GEOMETRY

Suppose that we require an uncontaminated sample of volume $V_s$. The smaller we make the sample probe radius $\alpha$, the sooner the fluid flowing into the sampling probe will be
Note that where the streamlines close to the axis. Setting \( t_0 = T(a - \alpha)_3^{1/2} \), as a function of nondimensional time \( t/T \), for \( \beta = 10 \). (a) Contamination of the total fluid entering the combined sample and guard probes, with \( t/T \) the asymptote predicted by the spherical far-field approximation (26b). The broken lines show the contamination of the fluid entering sample probes of radius \( \alpha = 0.2 \), (c) \( \alpha = 0.4 \), (d) \( \alpha = 0.6 \), and (e) \( \alpha = 0.8 \).

free of filtrate, but the longer it will take subsequently to collect a sample of volume \( V_s \). We seek to minimize the total time required to collect the sample. We first consider the case \( \alpha \ll 1 \), for which all streamlines of interest lie close to the axis of symmetry, so that we can use the analysis of Sec. III B. Then we consider the more general case, for which we use the spherical flow approximation of Sec. III C.

### A. Streamlines close to the axis, \( \alpha \ll 1 \)

The time \( t_0 \) required for the sample to become free of filtrate is obtained (for \( \alpha \ll 1 \)) by setting \( r_0 = \alpha \) in (23), and the time subsequently required to take a sample of volume \( V_s \) is \( V_s/Q_t \), where \( Q_t \) is given by (8). The total time taken to obtain the sample is therefore

\[
t_s = t_0 + \frac{V_s}{Q_t} = \frac{2\pi a^3 \beta \phi}{3Q_t} \left[ 1 + \frac{3}{2} \beta^2 + \frac{3 \alpha^2}{2} \left( 1 - \frac{1}{\beta^2} \right) + \cdots \right] + \frac{2V_s}{\alpha^2 Q_t}.
\]

This expression for \( t_s \) is minimized by the choice \( \alpha = \alpha_{\text{opt}} \), where

\[
\alpha_{\text{opt}}^2 = \frac{2V_s}{\pi a^3 \phi (\beta^3 - \beta)}.
\]

Note that (30) is only valid if \( \alpha_{\text{opt}} \) is small. We see that the \( \alpha_{\text{opt}} \) depends on the depth of invasion \( \beta a \), which is unknown but likely to be large compared to the probe radius \( a \). Consider a probe of radius \( a = 1 \) cm and a sample size \( V_s = 450 \) cm\(^3\). If \( \phi = 0.1 \) and \( \beta = 20 \), then (30) predicts \( \alpha_{\text{opt}} = 0.77 \), which is too large for the above analysis to be valid.

### B. Spherical flow approximation

When \( \beta \gg 1 \) we may use the spherical flow approximation of Sec. III C, rather than the analysis of Sec. III B for streamlines close to the axis. Setting \( \alpha = r_0 \) in (27) we conclude that a sample probe of radius \( \alpha \) will be free of filtrate after a time \( t_0 = T(1 - \alpha^2)_3^{1/2} \), so that the total time required to collect a sample of volume \( V_s \) is

\[
t_s = \frac{T}{(1 - \alpha^2)_3^{1/2}} + \frac{V_s}{Q_t[1 - (1 - \alpha^2)_3^{1/2}]}.
\]

and this time is minimized by the choice \( \alpha = \alpha_{\text{opt}} \), where \( \alpha_{\text{opt}} \) satisfies

\[
X = \left( \frac{3TQ_s}{V_s} \right)_3^{1/2} = \left( \frac{2\pi b^3 \phi}{V_s} \right)_3^{1/2} = \frac{1 - \alpha_{\text{opt}}^2}{1 - (1 - \alpha_{\text{opt}}^2)_3^{1/2}},
\]

and \( X^2 \) is a nondimensional sample volume. The quadratic Eq. (32) for \( (1 - \alpha_{\text{opt}}^2)_3^{1/2} \) has the solution

\[
\alpha_{\text{opt}}^2 = 1 - \frac{1}{2} [X^2 + 2X - X(X^2 + 4X)^3]^{1/2}.
\]

