Advanced Quantum Field Theory
Solutions to supplementary exercises to sheet 3
Lent Term 2018

Solutions by:

Alec Barns-Graham ab919@cam.ac.uk
Kai Roehrig kafr2@cam.ac.uk

Let's start this beautiful example sheet by doing this integral:

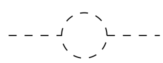
$$\begin{aligned} I(d,m,n,\alpha) &:= \int \mathrm{d}^d l \frac{l^m}{(l^2+\alpha)^n} \quad \text{for } \alpha \text{ some function independent of } l \\ &= \mathrm{Vol}(S^{d-1}) \int_0^\infty \mathrm{d}l \frac{l^{m+d-1}}{(l^2+\alpha)^n} \quad \text{because integrand is function of } |l| \\ &= \frac{\mathrm{Vol}(S^{d-1})}{\Gamma(n)} \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}t\, t^{n-1} l^{m+d-1} e^{-t(l^2+\alpha)} \quad \text{standard trick} \\ &= \frac{\mathrm{Vol}(S^{d-1})}{2\Gamma(n)} \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}u\, u^{(m+d)/2-1} t^{n-(m+d)/2-1} e^{-u-\alpha t} \quad \text{substituting } u = tl^2 \\ &= \frac{\mathrm{Vol}(S^{d-1})}{2\Gamma(n)} \alpha^{(m+d)/2-n} \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}s\, u^{(m+d)/2-1} s^{n-(m+d)/2-1} e^{-s-u} \quad \text{substituting } s = \alpha t \\ &= \frac{\mathrm{Vol}(S^{d-1})}{2\Gamma(n)} \alpha^{(m+d)/2-n} \Gamma\left(\frac{m+d}{2}\right) \Gamma\left(n - \frac{m+d}{2}\right) \quad \text{definition of the } \Gamma\text{-function} \\ &= \pi^{d/2} \alpha^{(m+d)/2-n} \frac{\Gamma\left(\frac{m+d}{2}\right)\Gamma\left(n-\frac{m+d}{2}\right)}{\Gamma(d/2)\Gamma(n)} \,. \end{aligned} \tag{1}$$

So note already for $m+d$ odd we have no divergence even for $m+d$ large (compared to n) where the integral does diverge. So first of all why is this possible? Well an analyst would be aghast at some of the algebraic manipulations that we did here, we swapped limits when we weren't necessarily allowed to.

This amounts to the fact that this is a regularisation scheme and we are seeing this here. It is making infinite quantities finite and thus giving us finite answers to actual physical observables. This may seem unsatisfactory, much like some people (myself included) can feel a bit uneasy about zeta-function renormalisation. To feel satisfied, just note that one can use other forms of regularization, such as using a cut-off, and one gets out the same final answer.

Exercise 1

a) This is the diagram:



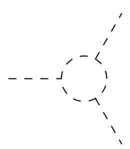
b) The amplitude for this diagram is

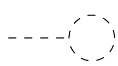
$$\begin{aligned} \frac{(-\mu^{\varepsilon/2} g)^2}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2+m^2} \frac{1}{(k+p)^2+m^2} \\ = \frac{\mu^\varepsilon g^2}{2(4\pi)^{d/2}} \Gamma(2-d/2) \int_0^1 \mathrm{d}x\, (x(1-x)p^2+m^2)^{d/2-2} \,. \end{aligned} \tag{2}$$

Putting in $d = 6 - \varepsilon$ we have that the $\mathcal{O}(1/\varepsilon)$ term is

$$-\frac{g^2}{(4\pi)^3}(m^2 + \frac{1}{6}p^2) \,, \tag{3}$$

as required.
We also need to calculate these diagrams





The diagram that changes g^3 has amplitude

$$\begin{aligned}
-\mu^{3\varepsilon/2} g^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2+m^2} \frac{1}{(k+p)^2+m^2} \frac{1}{(k+q)^2+m^2} \\
= -\frac{\mu^{3\varepsilon/2} g^3}{(2\pi)^d} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, I\big(d, 0, 3, m^2 + xp^2 + yq^2 - (xp+yq)^2\big) \,.
\end{aligned} \tag{4}$$

