Small acoustically forced symmetric bodies in viscous fluids

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The total force exerted on a small rigid body by an acoustic field in a viscous fluid is addressed analytically in the limit where the typical size of the particle is smaller than both the viscous diffusion length scale and the acoustic wavelength. In this low-frequency limit, such a force can be calculated provided the effect of the acoustic steady streaming is negligible. Using the Eulerian linear expansion of Lagrangian hydrodynamic quantities (velocity and pressure), the force on a small solid sphere free to move in an acoustic field is first calculated in the case of progressive and standing waves, and it is compared to past results. The proposed method is then extended to the case of more complex shapes with three planes of symmetry. For a symmetric body oriented with one of its axis along the wave direction, the acoustic force exerted by a progressive wave is affected by the particle shape at leading order. In contrast, for a standing wave (with the same orientation), the force experienced by the particle at leading order is the same as the one experienced by a sphere of same volume and density. © *2016 Acoustical Society of America*.
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I. INTRODUCTION

Over the last two decades, there has been a significant renewed interest in the study of small particles migrating in acoustic fields because of the wide array of possible applications at the intersection of medicine, micro-engineering, and microfluidics.[1] Future drug delivery techniques could for instance involve acoustically propelled vesicles, used as micro-carriers.[2,3] For both medical or engineering purposes, the use of micro-particles requires the ability to propel or organize them at a microscopic scale. Both purposes can be achieved by the mean of acoustic fields, using either progressive or standing waves.[4,5]

The force experienced by a rigid spherical particle in an acoustic field is an old problem, first addressed by King[6] in the case of an inviscid fluid. King's theory has shown good agreement with experiments,[7] but these experiments were conducted in conditions where viscosity in the fluid and elasticity of the solid where both negligible. The problem of the force experienced by an elastic particle in an acoustic field has been extensively investigated (theorically and experimentally) by Hasegawa and Yosioka.[8] The effects of viscosity have been first considered by Westervelt.[9,10] Westervelt found that, in the case of a viscous fluid and for a plane progressive wave, the force was several orders of magnitude larger than predicted by King. However, as pointed out by Doinikov,[11] Westervelt's conclusions, although they are correct, result from two wrong arguments. First, he considered the case of a fixed particle whereas the motion of the sphere, which is free to oscillate under the effect of the incident field, has a decisive contribution to the final value of the force. Second, Westervelt dealt with the case of large viscous diffusion lengths (relative to the radius of the sphere) while King's approach is valid for inviscid fluids, i.e., in the limit of asymptotically small viscous lengths. Westervelt's results and King's theory concern thus different physical regimes.

A number of other studies tackled the problem of the force experienced on a free sphere placed in a viscous fluid (kinematic viscosity ν) under the effect of an acoustic radiation (frequency $f = \omega/2\pi$). In most of these works, the effect of the acoustic streaming around the sphere is neglected.[11] For instance, Guz[12,13] neglected the acoustic streaming just in the domain $a/\delta \gg 1$, where a is the sphere radius and $\delta = (\nu/\omega)^{1/2}$ is the viscous diffusion length—a regime in which its contribution steps in at leading order. Danilov[14,15] attempted to fill the gap by taking into account the acoustic streaming in the case of a fixed sphere and completed his original work by deriving an expression for the force in the case of a free solid sphere. Although the method of derivation is quite involved, some of the conclusions agree qualitatively with more recent studies on the same topic. In particular, a change in the sign of the total force when switching from small to large viscous lengths was predicted.[14,15] Doinikov[11,16] then addressed the problem of rigid and deformable spheres free to move in a viscous fluid. The method used in each paper is an extension of King's method, leading to results valid in all situations where the amplitude of oscillation of the fluid particles is smaller than the sphere radius. Thermal effects were later taken into account.[17] The case of a plane progressive wave is treated as an example, and thermal effects are shown to introduce an additional

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term in the expression for the total force. This term turns out to be zero in the case when either sphere is rigid or when the heat capacity ratio of the fluid equals 1 (the speed of sound is a real constant). Doinikov's calculation has the advantage of being very general, but it is difficult to extend it to the case of more complex geometries.