When \( X \ll 1 \)

\[
\alpha_{\text{opt}} = 1 - \frac{X}{2} + \cdots, \quad X \to 0,
\]

and when \( X \gg 1 \)

\[
\alpha_{\text{opt}} = \left( \frac{2}{X} \right)_3^{1/2} + \cdots, \quad X \to \infty,
\]

which agrees with (30) in the limit \( \beta \to \infty \). When the sample volume \( V_s \) is large and \( X \) is small, it is more important to minimize the time \( V_s/Q_t \) to collect the sample, rather than the time required for the fluid arriving at the sample probe to become uncontaminated. The sample probe should therefore be large, with \( \alpha_{\text{opt}} \approx 1 \). Figure 8 shows \( \alpha_{\text{opt}} \) as a function of \( X \), as predicted by (33), together with the asymptotes (34) and (35). For the example given in Sec. V A above, \( X = 3.34 \) and, hence, \( \alpha_{\text{opt}} = 0.59 \).

The above analysis uses the spherical flow approximation of Sec. III C, and is therefore valid only for \( \beta \gg 1 \). Numerical computations could determine \( \alpha_{\text{opt}} \) for more general (smaller) \( \beta \). However, the geometry we are considering differs from the actual geometry of a probe pushed against a cylindrical wellbore, and it would be inappropriate to determine \( \alpha_{\text{opt}} \) exactly for an incorrect geometry.

Numerical reservoir simulators can compute the flow outside a cylindrical wellbore towards a guarded probe. The
filtrate and original pore fluid viscosities can be varied in such simulations, as can the ratio of vertical and horizontal permeabilities of the rock, though numerical dispersion must be controlled. We shall not discuss such simulations here, but the cylindrical wellbore geometry is sufficiently important that in Sec. VII we shall consider a very simple model in order to determine the dimensionless groups that govern flow in this geometry. However, before leaving the idealized plane geometry we consider the effect of dispersion.

VI. THE EFFECT OF DISPERSION

In real rock the sharp boundary between the filtrate and original pore fluid will be smeared by the effects of dispersion. We now estimate this effect in the axisymmetric geometry of Secs. II–V, assuming that dispersion can be modeled by a diffusive process with diffusivity $K_l$ in the direction of flow. We neglect the dispersion perpendicular to the flow, which is usually somewhat smaller than longitudinal dispersion. We take $c(r,t)$ to be the concentration of the filtrate in the pores, with $0 \leq c \leq 1$. For one-dimensional flow with fluid velocity $v=ut/\phi$ in the $x$ direction, a sharp concentration profile with initial filtrate concentration $c=0$ for $x > 0$ and $c=1$ for $x < 0$ at time $t=0$ evolves to become

$$c(x,t) = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{x - vt}{2(K_l)^{1/2}} \right) \right].$$

(36)

The midpoint of the front, $c=0.5$, reaches a position $x$ at time $t=x/v$, and since the front width varies slowly we approximate the concentration passing a fixed point $x_1$ at time $t$ by

$$c(x_1,t) = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{x_1 - d(t)}{2(K_l)^{1/2}} \right) \right].$$

(37)

where $d$ is the distance that the fluid has traveled when it arrives at $x_1$ at time $t$. If the flow rate is sufficiently high for dispersion to dominate molecular diffusion, we take the approximate diffusivity $K_l = \gamma \phi l/\phi$, where $l$ is a typical pore size and $\gamma = 0.5$ in unconsolidated grains. The dispersion of the filtrate depends only on the depth of invasion, and instead of a sharp front at $z = \beta a$ we now have a disperse front

$$c = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{z - \beta a}{2(\gamma \beta a l)^{1/2}} \right) \right].$$

(38)

where $(R, \theta)$ are spherical polar coordinates with $\theta = 0$ along the $z$ axis (Fig. 9). We now consider the flow along $\theta$ constant towards a sink at the origin $R=0$. The initial condition (38) for the sampling flow corresponds to a front at radial position $R = \beta a / \cos \theta$ which has a width of $2(\gamma \beta a l)^{1/2}/\cos \theta$ in the radial direction, as though it has already traveled a radial distance of $\beta a / \cos^2 \theta$.

