3 Evolution Equations on the Half-Line through a Unified Transform Method

3.1 The Heat Equation on the Half-Line

The new method involves three steps.

1. Given a domain, derive the Global Relation (GR), which is an equation coupling the function and its derivatives on the boundary of the domain.

For the domain
\[ \Omega = \{0 < x < \infty, \ t > 0\} \]
the GR is
\[ e^{\lambda^2 t} \hat{u}(\lambda, t) = \hat{u}_0(\lambda) - \hat{g}_1(\lambda^2, t) - i\lambda \hat{g}_0(\lambda^2, t), \ \Im \lambda \leq 0, \]
(3.2)
where
\[ \hat{u}(\lambda, t) = \int_0^\infty e^{-i\lambda x} u(x, t) dx, \ t > 0, \ \Im \lambda \leq 0, \]
(3.3)
\[ \hat{u}_0(\lambda) = \int_0^\infty e^{-i\lambda x} u_0(x) dx, \ t > 0, \ \Im \lambda \leq 0, \]
(3.4)
\[ \hat{g}_j(\lambda, t) = \int_0^t e^{\lambda \tau} g_j(\tau) d\tau, \ t > 0, \ j = 0, 1, \ \lambda \in \mathbb{C}, \]
(3.5)
with
\[ g_1(t) = u_x(0, t), \quad g_0(t) = u(0, t), \quad t > 0. \]
(3.6)
Regarding equations (3.3) and (3.4) we note that
\[ |e^{-i\lambda x}| = |e^{-i\lambda t x + \lambda t x}| = e^{\lambda t x}, \]
thus, this term is bounded as \( x \to \infty \), for \( \lambda t < 0 \).
The functions \( \tilde{g}_0 \) and \( \tilde{g}_1 \) are defined for all complex values of \( \lambda \), whereas \( \hat{u} \) and \( \hat{u}_0 \) are defined for \( \Im \lambda \leq 0 \), thus the global relation (3.2) is valid for \( \Im \lambda \leq 0 \).

Conceptually, the simplest way to derive the global relation is to use the half-Fourier transform, and to follow the same procedure used with the sine transform. Indeed, let the half-Fourier transform of \( u(x, t) \) be defined by considering a function \( f(x) \) which vanishes for \( x < 0 \). The Fourier transform pair equations (2.7) and (2.8) reduce to the equations

\[
\hat{f}(\lambda) = \int_{0}^{\infty} e^{-i\lambda x} f(x) dx, \quad \Im \lambda \leq 0,
\]

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \hat{f}(\lambda) d\lambda, \quad 0 < x < \infty.
\]

The first equation in comparison with equation (2.7) has the advantage that now \( \lambda \) can take any value in the lower half of the complex \( \lambda \)-plane. Indeed,

\[
e^{-i\lambda x} = e^{-i(\lambda_R + i\lambda_I)x} = e^{-i\lambda_R x} e^{i\lambda_I x},
\]

thus if \( \Im \lambda \leq 0 \), \( \exp\{-i\lambda x\} \) is bounded for all \( x > 0 \), including \( x \to \infty \). This means, that \( \hat{f}(\lambda) \) is an analytic function of the complex variable \( \lambda \) for \( \Im \lambda < 0 \).

Then,

\[
\hat{u}_t = \int_{0}^{\infty} e^{-i\lambda x} u_t dx = \int_{0}^{\infty} e^{-i\lambda x} u_{xx} dx
\]

\[
= u_x e^{-i\lambda x} \bigg|_{0}^{\infty} + i\lambda u e^{-i\lambda x} \bigg|_{0}^{\infty} - \lambda^2 \hat{u}.
\]

Thus,

\[
\hat{u}_t + \lambda^2 \hat{u} = -g_1(t) - i\lambda g_0(t).
\]

Hence,

\[
(\hat{u}e^{\lambda^2 t})_t = -e^{\lambda^2 t}(g_1(t) + i\lambda g_0(t)),
\]

or

\[
\hat{u}e^{\lambda^2 t} = \hat{u}_0 - \int_{0}^{t} e^{\lambda^2 \tau} [g_1(\tau) + i\lambda g_0(\tau)] d\tau,
\]

which is the GR.

2. Express the solution as an integral in the complex \( \lambda \)-plane involving \( \hat{u}_0(\lambda) \), as well as the \( t \)-transforms of all the relevant boundary values.

