Finite amplitude Kelvin-Helmholtz billows at high Richardson number

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Stratified shear flow

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + Ri_b b \mathbf{e}_z + \frac{1}{Re} \nabla^2 \mathbf{u} \\
\frac{\partial b}{\partial t} + \mathbf{u} \cdot \nabla b = \frac{1}{Re} \nabla^2 b \\
\nabla \cdot \mathbf{u} = 0
\]

\((Pr = 1)\)
Richardson Number

The Richardson number $Ri$ is the non-dimensional ratio of buoyancy to shear. It is important to distinguish the gradient (local) Richardson number $Ri_g = \frac{\partial b/\partial z}{(\partial u/\partial z)^2} Ri_b$ from the bulk Richardson number $Ri_b$, a parameter.
Miles-Howard Theorem

For a steady, one-dimensional, Boussinesq, inviscid, stratified shear flow, linear stability is guaranteed if $Ri_g > 1/4$ everywhere.

“Sufficiently strong stratification enforces stability.”

$Ri = 1/4$ is often seen as a magic number in oceanography, and is used in parameterisations, despite the limited scope of the theorem.
Forced equations

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u &= -\nabla p + Ri_b b e_z + \frac{1}{Re} \nabla^2 u + \frac{2}{Re} \tanh(z) \sech^2(z) e_x \\
\frac{\partial b}{\partial t} + u \cdot \nabla b &= \frac{1}{Re} \nabla^2 b + \frac{2}{Re} \tanh(z) \sech^2(z) \\
\nabla \cdot u &= 0
\end{align*}
\]

Steady solution

\[
\begin{align*}
u &= e_x \tanh z \\
b &= \tanh z
\end{align*}
\]
State tracking

Formally define $F$:

$$(u(T), b(T)) = F(u(0), b(0), T)$$

Look for steady states

$$F(u, b, T) = (u, b), \forall T.$$  

For simplicity, let $X = (u, b)$. Pick arbitrary $T$, find zeros of

$$G(X) \equiv F(X) - X$$
Newton-GMRES

To solve

\[ G(X) = 0 \]

use Newton iteration

\[ G_X(X_n) \cdot (X_{n+1} - X_n) = -G(X_n). \]

To solve linear system at each step using GMRES, need only know

\[ G_X(X) \cdot Y \approx \frac{G(X + \epsilon Y) - G(X)}{\epsilon} \]
Results ($Re = 1000$)

Stability analysis performed with Arnoldi iteration.
Results ($Re = 1000$)
At pitchfork/saddle-node bifurcation, attempt to solve

\[ F(X, Ri_b) = X \]
\[ F_X(X, Ri_b) \cdot Y = Y \]
\[ Y \cdot A = 1 \]

for \( X, Y \) and \( Ri_b \), where \( A \) is some fixed direction.

Similar for Hopf bifurcation, with 3 time integrations.
Conclusions

- Non-trivial steady states and complex behaviour are possible at $Ri > 1/4$.
- States only just go past $1/4$.

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Open questions

- How relevant is forced system?
- What are the effects of $Pr$?