More recently, Settnes and Bruus[18] proposed a method to derive an analytical expression for the force experienced by a solid sphere in a viscous fluid valid for any value of $\epsilon = a/\delta$. Indeed, Doinikov[11] only provides convenient expressions for the total force that are asymptotically valid in the limiting cases $\epsilon \gg 1$ and $\epsilon \ll 1$. Settnes and Bruus fill that gap and provide a simple expression of the total force that should be valid without restriction for any value of the ratio a/δ. Their method consists in solving a problem of far field (irrotational) scattering and then finding explicitly the unknown coefficients of the irrotational solution by matching far field solution to the viscous near field solution (i.e., the solution valid for $r < \delta$). The work is done in a general framework, without specifying the shape of the incident radiation (progressive or stationary). The general solution is then applied to the cases of stationary and progressive incident waves. The solutions obtained by Settnes and Bruus in the cases of progressive and standing incident waves are compared to the expressions derived in the present article in Sec. III C [Eqs. (50) and (54)].

In this paper, we propose a method to calculate the total force experienced by a non-spherical particle of typical size a (in this case, a can be seen as the radius of the sphere of equivalent volume) in an incident acoustic field (progressive or standing wave, see Fig. 1), for a particular class of symmetric shapes in the case of large viscous length scales, i.e., $\delta/a \gg 1$. Limiting the study to the large-δ limit enables us to evaluate the effect of a change in shape for various object such as ellipsoids, cylinders, disks, and, more generally, any shape having three planes of symmetry. We also stay in the so-called Rayleigh limit (or long-wave limit) where the acoustic wavelength is larger than the viscous wavelength and the particle radius, so that, in this hydrodynamic limit, the effects of fluid compressibility are not taken into account at leading order, but come in as quadratic corrections.

The paper is organized as follow. In Sec. II we calculate the first-order acoustic field, i.e., the solution to the inviscid nonlinear wave equation at first order in Mach number. The results, which can be found in several articles,[9,19,20] are written in a dimensionless form and expanded in Mach number. By doing this, we are able to identify the terms which have to be retained for the derivation of the total force. The calculation of the total force experienced by a small solid sphere is then presented in Sec. III. The first part is devoted to the calculation of the leading-order particle dynamics, required for the evaluation of the Lagrangian advective terms in the total force. We then calculate the force in the case of plane progressive and standing waves. The case of symmetric objects is dealt with in Sec. IV. As in Sec. III, the first part is devoted to the dynamics of the particle at leading order, and we then derive the explicit form of the acoustic pressure. Order-of-magnitude estimates and practical situations in which the limit $\delta/a \gg 1$ is relevant are discussed in Sec. V. Solutions to the first-order nonlinear wave equation are given in Appendix A while acoustic streaming is addressed in Appendix B.

Before proceeding, let us focus on the non-dimensionalization. In the sections where we are primarily interested in the acoustic field (e.g., in Sec. II), the acoustic wavelength will be chosen as typical length scale, a choice which of course impacts the scaling of stresses and forces. In the section where we focus on the physical processes occurring at the particle scale, which is the case when we are interested in the dynamics of the particle at leading order (Secs. III B and IV A), the particle radius (or its typical size) will be chosen as the non-dimensionalizing length scale, and the typical force will then scale accordingly. In the whole article, we use tildes to refer to dimensional field, force, and displacement variables. Corresponding dimensionless quantities are noted without a tilde while constants are always noted without a tilde.

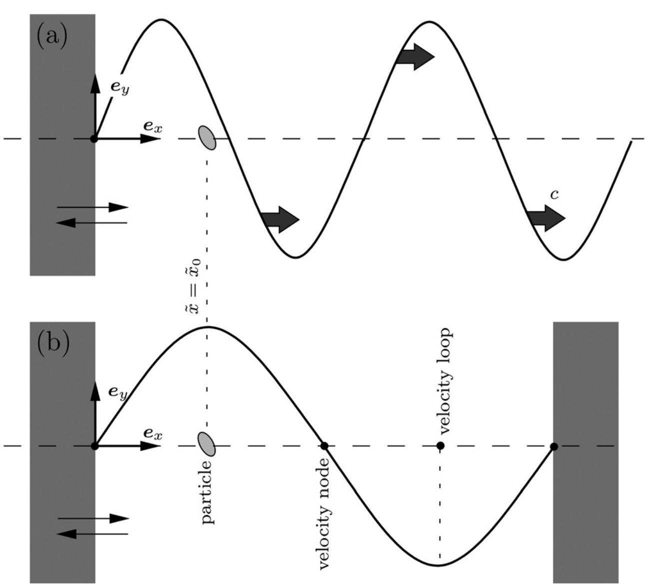

FIG. 1. Sketch of a particle acted on by (a) a plane progressive wave, (b) a standing wave. The position of the particle is referred to as $\tilde{x}_0$.