For the heat equation formulated on the half-line, we find

\[
u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda - \frac{1}{2\pi} \int_{0D^+} e^{i\lambda x - \lambda^2 t} \left[ \tilde{g}_1(\lambda^2, t) + i\lambda \tilde{g}_0(\lambda^2, t) \right] d\lambda,
\]

(3.7)
where the contour $\partial D^+$ is the boundary of the domain $D^+$ defined by

$$D^+ = \{ \Im \lambda \geq 0, \quad \Re \lambda^2 < 0 \}.$$  

(3.8)

Figure 3.2

Indeed, solving the global relation (3.2) for $\tilde{u}(\lambda, t)$ and then using the inverse Fourier transform formula, we find an expression similar to (3.7) but with the contour of integration along the real line instead of $\partial D^+$. In order to deform from the real line to $\partial D^+$ we use Cauchy’s theorem and Jordan’s Lemma. We first consider the function

$$e^{i\lambda x - \lambda^2 t} \tilde{g}_1(\lambda^2, t) = e^{i\lambda x} \int_0^t e^{-\lambda^2(t-\tau)} g_1(\tau) d\tau,$$

which is an analytic function of $\lambda$. This function involves the two exponentials

$$e^{i\lambda x} = e^{i\lambda x - \Re \lambda x}, \quad e^{-\lambda^2(t-\tau)} = e^{-\Re(\lambda^2)(t-\tau) - \Im(\lambda^2)(t-\tau)},$$

thus since $x \geq 0$ and $t - \tau \geq 0$, the above exponentials are bounded as $\lambda \to \infty$ if $\lambda$ satisfies $\Im \lambda \geq 0$ and $\Re \lambda^2 \geq 0$. Furthermore, integration by parts implies that the above function is of $O(1/\lambda^2)$ as $\lambda \to \infty$:

$$e^{-\lambda^2 t} \int_0^t e^{\lambda^2 \tau} g_1(\tau) d\tau \sim \frac{g_1(t)}{\lambda^2}, \quad \lambda \to \infty.$$

Thus, Cauchy’s theorem in the domain bounded by the real line and $\partial D^+$ implies that the integral of the above function can be deformed from $\Re$ to $\partial D^+$.

The situation is similar with the term $i\lambda \exp[i\lambda x - \lambda^2 t] \tilde{g}_0(\lambda^2, t)$, but now
because of the \(\lambda\) factor this function is of \(O(1/\lambda)\) as \(\lambda \to \infty\), thus we need to supplement Cauchy’s theorem with Jordan’s lemma.

3. For given boundary conditions, by employing the global relation as well as certain invariant transformations, eliminate from the integral representation obtained in step 2 the transforms of the unknown boundary values.

Consider for example the Dirichlet problem of the heat equation formulated on the half line, i.e., equation (2.18) supplemented with the initial and boundary conditions (2.19). In this case, the functions \(\hat{u}_0\) and \(\tilde{g}_0\) appearing in the global relation (3.2) are known but the functions \(\hat{u}\) and \(\tilde{g}_1\) are unknown.

The global relation is valid for \(\Im \lambda \leq 0\), whereas we need \(\tilde{g}_1\) for \(\Im \lambda \geq 0\). We note that the transformation \(\lambda \to -\lambda\) has two crucial properties: first, it maps the domain \(\Im \lambda \leq 0\) to the domain \(\Im \lambda \geq 0\), and also leaves \(\tilde{g}_0(\lambda^2, t)\) and \(\tilde{g}_1(\lambda^2, t)\) invariant. Using this transformation, the GR yields

\[
e^{\lambda^2 t} \hat{u}(-\lambda, t) = \hat{u}_0(-\lambda) - \tilde{g}_1(\lambda^2, t) + i\lambda \tilde{g}_0(\lambda^2, t), \quad \Im \lambda \geq 0.
\]  

(3.9)

Our strategy will be to use equation (3.7) to eliminate \(\tilde{g}_1\); in this procedure we ignore the fact that \(\hat{u}\) is unknown since it will turn out that its contribution to \(u(x, t)\) vanishes. Solving (3.9) for \(\tilde{g}_1(\lambda^2, t)\) we find

\[
\tilde{g}_1 = i\lambda \tilde{g}_0 + \hat{u}_0(-\lambda) - e^{\lambda^2 t} \hat{u}(-\lambda, t), \quad \Im \lambda \geq 0.
\]  

(3.10)

Replacing in equation (3.7) \(\tilde{g}_1\) with the RHS of (3.10) we find

\[
u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \left[2i\lambda \tilde{g}_0(\lambda^2, t) + \hat{u}_0(-\lambda)\right] d\lambda.
\]  