After calculation we find that the $\mathcal{O}(1/\varepsilon)$ term is

$$-\frac{g^3}{(4\pi)^3} \,. \tag{5}$$

The tadpole diagram has amplitude

$$\begin{aligned}
-\frac{\mu^{\varepsilon/2} g}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2+m^2} &= -\frac{\mu^{\varepsilon/2} g}{2(2\pi)^d} I(d,0,1,m^2) \\
&= -\frac{\mu^{\varepsilon/2} g m^{d-2}}{2^{d+1}\pi^{d/2}} \frac{\Gamma(1-d/2)}{\Gamma(1)} \,.
\end{aligned} \tag{6}$$

So the $\mathcal{O}(1/\varepsilon)$ term is

$$-\frac{1}{\varepsilon} \frac{g m^4}{2(4\pi)^3} \,. \tag{7}$$

c) We find by direct calculation that:

$$V''(\phi)^3 = m^6 + 3m^4 g \mu^{\varepsilon/2} \phi + 3m^2 g^2 \mu^{\varepsilon} \phi^2 + g^3 \mu^{3\varepsilon/2} \phi^3 \,. \tag{8}$$

The tadpole and the three vertex divergences are clearly accounted for. The propagator counter term contribution is

$$\frac{2}{\varepsilon} \left(\frac{1}{12(4\pi)^3} g^2 p^2 + \frac{1}{2(4\pi)^3} m^2 g^2 \mu^{\varepsilon/2} \right) = \frac{1}{\varepsilon} \frac{g^2}{(4\pi)^3} (m^2 + \frac{1}{6} p^2) \,. \tag{9}$$

Hence this renormalises the theory.
It is easy enough to check that $\mathcal{L}_{\mathrm{ct}}$ has dimension 6.
d) We write the dimensionful bare coupling as λ_0 and note that since the bare coupling is defined at a fixed μ_0 we have that $\frac{d\lambda_0}{d\log\mu} = 0$. The dimensionful bare coupling is

$$\lambda_0 = \mu^{\varepsilon/2} \left(g - \frac{g^3}{\varepsilon(4\pi)^3} \right) \,. \tag{10}$$

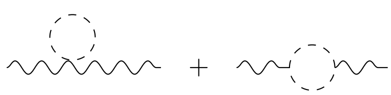
Differentiating both sides by $\log\mu$ and dividing by $\mu^{\varepsilon/2}$ gives

$$\begin{aligned}
0 = \frac{\varepsilon}{2} \left(g - \frac{g^3}{\varepsilon(4\pi)^3} \right) + \beta \left(1 - \frac{3g^2}{\varepsilon(4\pi)^3} \right) \,, \\
\implies \beta = -\frac{\varepsilon g}{2} - \frac{g^3}{(4\pi)^3} + \mathcal{O}\left(\frac{g^5}{\varepsilon} \right) \quad \xrightarrow{\varepsilon \to 0} \quad -\frac{g^3}{(4\pi)^3} \,.
\end{aligned} \tag{11}$$

Note that the order $1/\varepsilon$ terms are the divergent two loop terms, so it is absolutely essential that we *first* do the series expansion of β in g, and only after that take $\varepsilon \to 0$. (Otherwise we get a wrong result, as the order of limits really matters here.) In other words, we expect this to be divergent as we have not renormalised to two loops and hence throw them away to get our one-loop expression.

However we do not expect a continuum limit non-perturbatively as the potential is not bounded from below, so this theory will not be well-defined unless the coupling is zero and the theory is free. So this is not a quantum field theory. So what do our manipulations mean? One way to well define everything is via algebraic quantum field theory. Essentially one can multiply fields by compactly supported test functions then all our perturbative calculations are finite and well defined. We find that we can formally write our calculations in the way that we have written our calculations above. One can do this for any Lagrangian, for example ϕ^5 in 23 dimensions. However if there is no non-perturbative definition of the theory (which doesn't mean UV complete, it could be the effective theory of another theory) then the perturbative data that we calculate is essentially meaningless as it corresponds to no definite theory.