Subsequent dispersion during fluid sampling follows the same Gaussian process. In general, the path taken by the fluid returning to the probe is uncorrelated with that taken during invasion (apart from streamlines close to the axis of symmetry which retrace their path and for which the dispersion is partly reversible). An additional displacement $R$ in plane flow therefore leads to a front of width $2\left[ \gamma l(R + \beta a / \cos^2 \theta) \right]^{1/2}$. We adopt the spherical flow approximation of Sec. III C: if $R$ is large compared to the radius $a$ of the sampling probe the flow is approximated by that towards a point sink. Fluid velocities vary as $R^{-2}$ and the front will be rapidly stretched in space as it approaches the origin. However, we assume that this has no effect on the contamination at a fixed point as a function of time and that the contamination depends solely on the distance traveled, as given by (37).

We restrict our attention to the fluid flowing along a line of $\theta = \theta_1$. The midpoint of the front is initially at $R_1 = \beta a / \cos \theta_1$ and this arrives at the origin at time $t_1 = (2\pi \phi(\beta a)^2 / (3 \gamma a^3 Q_s))$, where $Q_s$ is the rate of extraction of the fluid. The concentration profile is reversed, in the sense that the fluid initially arrives at the sink with contamination $c=1$, and $c$ subsequently decreases towards zero. We use (37) to approximate the filtrate concentration arriving at the sink as

$$c(\theta_1,t) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{R_1 - R(t)}{2(\gamma l(R_1 + \beta a / \cos^2 \theta_1)^{1/2}} \right) \right].$$

(39)

where $R(t) = (3Q_s / 2\pi \phi)^{1/3}$ is the initial position of the fluid which arrives at the origin at time $t$. We now consider a probe that collects the fluid which arrives from within the cone of $\theta < \theta_2$ and therefore collects a fraction of $1 - \cos \theta_2$ of the total flow. This corresponds to a probe with a central sampling zone of $a = \sin \theta_2$. The mean concentration of the filtrate in the sample is $F_r = \bar{c}$, where
We require a filtrate-free sample of the original pore fluid, and we are therefore interested in the tail of the distribution (40) for large $t$. Using the asymptote

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt \sim 1 - \frac{\exp(-z^2)}{z^{1/2}}, \quad z \to \infty$$

(41)

the mean concentration $\bar{c}$ (40) at late time can be expressed as

$$\bar{c} \sim \frac{(\gamma \beta a \sqrt{\pi})^{1/2}}{1 - \cos \theta_2} \int_0^{\theta_2} \frac{(1 + \cos \theta)^{1/2}}{R(t) \cos \theta - \beta a} \exp\left(-\frac{[R(t) \cos \theta - \beta a]^2}{4 \gamma \beta a (1 + \cos \theta)}\right) \sin \theta d\theta. \quad (42)$$

The exponential is largest when $\theta = \theta_2$, and so by Laplace’s method,21

$$\bar{c} \sim \frac{4(\sqrt{\pi} \beta a)^{3/2}}{\pi^{1/2}(1 - \cos \theta_2)[R(t) \cos \theta_2 - \beta a]^2} \times \frac{[R(t) \cos \theta_2 - \beta a]^2}{4 \gamma \beta a (1 + \cos \theta_2)} \times \frac{(1 + \cos \theta_2)^{5/2}}{(2 + \cos \theta_2) R(t) + \beta a}. \quad (43)$$