(3.11)

The term \(e^{\lambda^2 t} \hat{u}(-\lambda, t)\) gives rise to the term

\[
\frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x} \hat{u}(-\lambda, t) d\lambda, \quad 0 < x < \infty, \quad t > 0,
\]

which vanishes, since both \(e^{i\lambda x}\) and \(\hat{u}(-\lambda, t)\) are bounded and analytic in the upper half of the complex \(\lambda\) plane, and furthermore \(\hat{u}(-\lambda, t)\) is of \(O(1/\lambda)\) as \(\lambda \to \infty\):

\[
\hat{u}(-\lambda, t) = \int_{0}^{\infty} e^{i\lambda x} u(x, t) dx \sim -\frac{u(0, t)}{i\lambda}, \quad \lambda \to \infty.
\]
Thus, Cauchy’s theorem supplemented with Jordan Lemma in the domain $D^+$ imply the desired result.

**An alternative way to derive the GR**

The heat equation (2.1) can be written in the form

$$
(e^{-i\lambda x + \lambda^2 t} u)_t - [e^{-i\lambda x + \lambda^2 t}(u_x + i\lambda u)]_x = 0, \quad \lambda \in \mathbb{C}. 
$$

(3.12)

Indeed, simplifying equation (3.12) we find

$$
e^{-i\lambda x + \lambda^2 t} \{ u_t + \lambda^2 u - (u_{xx} + i\lambda u_x) + i\lambda(u_x + i\lambda u) \} = e^{-i\lambda x + \lambda^2 t}[u_t - u_{xx}],
$$

which vanishes if and only if the heat equation is valid. The simplest way to derive the above equation is to consider the adjoint of the given equation, namely, the equation obtained from the given PDE by replacing $\partial_t$ and $\partial_x$ with $\partial_x$ and $-\partial_x$. The adjoint of the heat equation (2.1) is the equation

$$
-\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}. 
$$

(3.13)

Multiplying the heat equation (2.1) by $v$, the adjoint equation (3.13) by $u$, and then subtracting the resulting equation we find

$$
\frac{\partial}{\partial t}(uv) = \frac{\partial}{\partial x}(vu_x - u_x v).
$$

(3.14)

A particular solution of equation (3.14) is the exponential

$$
v = e^{-i\lambda x + \lambda^2 t}, \quad \lambda \text{ constant}.
$$

Employing this solution in (3.14) we find equation (3.12). Applying Green’s theorem at (3.12) yields

$$
\int_{\partial\Omega} (e^{-i\lambda x + \lambda^2 t} u)dx + [e^{-i\lambda x + \lambda^2 t}(u_x + i\lambda u)]dt = 0,
$$

where $\partial\Omega$ is the boundary of the domain $\Omega$ depicted in Figure 3.1.

Hence, we obtain

$$
-\int_0^\infty e^{-i\lambda x}u(x,0)dx + \int_0^t e^{\lambda^2 \tau}[u_x(0, \tau) + i\lambda u(0, \tau)]d\tau
+ \int_0^\infty e^{-i\lambda x + \lambda^2 t}u(x, t)dx = 0,
$$

which yields (3.2).
Exercise 3.1 Consider $u_t + u_{xxx} = 0$.
Rewrite it as
\[
(e^{-i\lambda x-\lambda^3 t}u)_t - \left[e^{-i\lambda x-\lambda^3 t}(\lambda^2 u - i\lambda u_x - u_{xx})\right]_x = 0,
\]
and determine the domain $D^+ \cup D^-$ as the one given by Figure 3.3.

![Figure 3.3: The domain $D^+ \cup D^-$ for the Exercise 3.1.](image)

Exercise 3.2 Consider $u_t + u_x + u_{xxx} = 0$.
Rewrite it as
\[
(e^{-i\lambda x+i(\lambda-\lambda^3) t}u)_t - \left[e^{-i\lambda x+i(\lambda-\lambda^3) t}(\lambda^2 - 1)u - i\lambda u_x - u_{xx})\right]_x = 0,
\]
and determine the domain $D^+ \cup D^-$ as the one given by Figure 3.4.

![Figure 3.4: The domain $D^+ \cup D^-$ for the Exercise 3.2.](image)
**Numerical Evaluations**

For the simple cases when the transforms of the given data can be computed explicitly, the numerical evaluation of the solution obtained by the new method reduces to the computation of a single integral in the complex $\lambda$-plane. Using simple contour deformations, it is possible to obtain an integrand which decays exponentially as $\lambda \to \infty$.