Exercise 2



a) The gauge invariant kinetic term for the scalar is

$$\partial_\mu \phi^* \partial^\mu \phi + ieA^\mu (\partial_\mu \phi^* \, \phi - \phi^* \partial^\mu \phi) + e^2 \phi^* A_\mu A^\mu \phi \, . \tag{12}$$

This means that we have two different couplings for the photon and the scalar (note that QED only has one coupling as there is only one covariant derivative in its kinetic term). They give rise to the two contributions to the vacuum polarization shown above. Since only the combination of all of these terms is gauge invariant, it is clear that we need both diagrams (with precisely the right relative coefficient) for a gauge invariant answer. This argument extends to all correlation functions: they will receive contributions from many Feynman diagrams, and generically, we need to sum up all of them to get a gauge invariant answer. This can be used e.g. as a sanity check on an obtained answer. In the case at hand we can actually refine this argument a little more: We will see below that the first diagram is independent of the external momentum, and its explicit role is to *cancel the mass renormalization* of the photon. Clearly, this serves the greater purpose of preserving gauge invariance, as, in order to preserve gauge invariance, we need the photon to stay massless and vice versa.

b) We introduce the dimensionless coupling

$$g := \mu^{-2+d/2} e \equiv \mu^{-\varepsilon/2} \, e \, . \tag{13}$$

The two Feynman diagrams lead to

$$\begin{aligned} \Pi^{\mu\nu}_{\text{1-loop}} &= -\mu^\varepsilon g^2 \int \frac{d^d p}{(2\pi)^d} \frac{(2p+q)^\mu (2p+q)^\nu}{(p^2+m^2)((p+q)^2+m^2)} + 2\,\mu^\varepsilon g^2 \int \frac{d^d p}{(2\pi)^d} \frac{\delta^{\mu\nu}}{p^2+m^2} \\ &= -\mu^\varepsilon g^2 \int \frac{d^d p}{(2\pi)^d} \frac{4\,p^\mu p^\nu - 2\,q^\mu p^\nu - 2\,p^\mu q^\nu + q^\mu q^\nu - 2\,\delta^{\mu\nu}((p+q)^2+m^2)}{((p+q)^2+m^2)(p^2+m^2)} \, . \end{aligned} \tag{14}$$

Now we shall use some a priori knowledge we have about the answer we expect in order to speed up our computation: We know that

$$\Pi^{\mu\nu}(q) = \left(\delta^{\mu\nu} - (1-\xi) \frac{q^\mu q^\nu}{q^2} \right) \frac{\pi(q^2)}{q^2} \, . \tag{15}$$

in the R_ξ-gauge, with π dimensionless. π can be written as a Taylor series in the coupling via loop expansion. Note that the tensor structure is completely fixed, by gauge invariance and Lorentz covariance – the only thing which we cannot put any more constraints on and hence have to compute is $\pi(q^2)$. This means that we can proceed e.g. by ignoring all terms in the correlator $\propto q^\mu q^\nu$, as gauge invariance dictates that at the end of the day they have to come with exactly the same coefficient as the terms $\propto \delta^{\mu\nu}$ (up to $(1-\xi)/q^2$). This means that we only look at

$$-\,\mu^\varepsilon g^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{4\,p^\mu p^\nu + 2\,q^\mu p^\nu + 2\,p^\mu q^\nu - 2\,\delta^{\mu\nu}((p+q)^2+m^2)}{((p+k)^2+m^2)(p^2+m^2)} \, . \tag{16}$$

We use the Feynman parametrisation trick

$$-\,\mu^\varepsilon g^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{4\,p^\mu p^\nu + 2\,q^\mu p^\nu + 2\,p^\mu q^\nu - 2\,\delta^{\mu\nu}((p+q)^2+m^2)}{(p^2 + 2xp\cdot q + xq^2 + m^2)^2} \, . \tag{17}$$