II. FIRST-ORDER MEAN ACOUSTIC FIELD

In this section, we first ignore the coupling with the solid particle and consider a one-dimensional harmonic acoustic field whose direction of propagation is aligned with the x-axis. The wavenumber $\boldsymbol{k}_0 = k_0\boldsymbol{e}_x$ satisfies the linear dispersion relation $k_0 = \omega/c_0$, where ω denotes the pulsation of the source and c_0 the speed of sound in the medium. The imaginary part of $\boldsymbol{k}_0$ is neglected, which means that the influence of viscosities (shear and bulk) on the incident acoustic field are not taken into account. For the sake of simplicity, the speed of sound and the viscosity are both assumed to be independent from the fluid density. For an ideal gas, the latter assumptions are equivalent to the condition $\gamma = 1$, where γ denotes the ratio of the specific heats of the medium. This assumptions means that the only nonlinearity is the one coming from the compressibility of the fluid (and thus no adiabatic nonlinearity). We finally assume that the velocity amplitude of the sound wave, say $\xi_0\omega$, is small compared to the speed of sound in the fluid. This last assumption enables us to consider the Mach number, $M = \xi_0\omega/c_0$, as the small acoustic parameter of the problem.

As we are interested in steady quadratic processes, solutions to the acoustic equations must be sought to the first order in M, in order to obtain the mean acoustic pressure and velocity quadratic in the displacements. The objective of this section is to present in a dimensionless form Westervelt's general method[9] to access the mean (first order) Eulerian acoustic field using dimensionless quantities.

Let us denote by $\tilde{\xi}$ the Lagrangian displacement of a fluid particle. The Lagragian wave equation, written to order $O(M)$, takes the form[9,20]

$$c_0^{-2}\tilde{\xi}_{\tilde{t}\tilde{t}} - \tilde{\xi}_{\tilde{x}\tilde{x}} = -[\tilde{\xi}_{\tilde{x}}^2]_{\tilde{x}}. \quad (1)$$

This equation can be made dimensionless, by choosing ξ_0, ω^{-1}, and k_0^{-1} for typical displacement, time, and distance, which yields

$$\xi_{tt} - \xi_{xx} = -M[\xi_x^2]_x, \quad (2)$$

where ξ is the dimensionless Lagrangian particle displacement. As expected, the nonlinear term vanishes for $c_0 \to \infty$, since it originates from the compressible nature of the fluid. Equation (2) must then be completed by the proper set of boundary conditions, depending on the practical situation of interest. We write now the Lagrangian displacement field ξ as a perturbation expansion in M, namely,

$$\xi = \xi^{(0)} + M\xi^{(1)} + O(M^2). \quad (3)$$

The leading order solution is the solution to the classical linear wave equation, whereas the first-order solution $\xi^{(1)}$ is forced by the nonlinear term, $-M[\xi_x^{(0)2}]_x$. Thus, the equations at order $O(1)$ and $O(M)$ are

$$\xi_{tt}^{(0)} - \xi_{xx}^{(0)} = 0, \quad (4)$$

$$\xi_{tt}^{(1)} - \xi_{xx}^{(1)} = -[\xi_x^{(0)2}]_x. \quad (5)$$

In principle, once the previous equations are solved, one can find any Eulerian quantity $\tilde{\mathcal{E}}(\tilde{x}, \tilde{t})$, provided the corresponding Lagrangian quantity is available to order $O(M)$. Indeed, the Lagrangian representation $\tilde{\mathcal{L}}$ of a physical quantity (temperature, pressure, velocity, etc.) is specified on a moving material point. In other words, $\tilde{\mathcal{L}}(\tilde{t})$ is the value of the quantity assessed at the actual position of the material particle at time $\tilde{t}$. If $\tilde{\zeta}(\tilde{t})$ denotes the small displacement of such a particle from a fixed position $\tilde{x}_0$, one can write $\tilde{\mathcal{L}}(\tilde{t}) = \tilde{\mathcal{E}}(\tilde{x}_0 + \tilde{\xi}(\tilde{t}), \tilde{t})$. Expanding the latter expression with respect to $\tilde{\xi}$ yields (the time dependence of $\tilde{\xi}$ has been omitted for notation convenience)