**Exercise 3.3** Consider the heat equation on the half line with

$$u(x, 0) = e^{-a^2x}, \quad u(0, t) = \cos(bt), \quad a, b \text{ real constants.}$$

Find the solution $u(x, t)$ and discuss the behaviour of the integrand.

**Additional Remarks**

(a) **Derivation of the sine transform.**

In deriving (3.7), the real line was deformed to $\partial D^+$. This deformation is always possible before using the global relation. However, after using the global relation we introduce $\hat{u}_0$ and then it is not always possible to return to the real axis. Actually, the cases where there does exist a usual transform, are precisely the cases where this “return” is possible.

In the particular case of (3.7), we note that $\hat{u}_0(-\lambda)$ is bounded and analytic in the upper half of the complex $\lambda$ plane, thus it is possible to return to the real axis:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} [\hat{u}_0(\lambda) - \hat{u}_0(-\lambda)] d\lambda - \frac{i}{\pi} \int_{-\infty}^{\infty} \lambda e^{i\lambda x - \lambda^2 t} \tilde{g}_0(\lambda^2, t) d\lambda.$$

Splitting the integral along $\mathbb{R}$ to an integral from $-\infty$ to 0 plus an integral from 0 to $\infty$, and letting $\lambda \to -\lambda$ in the former integral we obtain the representation obtained in section 2 via the sine transform. An easier way to obtain this representation is to recall that the global relation together with the equation obtained from the global relation after replacing $\lambda$ with $-\lambda$ are the following equations:

$$e^{\lambda^2 t} \hat{u}(\lambda, t) = \hat{u}_0(\lambda) - \tilde{g}_1 - i\lambda \tilde{g}_0, \quad \Re \lambda \leq 0,$$

$$e^{-\lambda^2 t} \hat{u}(-\lambda, t) = \hat{u}_0(-\lambda) - \tilde{g}_1 + i\lambda \tilde{g}_0, \quad \Re \lambda \geq 0. \quad (3.15)$$

If $\lambda$ is real, then both these equations are valid. Hence if $g_0$ is given, we subtract equations (3.15) and we obtain the equation for the sine transform.
of \( u(x,t) \). Similarly, if \( u_x(0,t) \) is given, we add equations (3.15) and we obtain
\[
e^{\lambda^2 t} \hat{u}_c(\lambda, t) = \hat{u}_0(\lambda) - \hat{g}_1(\lambda^2 t), \quad \lambda \in \mathbb{R},
\]
where \( \hat{u}_c \) and \( \hat{u}_0 \) denote the cosine transform of \( u(x,t) \) and \( u_0(x) \) respectively, namely:
\[
\hat{u}_c(\lambda, t) = \int_0^\infty \cos(\lambda x) u(x,t) dx, \quad \hat{u}_0(\lambda) = \int_0^\infty \cos(\lambda x) u_0(x) dx.
\]

(b) Causality

Suppose that the heat equation is valid for \( 0 < t < T \). Then we make the following substitutions
\[
\tilde{g}_0(\lambda, t) \rightarrow \tilde{g}_0(\lambda) = \tilde{g}_0(\lambda, T), \quad \tilde{g}_1(\lambda, t) \rightarrow \tilde{g}_1(\lambda) = \tilde{g}_1(\lambda, T). \quad (3.16)
\]
Then, the integral representation of the solution differs from (3.7) by a term which vanishes by using Cauchy’s theorem supplemented with Jordan’s lemma.

Indeed, the integral representation of the solution reads
\[
u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda
- \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \left[ \tilde{g}_1(\lambda^2) + i\lambda \tilde{g}_0(\lambda^2) \right] d\lambda. \quad (3.17)
\]
Hence, the RHS of equation (3.7) and the RHS of equation (3.17) differ by the term
\[
\frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x} \left[ \int_t^T e^{\lambda^2 (\tau-t)} u_x(0,\tau) d\tau + i\lambda \int_t^T e^{\lambda^2 (\tau-t)} u(0,\tau) d\tau \right] d\lambda,
\]
and Cauchy’s theorem supplemented with Jordan’s lemma imply that the above term vanishes.