Now we change variables to $\ell = p + xq$ and then ignore all terms $\propto q^\mu q^\nu$ for the same reason as above and we will also drop all terms linear in ℓ as the integral will be symmetric under $\ell \mapsto -\ell$, so any antisymmetric integrand will integrate to zero:

$$-\,\mu^\varepsilon g^2 \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{4\,\ell^\mu \ell^\nu - 2\,\delta^{\mu\nu}(\ell^2 + (1-x)^2\,q^2 + m^2)}{(\ell^2 + x(1-x)\,q^2 + m^2)^2} \tag{18}$$

Now we note that the integral over the unit sphere of $\ell^\mu \ell^\nu$ is invariant under rotations and thus by the uniqueness of the quadratic Casimir must be proportional to ℓ^2. By contracting e.g. with $\delta_{\mu\nu}$ we find the correct constant of proportionality and hence that the correct substitution is $\ell^\mu \ell^\nu \to \ell^2 \delta^{\mu\nu}/d$. This gives

$$2\,\mu^\varepsilon g^2\,\delta^{\mu\nu} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{(1-2/d)\,\ell^2 + (1-x)^2\,q^2 + m^2}{(\ell^2 + m^2 + x(1-x)\,q^2)^2} \, . \tag{19}$$

Using our master formula for these kind of integrals we find

$$-\frac{g^2 \mu^\epsilon}{(4\pi)^{d/2}}\, 2\left(m^2 + q^2 x(1-x)\right)^{\frac{d}{2}-2} \left(\frac{(d-2)\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2}+1)\left(m^2 + q^2 x(1-x)\right)}{d\,\Gamma(\frac{d}{2})} + \Gamma\left(2 - \frac{d}{2}\right)\left(m^2 + q^2(1-x)^2\right) \right) \tag{20}$$

Using the expansion of the Gamma function around $0, -1$ respectively, we find that the divergent piece of this expression is simply

$$-\frac{g^2}{24\pi^2}\,\delta^{\mu\nu} q^2 \tag{21}$$

Hence we find that the $\overline{\text{MS}}$-scheme the counter term is

$$-\frac{g^2}{48\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log 4\pi\right) \int \frac{1}{4} F \wedge *F \, . \tag{22}$$

Now, in order to use this result to determine the charge renormalization we need to use a trick, which comes in two variations: Either, we could change variables from $A_\mu \to A_\mu/e$, so the gauge covariant derivative becomes $D = \mathrm{d} + iA$, while the kinetic term becomes $\frac{1}{4\,e^2}F^2$. Since the vertices are now independent of e, but otherwise unchanged, we may easily adapt our result to this case and find in the action the physical+counterterm coefficient of $\int F \wedge *F$

$$\frac{1}{4}\left[\frac{1}{g^2} - \frac{2}{\varepsilon}\frac{1}{48\pi^2}\right]\mu^{-\epsilon}\ . \tag{23}$$

The bare coupling has no μ dependence, therefore the derivative of the whole factor is zero. This means that we have:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{1}{g^2}\right) = -\frac{2}{g^3}\beta(g) = -\frac{2}{48\pi^2} \tag{24}$$

Multiplying both sides by $-g^3/2$ gives the required answer. The second route to this result works without the change of variables, but realizing that the covariant derivative remains unchanged under renormalization. This requirement links the charge renomalization, which we're after, to be the inverse of the photon field renormalization, which we've calculated. Even though these two arguments might appear different at first glance, after a little thought it should be clear that they're completely equivalent. You are encouraged to practice by calculating the β-function using the second argument! (Be careful to expand everything as power series in g *before* taking the $\epsilon \to 0$ limit.) The upshot of this discussion is that the strength of the electric field and the size of the coupling constant are the same.

For the physical interpretation of these results we use the same argument as in chapter 5.2.4 of Skinner's notes. We use two different theories, one defined at $\mu \geqslant m$ the mass of the scalar being scalar QED and the other defined at $\mu \leqslant m$ being pure U(1) Yang Mills, i.e. the theory of a free photon. The low energy theory is a free theory and therefore the coupling does not run, we define it as being theory reached by integrating out the scalar from scalar QED, which is valid at energies less than the scalar as we have defined.