The analysis of Sec. III C predicted zero contamination of the sample in the absence of dispersion after a finite time $t_0$. At this same time $t_0$ the analysis above predicts that the filtrate contamination arriving at the sink on the streamline $\theta = \theta_2$ has reduced to $\epsilon = 0.5$ and contamination is lower on other streamlines closer to the axis of symmetry. The mean filtrate contamination is therefore $F_c = \bar{c} < 0.5$. The exponential in (43) that governs the tail of filtrate contamination decays once $R(t) \cos \theta_2 - \beta a \gg 2 [\gamma \beta a (1 + \cos \theta_2)]^{1/2}$ and we recall that the typical grain size $l$ is small compared to the probe radius $a$. The exponential decay of contamination becomes slower as $\cos \theta_2$ becomes small. Dispersion, not surprisingly, has prevented the achievement of zero contamination in finite time. Nevertheless, the exponential decay of contamination predicted by (43) is a marked improvement over the algebraic decay predicted for an unguarded probe.2

VII. APPROXIMATE ANALYSIS OF A PROBE AGAINST A CYLINDRICAL WELBORE

As before, we nondimensionalize all lengths by the radius $a$ of the probe. We consider a cylindrical wellbore of radius $r_w$, surrounded by a rock with isotropic permeability $k$. Filtrate has invaded the rock out to a radius $r_w$. The presence of the wellbore, with walls coated by filtercake, ensures that close to the probe the geometry is similar to that of the plane impermeable wellbore wall. We follow Hammond3 and assume that the spherical sink flow in the far field is perturbed only slightly by the presence of the wellbore. This hypothesis will always be false for some fluid particles, trapped in stagnation points caused by the presence of the wellbore, but numerical computations suggest that the corresponding streamlines contribute little to the total flow into the probe. We neglect the fact that the probe is not placed on the axis of the wellbore. The geometry is sketched in Fig. 9, and fluid particle trajectories are sketched in Fig. 10.

We nondimensionalize time by the timescale

$$T_1 = \frac{4 \pi a^3 \phi}{3 Q_t}, \quad (44)$$

i.e., by the time required to pump the pore volume of fluid within a sphere of radius $a$. Flows induced by density differences between the filtrate and original pore fluid will be slow,9 and will be negligible over the timescales for pumping. Predictions for fluid contamination therefore depend on the volume pumped, rather than time, as discussed in Sec. III.
At time \( \hat{t} \) fluid has been withdrawn from a sphere of radius \( R \). After accounting for the volume occupied by the wellbore of radius \( r_w \), we find that \( R \) satisfies

\[
(R^2 - r_w^2)^{3/2} = \hat{t}.
\]

The surface of this sphere (excluding its intersection with the wellbore) is

\[
S = 4\pi a^2 R^2 [1 - (r_w/R)^2]^{1/2}.
\]

The surface of the spherical cap which subtends a half-angle \( \theta_i = \cos^{-1}(r_m/R) \) around the probe axis lies solely within the original pore fluid (Fig. 9). This spherical cap has an area of

\[
2\pi a^2 R^2 (1 - \cos \theta_i) = 2\pi a^2 R (R - r_m),
\]

and hence contributes a fraction

\[
f_i = \frac{R - r_m}{2(R^2 - r_w^2)^{1/2}} = \frac{\hat{t}^{-1/3}}{2} \left( (r_m^2 + \hat{t}^{2/3})^{1/2} - r_m \right)
\]

of the total flow into the probe at time \( \hat{t} \). We assume that \( \hat{t}^{1/2} \gg r_w \) (i.e., that we have pumped fluid from a region that extends far beyond the wellbore), so that we can neglect \( r_w \) in (47). This filtrate-free fraction (47) enters the probe over its central zone \( 0 < r < r_0 \), where \( r_0 \) is given by the local, plane analysis of Sec. III A. Hence, by (14) and (47)

\[
1 - (1 - r_0^3)^{1/2} = \frac{1}{2} (1 - r_m) \hat{t}^{-1/2}.
\]

If the sample probe has radius \( \alpha \) it will therefore be free of filtrate after a time

\[
i_0 = \frac{r_m^3}{[2(1 - \alpha^2)^{1/2} - 1]}.
\]

Any spherical cap which subtends a half-angle greater than \( \theta_i \) will intersect the filtrate-invaded zone, and at time \( t_0 \) a mixture of filtrate and original pore fluid will arrive at the outer part of the probe \( \alpha < r < 1 \).