Similarly, equation (3.11) is equivalent to the equation
\[
u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda
- \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \left[ 2i\lambda \tilde{g}_0(\lambda^2) + \hat{u}_0(-\lambda) \right] d\lambda. \quad (3.18)
\]
The advantage of (3.18) is that the only \((x,t)\) dependence of the RHS of this equation is of the form \( e^{i\lambda x - \lambda^2 t} \), thus it immediately follows that the
function \( u \) defined in (3.18) satisfies the heat equation. On the other hand, (3.7) is consistent with causality, since the function \( u(x, t) \) cannot depend on the values of \( g_0(\tau) \) for \( \tau > t \).

(**c**)-Verification

For the above formulation, it is immediate that \( u(x, t) \) satisfies the heat equation: the only \((x, t)\) dependence is of the form \( e^{i\lambda x - \lambda^2 t} \).

The verification of the initial \((t = 0)\) and the boundary \((x = 0)\) conditions is given by contour deformation, using Cauchy’s theorem supplemented with Jordan’s lemma.

Indeed, evaluating (3.11) at \( t = 0 \) we find

\[
u(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \hat{u}_0(\lambda) d\lambda - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x} \hat{u}_0(-\lambda) d\lambda, \quad x > 0.
\]

Jordan’s lemma implies that the second integral in the above expression vanishes and hence by recalling the definition of \( \hat{u}_0(\lambda) \) and employing the inverse Fourier transform formula we find \( u(x, 0) = u_0(x) \).

Evaluating (3.18) at \( x = 0 \) we find

\[
u(0, t) = \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} e^{-\lambda^2 t} \hat{u}_0(\lambda) d\lambda - \int_{\partial D^+} e^{-\lambda^2 t} \hat{u}_0(-\lambda) d\lambda \right]
- \frac{1}{2\pi} \int_{\partial D^+} 2\lambda e^{-\lambda^2 t} \hat{g}_0(\lambda^2) d\lambda.
\]  

By deforming the second integral to the real axis and then replacing \( \lambda \) with \(-\lambda\) we find that the first two terms in the RHS of (3.19) cancel. Furthermore, letting \( i\lambda^2 = l \) in the last integral in the RHS of (3.19) we find

\[
u(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itl} \left( \int_0^T e^{-it\tau} g_0(\tau) d\tau \right) dl = g_0(t).
\]  

**3.1.1 Inhomogeneous PDEs**

The representations for the solutions of the homogeneous PDEs obtained in the above section can be immediately extended to the case of inhomogeneous PDEs: the solution of an inhomogeneous PDE with forcing \( f(x, t) \) is obtained from the representation of the corresponding homogeneous PDE by replacing \( \hat{u}_0(\lambda) \) with

\[
\hat{v}(\lambda, t) = \hat{u}_0(\lambda) + \int_0^\infty dx e^{-i\lambda x} \int_0^t d\tau e^{-\omega(\lambda)\tau} f(x, \tau),
\]  

(3.20)
where \( \exp(i\lambda x + \omega(\lambda)t) \) is a solution of the homogeneous PDE.

For example, consider the inhomogeneous heat equation

\[
    u_t - u_{xx} = f(x, t).
\]

The first step of the unified transform yields

\[
    e^{2t} \hat{u}(\lambda, t) = \hat{u}_0(\lambda) + \int_0^\infty dx e^{-i\lambda x} \int_0^t d\tau e^{i\lambda x} f(x, \tau) - \tilde{g}_1(\lambda^2, t) - i\lambda \tilde{g}_0(\lambda^2, t), \quad \text{Im}\lambda \leq 0.
\]

Then, since steps 2 and 3 are identical with those for the heat equation, we find

\[
    u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - \lambda^2 t} \hat{v}(\lambda, t) d\lambda - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \left[ \tilde{v}(-\lambda, t) + 2i\lambda \tilde{g}_0(\lambda^2) \right] d\lambda,
\]

where

\[
    \hat{v}(\lambda, t) = \int_0^\infty dx e^{-i\lambda x} \int_0^t d\tau e^{i\lambda x} f(x, \tau).
\]

### 3.1.2 Robin Boundary conditions

Steps 1 and 2 of the unified transform yield the global relation and the integral representation. Then, employing transformations \( \nu(\lambda) \) which leave \( \omega(\lambda) \) invariant, it is straightforward to solve a variety of boundary value problems.

For concreteness we consider the case of the so-called Robin boundary condition, namely we consider the boundary condition

\[
    u_x(0, t) - \gamma u(0, t) = g_R(t), \quad t > 0,
\]

where \( g_R \) is a given function with sufficient smoothness and \( \gamma \) is a real constant.