Exercise 3

a) First the γ matrix identities

$$\begin{aligned}\gamma_5^2 &= \gamma_1\gamma_2\gamma_3\gamma_4\gamma_1\gamma_2\gamma_3\gamma_4 \\ &= (-)^2\gamma_1^2\gamma_2^2\gamma_3^2\gamma_4^2 = 1\,,\end{aligned} \tag{25}$$

and

$$\gamma_1\gamma_2\gamma_3\gamma_4\gamma_\mu = -\gamma_\mu\gamma_1\gamma_2\gamma_3\gamma_4\,, \tag{26}$$

as μ is one of 1,2,3 or 4.

As for the global symmetry, the only non-obvious bits are the kinetic term for the fermion and Yukawa coupling. Now we have that $\psi^\dagger \mapsto \psi^\dagger e^{i\pi\gamma_5/2}$, hence $\overline{\psi} \mapsto \overline{\psi}e^{-i\pi\gamma_5/2}$. Hence

$$\begin{aligned}\overline{\psi}(i\partial\!\!\!/)\psi &\mapsto \overline{\psi}e^{-i\pi\gamma_5/2}i\partial\!\!\!/ e^{-i\pi\gamma_5/2}\psi \\ &= \overline{\psi}i\partial\!\!\!/\psi \\ &= \overline{\psi}i\partial\!\!\!/\psi\,,\end{aligned} \tag{27}$$

because $\gamma_5^2 = 1$ means that $e^{i\theta\gamma_5} = \mathbb{1}\cos\theta + i\gamma_5\sin\theta$. Note that the theory is indeed invariant. The Yukawa coupling transforms as

$$\begin{aligned}\phi\overline{\psi}\gamma_5\psi &\mapsto -\phi\overline{\psi}e^{-i\pi\gamma_5/2}\gamma_5 e^{-i\pi\gamma_5/2}\psi \\ &= -\phi\overline{\psi}\gamma_5 e^{-i\pi\gamma_5}\psi \\ &= \phi\overline{\psi}\gamma_5\psi\,.\end{aligned} \tag{28}$$

Hence the action is invariant under this symmetry.

If the path integral measure is also invariant under this symmetry then there is no anomaly and hence this is a symmetry of the quantum theory[1]. One consequence of this symmetry is that correlators which are schematically of the form

$$\langle\ \phi^n\ (\bar{\psi}\gamma_g\psi)^m\ \rangle \tag{29}$$

are non-zero only if $m + n$ is even.

b) Now for a lot of algebra. We will do the actual renormalisation only for the two point function of the scalar. The rest of the calculations should hopefully be clear, all that is left to do is Feynman integral manipulations.

[1] This is in fact (for massless fermions) the non-anomalous residual symmetry to the anomalous chiral symmetry, general $U(1)$ rotations of the type described here *are* anomalous, but generically a finite group of the chiral symmetry is non-anomalous. The residual group depends on the fermion content of the theory. In this case it is $\mathbb{Z}_4 \leqslant U(1)$. See, for example, Bilal's lectures on anomalies

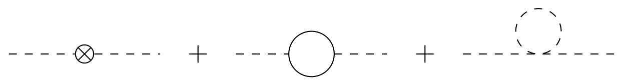
We put subscript 0's on the scalars, fermions, masses and couplings in the Lagrangian to signify that these are bare. We then renormalise the fields with $\phi_0 = Z^{1/2}\phi$, $\psi_0 = Z_2^{1/2}\psi$, $m_0^2 = m^2 + Z^{-1}\delta m^2$, $\mu_0 = \mu + Z_2^{-1}\delta\mu$, $g_0 = Z^{-1/2}Z_2^{-1}(g+\delta g)$ and $\lambda_0 = Z^{-2}(\lambda+\delta\lambda)$. Thus we can write the Lagrangian as

