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SHORTER CONTRIBUTION

Turbulence in a strongly stratified fluid—is it unstable?

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Abstract—It is shown that if the buoyancy flux is a local property of turbulence in a stratified fluid that decreases sufficiently rapidly as the local Richardson number increases, then an initially linear density profile in a turbulent flow far from boundaries may become unstable with respect to small variations in the vertical density gradient. An initially linear profile will then become ragged; this possible instability may be associated on occasions with the formation of density microstructure in the ocean.

RECENT observations, both visual and with rapid response thermistors (see, for example, WOODS, 1968 and NEAL, NESHYBA and DENNER, 1969) have demonstrated the widespread occurrence of microstructure in the density field in regions of the ocean where the overall stratification is stable. On occasion, this microstructure is in the form of a succession of sheets and layers; the density (and temperature) being almost uniform in each layer and jumping nearly discontinuously across the sheets that separate them. On other occasions, this structure is less distinct, the profile of density gradient being merely ragged with an apparently continuous distribution of density gradient ranging from large negative values (strongly stable) to zero.

The apparently common occurrence of this microstructure has led to several speculations concerning its origin. GARRETT and MUNK (1971) have suggested that it arises as a consequence of sporadic local instability of large-scale internal gravity waves that produces a locally mixed patch or ribbon if the breaking continues as the wave passes through the fluid. Each patch will spread outwards to some extent, interleaving with others that may previously have been produced in the same way. Herbert Huppert, in unpublished work, has shown that many of the characteristics of the microstructure, particularly as observed by NEAL, NESHYBA and DENNER (1969) beneath an arctic ice island, are similar to those of the doubly diffusive layers (TURNER, 1967) that are set up when two properties of the water (temperature and salinity) that influence the density in opposite senses are associated with fluxes (of heat and salt) in opposite directions. Again, John Lazier, in unpublished observations in a Welsh lake, has shown fairly convincingly that a pattern of sheets and layers moving slowly vertically through the fluid can be interpreted in terms of low frequency plane internal gravity waves with their wave-numbers nearly vertical.

It is almost certain that not all of the microstructure found in the ocean can be ascribed to a single mechanism, and quite likely that, in the appropriate circumstances, each of these three processes can produce structures similar to those observed. The purpose of this note is to point out the possibility of yet another process that may be involved, at least on occasion. The matter of sorting out which if any of these mechanisms is responsible for the structure observed on any given occasion may require measurements more detailed than those needed to establish the existence of the structure itself. For example, Lazier's mechanism is characterized by slow vertical movements of the structure through the fluid; Huppert's by the existence of counter-gradients of temperature and salinity within certain ranges and that of Garrett and Munk by the sufficiently frequent occurrence of shear in the internal wave field large enough to produce local instability.

The suggestion made is that a field of turbulence in a stratified fluid far from boundaries, being supported by a Reynolds stress and having a non-zero vertical flux of buoyancy (or equivalently, density) may be unstable to variations in the vertical density gradient. We are concerned in this note only with the possible instability of a flow with initially constant gradients of buoyancy and velocity; if the instability in the mean profile develops, its subsequent evolution into a pattern of sheets and layers (which is beyond the scope of this analysis) may take place as described by LONG (1970). Long's analysis is based on a specific model of wave breaking and consequent mixing and in it the

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curvature of the buoyancy profile plays an essential role. The present note suggests a way in which the initial curvature can be developed. The possible instability can be described qualitatively in the following way. Suppose that the buoyancy flux is initially constant and that the initially uniform mean density gradient is disturbed locally, becoming slightly more negative (more strongly statically stable). If this *reduces* the local 'exchange coefficient' for buoyancy more rapidly than the local gradient increases, then in order to maintain the mean flux, the magnitude of the local gradient will tend to increase further; the perturbation will amplify. The condition under which this can occur is found simply.

Consider a turbulent layer supporting and supported by a constant stress $\tau = \varrho u_*^2$, initially steady in the mean. The buoyancy is defined by $\hat{B} = -g\delta\varrho/\varrho_0$ where $\delta\varrho$ is the difference between the local density and the reference density ϱ_0. The velocity and buoyancy fields can be represented as the sum of the mean and fluctuating (turbulent) components

$$U_0(z)+u,v,w; \quad B_0(z)+b$$

with the choice of the x-axis as that of the mean velocity and with the z-axis vertical. The conversation of horizontal momentum and of buoyancy require that the vertical fluxes are independent of height

$$\frac{\partial}{\partial z}(\overline{uw})_0 = 0, \quad \frac{\partial}{\partial z}(\overline{bw})_0 = 0. \tag{1}$$