The \( t \)-transform of (3.25) yields

\[
    \tilde{g}_1(\lambda, t) = \gamma \tilde{g}_0(\lambda, t) + \tilde{g}_R(\lambda, t); \quad \tilde{g}_R(\lambda, t) = \int_0^t e^{i\lambda \tau} g_R(\tau) d\tau, \quad t > 0.
\]

The solution representation formula (3.7) involves the term

\[
    \tilde{g}_1(\lambda^2, t) + i\lambda \tilde{g}_0(\lambda^2, t) = (i\lambda + \gamma) \tilde{g}_0(\lambda^2, t) + \tilde{g}_R(\lambda^2, t).
\]
Using (3.26) in the global relation (3.9) we find
\[ e^{\lambda_2 t} \hat{u}(-\lambda, t) = \hat{u}_0(-\lambda) + (i\lambda - \gamma) \tilde{g}_0(\lambda^2, t) - \tilde{g}_R(\lambda^2, t). \] (3.28)

Solving this equation for \( \tilde{g}_0 \) and then substituting the resulting expression in the rhs of (3.27) we find that (3.7) becomes
\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda \]
\[ - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \frac{e^{i\lambda x - \lambda^2 t}}{\lambda + i\gamma} \left\{ 2\lambda \tilde{g}_R(\lambda^2, t) + (\lambda - i\gamma) \left[ e^{\lambda_2 t} \hat{u}(-\lambda, t) - \hat{u}_0(-\lambda) \right] \right\} d\lambda. \] (3.29)

The factor \( \lambda + i\gamma \) vanishes at the point \( \lambda_0 = -i\gamma \). We distinguish two cases:

(i) \( \gamma > 0 \)
\( \lambda_0 \) is outside \( D^+ \), thus employing Cauchy’s theorem and Jordan’s lemma in \( D^+ \) we find that the contribution of \( \hat{u}(-\lambda, t) \) vanishes, and hence (3.29) becomes
\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda \]
\[ - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \frac{e^{i\lambda x - \lambda^2 t}}{\lambda + i\gamma} \left\{ 2\lambda \tilde{g}_R(\lambda^2, t) + (\lambda - i\gamma) \hat{u}_0(-\lambda) \right\} d\lambda, \quad \gamma > 0. \] (3.30)

(ii) \( \gamma < 0 \)
\( \lambda_0 \) is removable singularity in \( D^+ \). Indeed, the curly bracket in (3.29) vanishes at \( \lambda = \lambda_0 \):
\[ \left\{ 2\lambda \tilde{g}_R(\lambda^2, t) + (\lambda - i\gamma) \left[ e^{\lambda_2 t} \hat{u}(-\lambda, t) - \hat{u}_0(-\lambda) \right] \right\}_{\lambda = \lambda_0} \]
\[ = -2i\gamma \left\{ \tilde{g}_R(-\gamma^2, t) + \left[ e^{-\gamma_2 t} \hat{u}(i\gamma, t) - \hat{u}_0(i\gamma) \right] \right\}, \]
and the above curly bracket vanishes, as can be seen by evaluating (3.28) at
\[ \lambda = -i\gamma: \]
\[ e^{-\gamma_2 t} \hat{u}(i\gamma, t) = \hat{u}_0(i\gamma) - \tilde{g}_R(-\gamma^2, t). \] (3.31)

Using equation (3.31), and employing the residue theorem, we can compute the contribution of \( \hat{u}(-\lambda, t) \):
\[ - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} e^{i\lambda x - \lambda^2 t} \hat{u}(-\lambda, t) d\lambda = \frac{2\pi i e^{i\pi}(2\gamma) \hat{u}(i\gamma, t)}{2\pi e^{i\pi}(2\gamma) \hat{u}(i\gamma, t)} \]
\[ = -2\gamma e^{\gamma x + \gamma^2 t} [\hat{u}_0(i\gamma) - \tilde{g}_R(-\gamma^2, t)]. \]
Thus, (3.29) becomes

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda - 2\gamma e^{\gamma x + \gamma^2 t} \hat{u}_0(i\gamma) - \hat{g}_R(-\gamma^2, t) \]
\[ - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \left[ 2\lambda \hat{g}_R(\lambda^2, t) - (\lambda - i\gamma) \hat{u}_0(-\lambda) \right] d\lambda, \quad \gamma < 0. \]

(3.32)

**Derivation of ‘classical’ transform**

The heat equation with Robin boundary conditions can be solved via a traditional transform pair, but now the derivation of this pair cannot be guessed. Particularly for the case of \( \gamma < 0 \), where the relevant transform pair involves an additional term (arising from the so-called ‘point spectrum’). Taking into consideration that the representations obtained via the unified transform have several advantages in comparison with the representations obtained via traditional transform pairs, it follows that the use of traditional transforms is obsolete. Furthermore, if, for whatever reason, one requires a traditional transform pair, the unified transform provides a much easier way for constructing this pair than the usual Green’s function approach. This was already discussed earlier in this section for the simple cases when either \( u_x(0, t) \) or \( u(0, t) \) are given. In what follows, we discuss the case of the Robin boundary condition (3.25) with \( \gamma < 0 \).