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + i\overline{\psi}\partial\!\!\!/\psi + g\phi\overline{\psi}\gamma_5\psi + \frac{\lambda}{4!}\phi^4 \\ &+ \frac{1}{2}(Z-1)((\partial\phi)^2 + m^2\phi^2) + \frac{1}{2}\delta m^2\phi^2 + i(Z_2-1)\overline{\psi}\partial\!\!\!/\psi + \delta g\phi\overline{\psi}\gamma_5\psi + \frac{\delta\lambda}{4!}\phi^4 \,.\end{aligned} \tag{30}$$

We define the 1-loop self-energy of the scalar as $\mathcal{M}^2(p^2)$ and the 1-loop self energy of the electron as $\Sigma(p)$. We demand that:

1. $\mathcal{M}^2(p^2)|_{p^2+m^2=0} = 0$ and $\frac{\partial\mathcal{M}^2(p^2)}{\partial p^2}|_{p^2+m^2=0} = 0$.
2. $\det\Sigma(p)|_{p^2+\mu^2=0} = 0$ and $\det\frac{\partial\Sigma(p^2)}{\partial p\!\!\!/}|_{p^2+\mu^2=0} = 0$.
3. $\left\langle\tilde{\phi}(0)\tilde{\overline{\psi}}(0)\tilde{\psi}(0)\right\rangle_{\text{1-loop}} = 0$.
4. $\left\langle\tilde{\phi}(0)\tilde{\phi}(0)\tilde{\phi}(0)\tilde{\phi}(0)\right\rangle_{\text{1-loop}} = 0$.

1. The self-energy of the scalar is given by the following Feynman diagrams



We calculate this to one-loop to be

$$\mathcal{M}^2(p^2) = \left(-(Z-1)(p^2+m^2) + \delta m^2 + g^2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\operatorname{tr}\left((-ik\!\!\!/)\gamma_5(-ip\!\!\!/ - ik\!\!\!/)\gamma_5\right)}{k^2(p+k)^2} + \frac{\lambda}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2+m^2}\right)\,. \tag{31}$$

We will evaluate these two momenta integrals. Let's do the one corresponding to the Yukawa coupling first. We use that, for $l = k + xp$:

$$\begin{aligned}\frac{1}{k^2(k+p)^2} &= \int_0^1 \mathrm{d}x\frac{1}{(k^2 + x(2k.p + p^2))^2} \\ &= \int_0^1 \mathrm{d}x\frac{1}{(l^2 + x(1-x)p^2)^2}\,.\end{aligned} \tag{32}$$

Then looking at the numerator, doing some gamma matrix algebra:

$$\operatorname{tr}((-ik\!\!\!/)\gamma_5(-ip\!\!\!/ - ik\!\!\!/)\gamma_5) = dl^2 + dx(1-x)p^2 + \text{terms linear in } l\,. \tag{33}$$

We use that terms linear in l will be zero in the integral. We end up with

$$\text{Yukawa integral} = \frac{g^2 d}{(2\pi)^d}\int_0^1 \mathrm{d}x\big(I(d,2,2,\alpha) + \alpha I(d,0,2,\alpha)\big)\,, \tag{34}$$

for $\alpha := x(1-x)p^2$.

The ϕ^4-integral is much more easy. It is given by

$$\phi^4\text{-integral} = \frac{\lambda}{2}I(d,0,1,m^2)\,. \tag{35}$$

This means that we have

$$\frac{\partial\mathcal{M}^2(p^2)}{\partial p^2} = -(Z-1) + \frac{g^2 d}{2(2\pi)^d}\int_0^1 \mathrm{d}x\left(\frac{x(1-x)}{2\alpha}I(d,2,2,\alpha) - \frac{x(1-x)}{2}I(d,0,2,\alpha) + x(1-x)I(d,0,2,\alpha)\right)\,. \tag{36}$$

Defining $\beta := \alpha|_{p^2=0} = 1 + x^2 - x$, this gives

$$Z = 1 - \frac{g^2 d}{2(2\pi)^d}\int_0^1 \mathrm{d}x\left(\frac{x(1-x)}{\beta}I(d,2,2,\beta) - \frac{x(1-x)}{2}I(d,0,2,\beta) + x(1-x)I(d,0,2,\beta)\right)\,. \tag{37}$$