An analysis analogous to that of Sec. V gives the non-dimensional time \( i_1 \) required to collect a filtrate-free sample of volume \( V_s \):

\[
i_1 = i_0 + V_s T_1 = \frac{r_m^3}{[2(1 - \alpha^2)^{1/2} - 1]^{1/3}} + \frac{V_s}{T_1 Q_s (1 - (1 - \alpha^2)^{1/2})},
\]

and this time is minimized by the choice \( \alpha = \alpha_{opt} \), where \( \alpha_{opt} \) satisfies

\[
Y = \left( \frac{\pi a^3 r_m^3 \phi}{2 V_s} \right)^{1/2} = \frac{[2(1 - \alpha_{opt}^2)^{1/2} - 1]^2}{4[1 - (1 - \alpha_{opt}^2)^{1/2}]}. \tag{51}
\]

This is a quadratic equation for \((1 - \alpha_{opt}^2)^{1/2}\), with solution

\[
\alpha_{opt}^2 = \frac{3}{4} - \frac{\sqrt{2}}{2} - \frac{1}{2} (1 - Y)(Y^2 + 2Y)^{1/2}. \tag{52}
\]

Figure 11 shows \( \alpha_{opt} \) as a function of \( Y \), together with the asymptotes

\[
\alpha_{opt} \approx \left( \frac{3}{4} \right)^{1/2} - \left( \frac{Y}{2} \right)^{1/2} + \cdots, \quad Y \to 0
\]

and

\[
\alpha_{opt} \sim \left( \frac{2}{Y} \right)^{1/2}, \quad Y \to \infty. \tag{54}
\]

\[\text{FIG. 11. The sample probe radius } \alpha_{opt} \text{ which minimizes the time to obtain a sample in a cylindrical wellbore. (a) } \alpha_{opt} \text{ as a function of } Y = (\pi a^3 r_m^3 \phi/2 V_s)^{1/2}, \text{ as given by (52). Curves (b) and (c) show asymptotes for } Y \gg 1 \text{ (54) and } Y \ll 1 \text{ (53), respectively.} \]

\[\alpha_{opt} \sim \left( \frac{2}{Y} \right)^{1/2}, \quad Y \to \infty. \quad \tag{54}\]

\[\text{VIII. CONCLUDING REMARKS} \]

The above analysis assumes that the filtrate and original pore fluid are identical. If the viscosity \( \mu_f \) of the filtrate is less than the viscosity \( \mu_p \) of the original pore fluid, breakthrough of the original pore fluid to the probe will be delayed, though in this case the interface is unlikely to be initially sharp, due to fingering of low-viscosity fluid as it enters the rock. If \( \mu_p < \mu_f \) the original pore fluid will reach the probe earlier, because of fingering, but cleanup of the filtrate will be slower. Diffusion and dispersion will similarly delay the cleanup of the filtrate. Any estimates of the time at which the sample is clear of filtrate are likely to be underestimated. Such estimates are, in any case, difficult since the depth of invasion is not always known.

The analysis presented here has similarities with studies of water coning,10–14 in which an oil/water interface moves towards a production well (a point sink). However, gravity, unimportant in the sampling problem, usually plays an important role in coning, for which the interesting case is that of the strongest flow which allows gravitational and viscous forces to balance without the resulting deformed interface being drawn into the well. If we consider the flow of an unbounded homogeneous fluid towards a point sink, we conclude that in the absence of interfacial tension and gravity an initially plane interface develops a nearly conical finger which becomes cusped as it approaches the sink. The final singular cusp is avoided when, as here, the point sink is replaced by a disc of nonzero radius.

In most studies of fluid withdrawal and fingering there is a single sink (or production well). The composite probe/sink considered here adds an extra dimension of interest to such fingering problems.

21F. W. J. Olver, Asymptotics and Special Functions (Academic, San Diego, 1974).