The Granular Blasius Problem
Boundary layers in granular flows

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Modelling granular currents is important

- > 7,600 deaths from landslides annually (Perkins 2012)
- Usually in developing countries

Common models are depth-averaged (‘shallow water’)

Ad hoc description of depthwise velocity profile

Want to understand internal dynamics better
Depth-averaged models

- Shallow water equations on a slope

\[
\frac{\partial h}{\partial t} + \frac{\partial (h \bar{u})}{\partial t} = 0
\]

\[
\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial t} \left( \frac{1}{2} hu^2 + \frac{1}{2} gh^2 \cos \theta \right) = gh \sin \theta
\]

- Depth \( h \), depth-averaged velocity \( \bar{u} \)

- Closure relation \( \bar{u}^2 = \chi \bar{u}^2 \) for shape factor \( \chi \geq 1 \)

- Shape factor characterises depthwise velocity profile
Depthwise velocity profile

\[ \chi = \frac{\bar{u}^2}{\bar{u}^2} \]

- Usually assume constant \( \chi \), e.g. plug flow, \( \chi = 1 \)
- Reasonable assumption over long lengthscales
- But \( \chi \) is not constant when topography is present
- Difficult to measure velocity profile experimentally
- Can be measured in DPM simulations
Granular Blasius problem

$H$

$U$

$x < 0$

smooth surface (possibly frictional)

$x > 0$

bumpy surface

depth profile of a steady flow

current flows off end of surface

flow introduced upstream

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Granular boundary layer problem

- Model of increasing topographical resistance
- \( x < 0 \): Smooth, slip allowed
- \( x > 0 \): No-slip condition creates **boundary layer**
- BL grows and eventually takes over
Granular vs. classical Blasius problems

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The Granular Blasius Problem Boundary layers in granular flows
From classical to granular:

blade (no-slip)
From classical to granular: Free surface, finite depth
From classical to granular: Slope
From classical to granular: Granular rheology

smooth bumpy

\( \dot{\theta} \)

\( g \)

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The Granular Blasius Problem Boundary layers in granular flows
From classical to granular: Free surface, finite depth

- BL induces flow in outer layer, which affects BL
- Behaviour as $Re \to \infty$ depends on $Fr$
- Tsang et al. submitted to *JFM Rapids*
From classical to granular: Slope

- Evolution towards far-field profile
- Nusselt film for laminar Newtonian fluid
- Bagnold profile for granular flow
From classical to granular: Granular rheology

- $\mu(I)$ rheology (Jop et al. 2006)
- high $\dot{\gamma} \implies$ high $I$ in BL
The BL equation has the same structure

- **Classical:**
  \[ u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{\partial p}{\partial x} + \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial z^2} \]

- **Under \( \mu(I) \):**
  \[ u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = \sin \theta + \frac{\partial}{\partial z} (\mu(I)p) \]
  \[ \sim \sin \theta + \mu \frac{\partial p}{\partial z} + p \frac{d\mu}{dl} \frac{\partial l}{\partial z} \]
  \[ \sim (\cdots) + (\cdots) \frac{\partial^2 u}{\partial z^2} \]
Analysing the granular BL equation

- Solutions depend on behaviour of $\mu(I)$ as $I \to \infty$

$$\mu(I) \sim \mu_1 + \frac{\mu_2 - \mu_1}{l_0/I + 1}$$

- Generalise $\mu(I)$

$$\mu(I) \sim \mu_2 - \frac{m}{\alpha - 1} \left(\frac{l_0}{I}\right)^{\alpha - 1}$$

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \sim \frac{\partial^2 u/\partial z^2}{(\partial u/\partial z)^\alpha}$$

- Problems with well-posedness for high $I$? (Barker et al. 2017)
Analysing the granular BL equation

- Approximate similarity solutions

\[ u \propto f'(z/\beta(x)), \quad f''' + \frac{u_s^{1+\alpha}}{2 - \alpha} ff''^{1+\alpha} = 0 \]

- Singular behaviour as \( \alpha \to 2^- \)?
Realisation in DPM (MercuryDPM)

Topography: ior-ballotini-slope16-run2 : v200-h020 : 150

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Realisation in DPM (MercuryDPM)

- Is the no-slip condition realistic?
- Rolling resistance
- Restitution coefficient
- . . .
Summary

- Study how granular currents respond to topography
- Similar to classical Blasius problem
- BLs dynamics governed by high $I$
- Generalisations of $\mu(I)$
- DPM realisation has some subtleties