Then we put $\mathcal{M}^2$ on-shell to work out δm^2, we get

$$\delta m^2 = -\frac{1}{Z}\left(\text{Yukawa-integral} + \phi^4\text{-integral}\right)\,. \tag{38}$$

In these equations we use $d = 4 - \varepsilon$, and that for $|z|$ around zero

$$\Gamma(z) \approx 1/z - \gamma\,. \tag{39}$$

2. The self-energy of the fermion is given by the following Feynman diagrams:
   We calculate this to one-loop to be

$$\Sigma(p) = -i(Z_2 - 1)\not{p} + g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{-i\not{k}}{k^2((p+k)^2 + m^2)} \,. \tag{40}$$

   We need to evaluate the integral

$$\begin{aligned}\int \mathrm{d}^d k \frac{-i\not{k}}{p^2((p+k)^2+m^2)} &= \int \mathrm{d}^d k \int_0^1 \mathrm{d}x \frac{-i\not{k}}{(k^2 + 2xp\cdot k + xp^2 + xm^2)^2} \\ &= \int \mathrm{d}^d \ell \int_0^1 \mathrm{d}x \frac{-i(\not{\ell} - \not{p})}{(\ell^2 + x(1-x)p^2 + xm^2)^2} \\ &= \int \mathrm{d}^d \ell \int_0^1 \mathrm{d}x \frac{i\not{p}}{(\ell^2 + x(1-x)p^2 + xm^2)^2} \\ &= \int_0^1 \mathrm{d}x i\not{p} I(d, 0, 2, x(1-x)p^2 + xm^2) \,.\end{aligned} \tag{41}$$

   So we have

$$\Sigma(p) = -i(Z_2 - 1)\not{p} + \frac{g^2}{(2\pi)^d} \int_0^1 \mathrm{d}x i\not{p} I(d, 0, 2, x(1-x)p^2 + xm^2) \,. \tag{42}$$

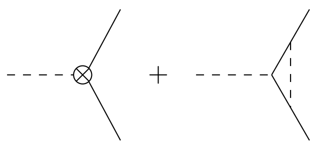
   The first condition $\det \Sigma|_{p^2=0}$ is guaranteed, which is good because we have one more equation that we have degrees of freedom (this wouldn't have been a problem if we could have had a fermion mass term and the reason why we can't is why this vanishes). Then we have the other condition

$$\det \frac{\partial \Sigma}{\partial \not{p}}\bigg|_{p^2=0} = -i(Z_2 - 1) + \frac{g^2}{(2\pi)^d} \int_0^1 \mathrm{d}x i I(d, 0, 2, x(1-x)p^2 + xm^2) = 0 \,, \tag{43}$$

   and so

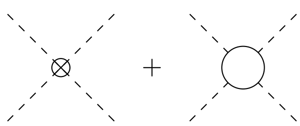
$$Z_2 = 1 + \frac{g^2}{(2\pi)^d} \int_0^1 \mathrm{d}x I(d, 0, 2, x(1-x)p^2 + xm^2) \,. \tag{44}$$

3. The correlator is given by the following Feynman diagrams: We calculate the counter term to one loop to get



$$\begin{aligned}\delta g &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{tr}\left((-i\not{k}\gamma_5)^2\right)}{(k^2+m^2)(k^2)^2} \\ &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2+m^2)(k^2)^2} \\ &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{k^2}{(k^2+xm^2)^3} \\ &= \frac{g^2}{(2\pi)^d} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, I(d, 2, 3, xm^2) \,.\end{aligned} \tag{45}$$

4. The correlator is given by the following Feynman diagrams



We calculate the counter-term to one-loop to get

$$\begin{aligned}
\delta\lambda &= g^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\operatorname{tr}\left((-i\not{k}\gamma_5)^4\right)}{k^8} \\
&= g^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{d}{k^4} \\
&= \frac{dg^4}{(2\pi)^d} I(d, -4, 0, 0) = 0 \,.
\end{aligned} \tag{46}$$

The fact that this is zero is a phenomenon known as Veltman's formula. Two references that talk about this are J.C. Collins, Renormalization, Cambridge University Press, Cambridge, 1984 and G. Leibbrandt, Rev. Mod. Phys. 74, 849 (1975).