Now, suppose that the *mean* velocity and buoyancy fields, together with the mean turbulent fluxes are distorted slightly in the vertical from their initial states, becoming

$$U_0(z)+U_1(z,t); \quad B_0(z)+B_1(z,t),$$

where the perturbations U_1 and B_1 are infinitesimal. The mean horizontal momentum and buoyancy equations are

$$\left.\begin{aligned}\frac{\partial U_1}{\partial t}+\frac{\partial}{\partial z}\overline{uw} = 0\\ \frac{\partial B_1}{\partial t}+\frac{\partial}{\partial z}\overline{bw} = 0.\end{aligned}\right\} \tag{2}$$

In a turbulent shear flow, the Reynolds stress itself is not a local property of the mean motion, but the stress *gradient* probably is more nearly so; it has been argued (PHILLIPS, 1967) that in a flow of uniform density

$$\frac{\partial}{\partial z}\overline{uw} = -A\,\overline{w^2}\,\Theta\frac{\partial^2 U}{\partial z^2},$$

where A is numerical constant and Θ integral time scale of the turbulent eddies in a frame of reference moving with the local mean flow. In a stratified flow, the coefficient of $\partial^2U/\partial z^2$ (a kind of 'eddy viscosity') will depend presumably on the Richardson number; whatever its form it is expected to be positive in turbulent flow. One might assume on similarity grounds that

$$\frac{\partial}{\partial z}\overline{uw} = -u_*^2 N^{-1} F(J)\frac{\partial^2 U}{\partial z^2} \tag{3}$$

where $N = (\partial B/\partial z)^{\frac{1}{2}}$ is the local Brunt–Väisälä frequency, $J = N^2/U'^2$ the local Richardson number and the function F is positive but otherwise undetermined. Substitution of (3) into the first of (2) shows that $U_1(z,t)$ obeys a diffusion-type equation; initial small perturbations in the velocity profile always decay. Note also that from (1) the initial velocity gradient is constant in the region under consideration.

It is in the buoyancy distribution that the possibility of instability arises. Under the assumption that the buoyancy flux is a *local** property of the mean field, dependent on the external parameters u_*, J and dU/dz (or N) that specify the undisturbed turbulent motion, it follows on similarity grounds that

$$\begin{aligned}\overline{bw} &= -u_*^2 N^{-1} G(J)\frac{\partial B}{\partial z} = -u_*^2 N\,G(J)\\ &= -u_*^2 U' J^{\frac{1}{2}} G(J),\end{aligned} \tag{4}$$

*This assumption is, in fact, essential. TOWNSEND (1956, p. 110) argues that the flux of heat in turbulent flow can realistically be expressed in terms of the local gradient of temperature, consistently with this assumption. Momentum flux is certainly not local, because of the long-range influence of pressure forces.

where G is another positive function that is otherwise undetermined. As $J \to \infty$ as u_* is held fixed, the buoyancy flux must remain finite or approach zero; thus $G(J)$ must decrease at least as rapidly as $J^{-\frac{1}{2}}$ as $J \to \infty$.

If the second equation of (2) is differentiated with respect to z and the expression (4) substituted, there results

$$2N\frac{\partial N}{\partial t} - u_*^2 \frac{\partial^2}{\partial z^2}\{N\,G(J)\} = 0.$$

On substitution for N and U their mean values, N_0, U_0 plus a small perturbation $N_1(z,t)$, $U_1(z,t)$, it follows after a little algebra that to the first order in the perturbation quantities,

$$\frac{\partial N_1}{\partial t} - N_0^{-1} J_0^{\frac{1}{2}} u_*^2 \left\{\frac{\partial}{\partial J}[J^{\frac{1}{2}}\,G(J)]\right\}_0 \frac{\partial^2 N_1}{\partial z^2} = f(U_1), \tag{5}$$

where f is a linear function of the decaying velocity perturbation. If the quantity in the curly brackets is positive, this also is a diffusion equation and the perturbation in N (or density gradient) decays, but if

$$\frac{\partial}{\partial J}[J^{\frac{1}{2}}\,G(J)] < 0 \tag{6}$$

equation (5) becomes in essence a diffusion equation with time reversed, and initially small disturbances in the density gradient amplify.

It is not clear whether this condition for instability can be satisfied for sufficiently large J. If $G(J)$ decreases precisely as $J^{-\frac{1}{2}}$ as $J \to \infty$, the inequality is never achieved, but if it decreases faster than this, then there will exist a critical value of J beyond which an initially uniform density gradient in turbulent flow will tend to break up. Small-scale disturbances amplify most rapidly, but the minimum vertical wave length amplified in nature presumably depends on the smallest scale over which the buoyancy flux can be regarded as a local function; the analysis can say nothing about this.

The results of atmospheric and laboratory measurements on turbulence in strongly stratified fluids are insufficiently clear to determine whether in fact (6) is satisfied, though the laboratory experiments of Moore and Long (1969) in which such a flow developed a sharp density gradient and a much more diffuse velocity gradient, tend to support the notion that the initial state described here is unstable in this way. The purpose of this note, however, is not to resolve the question but to raise it.

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