Evaluating (3.32) at \( t = 0 \) and deforming \( \partial D^+ \) back to the real axis, we find

\[ u_0(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x} \hat{u}_0(\lambda) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x} \frac{\lambda - i\gamma}{\lambda + i\gamma} \hat{u}_0(-\lambda) d\lambda - 2\gamma e^{\gamma x} \hat{u}_0(i\gamma). \]

Using the definition of \( \hat{u}_0(\lambda) \), the above equation becomes

\[ u_0(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x} \left[ \int_{0}^{\infty} \left( e^{-i\lambda \xi} + \frac{\lambda - i\gamma}{\lambda + i\gamma} e^{i\lambda \xi} \right) u_0(\xi) d\xi \right] - 2\gamma e^{\gamma x} \int_{0}^{\infty} e^{\gamma \xi} \hat{u}_0(\xi) d\xi. \]

Thus, renaming \( u_0(x) \) as \( f(x) \), we obtain the following transform pair:

\[ \hat{f}_R(\lambda) = \int_{0}^{\infty} \left( e^{-i\lambda x} + \frac{\lambda - i\gamma}{\lambda + i\gamma} e^{i\lambda x} \right) f(x) dx \]
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x} \hat{f}_R(\lambda) d\lambda - 2\gamma e^{\gamma \xi} \int_{0}^{\infty} e^{\gamma \xi} f(\xi) d\xi. \]

(3.33)
3.2 Third order PDEs on the half line

For simplicity we consider the PDE
\[ u_t + u_{xxx} = 0. \]  
(3.35)

Similar consideration apply to the Stokes equation, see Exercise 3.2.

1. Employing the Fourier transform, in analogy with equations (3.2) we find
\[ e^{i \lambda^3 t} \hat{u}(\lambda, t) = \hat{u}_0(\lambda) + \tilde{g}_2(-i \lambda^3, t) + i \lambda \tilde{g}_1(-i \lambda^3, t) - \lambda^2 \tilde{g}_0(-i \lambda^3, t), \quad \Im \lambda \leq 0, \]  
(3.36)

where \( \hat{u}_0(\lambda), \tilde{g}_0, \tilde{g}_1 \) are defined in (3.4), (3.5) and
\[ \tilde{g}_2(\lambda, t) = \int_0^t e^{\lambda^2 \tau} u_{xx}(0, \tau) d\tau. \]  
(3.37)

2. The expression \( e^{i \lambda x + i \lambda^3 t} \) is a particular solution of (3.35). Also, if \( \lambda = |\lambda| e^{i \theta}, \) then
\[ \Re(i \lambda^3) = \Re(i |\lambda|^3 e^{3i \theta}) = -|\lambda|^3 \sin 3 \theta. \]

Thus,
\[ D \doteq \left\{ \lambda = |\lambda| e^{i \theta}, \quad \frac{\pi}{3} < \theta < \frac{2\pi}{3}, \quad \pi < \theta < \frac{4\pi}{3}, \quad \frac{5\pi}{3} < \theta < 2\pi \right\}, \]  
(3.38)

see figure 3.5. Thus,
\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \lambda x + i \lambda^3 t} \hat{u}_0(\lambda) d\lambda \]
\[ + \frac{1}{2\pi} \int_{\partial D^+} e^{i \lambda x + i \lambda^3 t} \left[ \tilde{g}_2(-i \lambda^3, t) + i \lambda \tilde{g}_1(-i \lambda^3, t) - \lambda^2 \tilde{g}_0(-i \lambda^3, t) \right] d\lambda. \]
(3.39)

3. Let \( v(\lambda) \) be the transformations which leave \( i \lambda^3 \) invariant. Then,
\[ v_1(\lambda) = e^{\frac{2i\pi}{3} \lambda}, \quad v_2(\lambda) = e^{\frac{4i\pi}{3} \lambda}. \]