We expect all of one loop amplitudes to be finite now because any loop contributing to the one-loop amplitude must be of the form of figure 1.

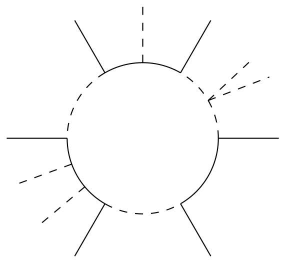
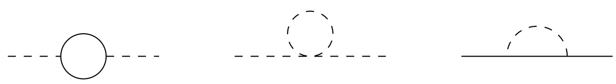


Figure 1: Generic one loop diagram. This one corresponds to a contribution to the correlator $\langle (\psi\overline{\psi})^3 \phi^5 \rangle$. Note that there are finitely many one loop diagrams for a given correlator.

Each external scalar propagator that we stick to the loop will change it to a value $\leqslant g/m^2$ the previous value. While each time we put a fermion bit within a scalar part of the loop or a scalar part within a fermion part then we add two external fermion propagators and change the value to $\leqslant \lambda/\mu^2$ of the previous value. Therefore we can never reach an infinite value through adding more external propagators. We consider the minimal diagrams, there are three:



We have made all of these finite via our renormalisation scheme.

Exercise 4

a) Scalar QED has a kinetic term

$$(D_\mu \varphi)^\dagger D^\mu \varphi \,, \tag{47}$$

with $D_\mu = \partial_\mu - eA_\mu$. This must remain invariant under the action of charge conjugation, $\mathcal{C}$. We have that

$$\begin{aligned}
\partial_\mu \varphi^\dagger + ieA_\mu \varphi^\dagger &\mapsto \partial_\mu \varphi + ie\mathcal{C}A_\mu \varphi \\
\partial^\mu \varphi - ieA^\mu \varphi &\mapsto \partial^\mu \varphi^\dagger - ie\mathcal{C}A^\mu \varphi^\dagger \,,
\end{aligned} \tag{48}$$

whence we can see that we must have that $\mathcal{C}A_\mu = -A_\mu$.

b) Suppose we have a theory with an action Sthat is invariant under $\mathcal{C}$, for ϕ representing all the other fields of the theory and with $\mathcal{C}A_\mu = -A_\mu$. While we can also see that $D\phi_{\text{real}} D\phi_{\text{imaginary}} DA \mapsto -D\phi_{\text{real}} D\phi_{\text{imaginary}} DA$ and might worry about this, the domain of integration for $\phi_{\text{imaginary}}$ flips and this fixed up the minus sign. Finally we need to worry about the boundary conditions and wonder if they are invariant under $\mathcal{C}$. The boundary conditions are the vacuum, where everything goes to zero at ∞ and this is clearly invariant.

Then we have that (suppressing the normalisation factor)

$$\langle \tilde{A}_{\mu_1}(k_1) \dots \tilde{A}_{\mu_n}(k_n) \rangle = \int DAD\phi e^{-S} \tilde{A}_{\mu_1}(k_1) \dots \tilde{A}_{\mu_n}(k_n) \,. \tag{49}$$

The trick is to then redefine all fields in the integral as $\phi \mapsto \mathcal{C}\phi$ and $A_\mu \mapsto \mathcal{C}A_\mu$ giving, by the invariance of $DAD\phi e^{-S}$,

$$\langle \tilde{A}_{\mu_1}(k_1) \dots \tilde{A}_{\mu_n}(k_n) \rangle = \int DAD\phi e^{-S} (-1)^n \tilde{A}_{\mu_1}(k_1) \dots \tilde{A}_{\mu_n}(k_n) = (-1)^n \langle \tilde{A}_{\mu_1}(k_1) \dots \tilde{A}_{\mu_n}(k_n) \rangle \,. \tag{50}$$

This means that the correlator has to be zero for n odd.

c) There is nothing in the argument that requires on-shell photons.