Gravity and Entanglement

Josh
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This seminar is a very quick introduction to the role played by entanglement in quantum gravity and holography. If you want to learn more, the following three sets of lecture notes are highly recommended:


1. Entanglement
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2. Entanglement entropy
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2. Entanglement entropy
3. Entanglement entropy in QFT
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2. Entanglement entropy
3. Entanglement entropy in QFT
4. Holographic entanglement entropy
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2. Entanglement entropy
3. Entanglement entropy in QFT
4. Holographic entanglement entropy
5. Geometry from entanglement
Entanglement
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The states $|i, j\rangle = |i\rangle_A \otimes |j\rangle_B$ comprise an orthonormal basis of the product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. 
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product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

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If a state $|\psi\rangle \in \mathcal{H}$ can be written as the product of two states $|\psi_A\rangle \in \mathcal{H}_A$, $|\psi_B\rangle \in \mathcal{H}_B$, i.e. $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, then there is no entanglement.
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Otherwise, it is an entangled state.

For example, $|1, 1\rangle$ is not entangled, while $\frac{1}{\sqrt{2}} (|1, 2\rangle + |2, 1\rangle)$ is.
This definition of entanglement is qualitative. Either a state is entangled, or it isn’t.
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It would be desirable to have a quantitative measure of entanglement, i.e. one that can tell us the degree to which a state is entangled.
Entanglement entropy
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We should therefore be able to quantify entanglement in terms of the amount of information available. Information is most naturally measured in terms of entropy.
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Density matrices can account for both quantum uncertainty and statistical uncertainty.
Whenever there is statistical uncertainty in a state, measuring that state will give us some information about the real world.

In terms of density matrices, the amount of information we can expect to obtain from such a measurement is given by the von Neumann entropy

$$S = \text{tr}(\log \rho)$$

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On the other hand, consider for example the density matrix $\rho = \frac{1}{2} (|\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|)$, where $|\psi_1\rangle, |\psi_2\rangle$ are orthonormal.
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and the components of the reduced density matrix are

$$\langle i | A \rho_A | j \rangle_A = \sum_k \langle i, k | \rho | j, k \rangle . \quad (4)$$
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In general, even if the original density matrix $\rho$ is pure, the reduced density matrix $\rho_A$ may be mixed. In fact, this occurs exactly when $\rho$ contains entanglement.
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The entanglement entropy \( S_A \) of a density matrix \( \rho \) in the factor \( \mathcal{H}_A \) is defined as the von Neumann entropy of \( \rho_A \).

\[
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- \( \rho = |\psi\rangle \langle \psi |, \) where \( |\psi\rangle = \frac{1}{\sqrt{2}}(|1, 2\rangle + |2, 1\rangle). \) Then \( \rho_A = \frac{1}{2}(|1\rangle_A \langle 1|_A + |2\rangle_A \langle 2|_A), \) and \( S_A = \log 2. \)
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Entanglement entropy in QFT
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Under reasonable locality assumptions, the full Hilbert space decomposes as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ contains the degrees of freedom living in $A$, and $\mathcal{H}_B$ contains the degrees of freedom living in $B$. 
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If we are given a density matrix $\rho$ on the full Hilbert space $\mathcal{H}$, we can as before compute the reduced density matrix $\rho_A$ on the factor $\mathcal{H}_A$, and then find the entanglement entropy $S_A = - \text{tr}_A \rho_A \log \rho_A$. 
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We can similarly associate an entanglement entropy $S_A$ with every possible subregion $A \subset \Sigma$.

$S_A$ tells us how much information we can expect to learn about the state outside of $A$, after measuring the state inside of $A$. 
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For example, in a 2D CFT with central charge $c$, consider the vacuum state $\rho = |0\rangle \langle 0|$. Let $A$ be a subregion of length $L$. Then

$$S_A = \frac{c}{3} \log \left( \frac{L}{\epsilon} \right),$$

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It is characteristic of local QFTs that the vacuum state is highly entangled.
Holographic entanglement entropy
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All we will need to know about AdS/CFT is that it provides a one-to-one mapping between degrees of freedom living in a CFT on the boundary of a spacetime, and those living in the gravitational bulk.
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$$\rho_A = \text{tr}_{\overline{A}} \rho.$$
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Precisely, let $T$ be a surface in the bulk which shares its boundary with $A$, $\partial T = \partial A$. Let $T_{\text{min}}$ be the surface with this property whose area is minimal, and let $A_{\text{min}}$ be the area of $T_{\text{min}}$. 

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Then the entanglement entropy at leading order is given by

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The Ryu-Takayanagi conjecture applies to holographic theories where the bulk theory of gravity is general relativity, but there are generalisations to higher derivative theories of gravity. In these cases the area is replaced by some other geometric quantity.
It can be shown that the Ryu-Takayanagi formula implies the Bekenstein-Hawking formula.
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Thus the Ryu-Takayanagi conjecture is a vast generalisation of black hole entropy.
Spacetime from entanglement
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But the Ryu-Takayanagi conjecture implies that much of the bulk geometry is encoded in the entanglement structure of the density matrix $\rho$.

Entanglement is an absolutely fundamental feature of a quantum theory. Geometry is not.

This has lead to the fascinating suggestion that, in quantum gravity, geometry does not exist a priori, but emerges from the entanglement structure of the quantum state of the universe.
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In other words, by modifying the entanglement structure of the state of universe, we have modified the geometry of spacetime.
The idea of emergent geometry allows us to reason about gravitational physics by using mathematical facts about entanglement. For example:

- The 'first law of entanglement' implies the linearised Einstein equations.
- 'Relative entropy inequalities' allow one to define a well-behaved completely general notion of perturbative gravitational energy.
- 'Strong subadditivity' is related to the null energy condition.
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These sound very interesting, but I haven’t yet been able to fully understand how they work. See those lecture notes for more on them.
The end