Thus, if \( \lambda \) is in \( D^+ \) then \( v_1(\lambda) \) is in \( D_1^+ \) and \( v_2(\lambda) \) is in \( D_2^+ \). Hence, if we replace in the global relation (3.36) \( \lambda \) by \( v_1(\lambda) \) and by \( v_2(\lambda) \) we obtain two equations which are valid for \( \lambda \) in \( D^+ \):
\[ e^{-i \lambda^3 t} u\left( e^{\frac{2i\pi}{3} \lambda}, t \right) = \hat{u}_0\left( e^{\frac{2i\pi}{3} \lambda} \right) + \tilde{g}_2 + ie^{\frac{2i\pi}{3} \lambda} \tilde{g}_1 - e^{\frac{4i\pi}{3} \lambda^2} \tilde{g}_0, \]  
(3.40)
\[ e^{-i \lambda^3 t} u\left( e^{\frac{4i\pi}{3} \lambda}, t \right) = \hat{u}_0\left( e^{\frac{4i\pi}{3} \lambda} \right) + \tilde{g}_2 + ie^{\frac{4i\pi}{3} \lambda} \tilde{g}_1 - e^{\frac{2i\pi}{3} \lambda^2} \tilde{g}_0, \quad \lambda \in D^+, \]  
(3.41)
Figure 3.5: The domain $D$ and the contour $\partial D^+$.

where for convenience of notation we have suppressed the dependance of $\{\tilde{g}_j\}_{j=0,1,2}$.

As with the earlier examples we expect that the contribution of the terms $\tilde{u} \left( e^{\frac{i\pi}{3}} \lambda, t \right)$ and $\tilde{u} \left( e^{\frac{2i\pi}{3}} \lambda, t \right)$ vanishes, thus equations (3.40)-(3.41) are 2 algebraic equations coupling $\tilde{g}_0$, $\tilde{g}_1$ and $\tilde{g}_2$. Thus, for a well posed problem, one needs to specify one boundary condition at $x = 0$.

For example, consider the initial and boundary conditions specified in (2.19). In this case, equations (3.40)-(3.41) can be solved for the unknown functions $\tilde{g}_1$ and $\tilde{g}_2$, and then the resulting expressions can be substituted to the bracket of (3.39). Alternatively, we can compute directly this bracket as follows: we supplement equations (3.40)-(3.41) with the equation

$$\tilde{g}(\lambda, t) = \tilde{g}_2 + i\lambda \tilde{g}_1 - \lambda^2 \tilde{g}_0.$$  \hfill (3.42)

We multiply equation (3.40) by $e^{\frac{2i\pi}{3}}$, equation (3.41) by $e^{\frac{4i\pi}{3}}$ and add the resulting equations to (3.42):

$$\tilde{g}(\lambda, t) = -e^{-i\lambda t} \left[ e^{\frac{2i\pi}{3}} \tilde{u} \left( e^{\frac{2i\pi}{3}} \lambda, t \right) + e^{\frac{4i\pi}{3}} \tilde{u} \left( e^{\frac{4i\pi}{3}} \lambda, t \right) \right]$$

$$+ e^{\frac{2i\pi}{3}} \tilde{u}_0 \left( e^{\frac{2i\pi}{3}} \lambda \right) + e^{\frac{4i\pi}{3}} \tilde{u}_0 \left( e^{\frac{4i\pi}{3}} \lambda \right) - 3\lambda^2 \tilde{g}_0,$$  \hfill (3.43)

where we have used the identity

$$1 + e^{\frac{2i\pi}{3}} + e^{\frac{4i\pi}{3}} = 0.$$
Substituting the expression of $\tilde{g}(\lambda, t)$ into (3.39) and noting that Cauchy’s theorem in $D^+$ implies that the contributions of the terms $\hat{u}(e^{\frac{2\pi}{3} \lambda}, t)$ and $\hat{u}(e^{\frac{4\pi}{3} \lambda}, t)$ vanish, we find

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x + i\lambda^3 t} \hat{u}_0(\lambda) d\lambda$$

$$+ \frac{1}{2\pi} \int_{\partial D^+} \left\{ e^{i\lambda x + i\lambda^3 t} \left[ e^{\frac{2\pi}{3} \lambda} \hat{u}_0 \left( e^{\frac{2\pi}{3} \lambda} \right) + e^{\frac{4\pi}{3} \lambda} \hat{u}_0 \left( e^{\frac{4\pi}{3} \lambda} \right) \right] - 3\lambda^2 \tilde{g}_0(-i\lambda^3, t) \right\} d\lambda.$$

(3.44)