$$\tilde{\mathcal{L}}(\tilde{t}) = \tilde{\mathcal{E}}(\tilde{x}_0, \tilde{t}) + \tilde{\xi}\tilde{\mathcal{E}}_{\tilde{x}} + \frac{1}{2}\tilde{\xi}^2\tilde{\mathcal{E}}_{\tilde{x}\tilde{x}} + \cdots, \quad (6)$$

where $\tilde{\mathcal{E}}_{\tilde{x}} = \partial_{\tilde{x}}\tilde{\mathcal{E}}|_{\tilde{x}_0}$ and $\tilde{\mathcal{E}}_{\tilde{x}\tilde{x}} = \partial_{\tilde{x}\tilde{x}}\tilde{\mathcal{E}}|_{\tilde{x}_0}$. Provided that ξ is small enough, only the first-order term can be kept in Eq. (6), which simplifies to

$$\tilde{\mathcal{L}}(\tilde{t}) = \tilde{\mathcal{E}}(\tilde{x}_0, \tilde{t}) + \tilde{\xi}\tilde{\mathcal{E}}_{\tilde{x}}. \quad (7)$$

Denoting by $\tilde{\mathcal{E}}$ the Eulerian quantity $\tilde{\mathcal{E}}(\tilde{x}_0, \tilde{t})$ and $\tilde{\mathcal{L}}$ the Lagrangian quantity $\tilde{\mathcal{L}}(\tilde{t})$, one gets

$$\tilde{\mathcal{E}} = \tilde{\mathcal{L}} - \tilde{\xi}\tilde{\mathcal{E}}_{\tilde{x}}, \quad (8)$$

which, to first order in $\tilde{\xi}$ can be rewritten as

$$\tilde{\mathcal{E}} = \tilde{\mathcal{L}} - \tilde{\xi}\tilde{\mathcal{L}}_{\tilde{x}}, \quad (9)$$

since from Eq. (8) itself, $\tilde{\mathcal{E}}_{\tilde{\xi}} = \tilde{\mathcal{L}}_{\tilde{\xi}}$ at zeroth-order in $\tilde{\xi}$.

Equation (9) can be written in the dimensionless form

$$\mathcal{E} = \mathcal{L} - M\xi\mathcal{L}_x. \quad (10)$$

By expanding $\mathcal{L}$ in a power series of the Mach number, Eq. (10) eventually takes the form

$$\mathcal{E} = \mathcal{L}^{(0)} + M[\mathcal{L}^{(1)} - \xi^{(0)}\mathcal{L}_x^{(0)}]. \quad (11)$$

We now use relation (11) to derive the expressions of the steady Eulerian components for velocity, density, and pressure. Each quantity will be further evaluated in the case of plane progressive and standing waves.

A. Mean velocity

If u denotes the dimensionless Eulerian velocity, Eq. (11) takes the form

$$u = \xi_t^{(0)} + M[\xi_t^{(1)} - \xi^{(0)}\xi_{tx}^{(0)}], \quad (12)$$

since ξ_t is the Lagrangian velocity.

The solution to Eqs. (4) and (5) in the case of a standing wave have been given by Fubini, who used a method due to Airy.[21] A broad outline of classical Fubini's result is presented in Appendix A. Using Fubini's solution for $(\xi^{(0)}, \xi^{(1)})$ in Eq. (12) and taking the time average leads to the following result for the mean velocity in a plane progressive wave:

$$\langle u\rangle_{\rm pw} = M\langle u^{(1)}\rangle_{\rm pw} = -\frac{1}{2}M, \quad (13)$$

where we used the subscript pw to denote *progressive wave*. The counterintuitive minus sign in the mean velocity Eq. (13) comes from the Eulerian nature of u. A comprehensive explanation of this apparent paradox is provided in Ref. 20. Note that if the Lagrangian velocity is bounded in time, the first two terms in Eq. (12) vanish when time-averaged so that the only contribution to the Eulerian velocity comes from the convective term.

The case of a plane standing wave can be analyzed in the same fashion (see Appendix A). For such a wave, the mean velocity turns out to be zero everywhere, that is,

$$\langle u\rangle_{\rm sw} = \langle u^{(1)}\rangle_{\rm sw} = 0, \quad (14)$$

where now sw stands for *standing wave*.

B. Mean pressure

While the application of Eq. (11) is immediate for the velocity, the explicit form of the Lagrangian pressure must be derived to obtain the mean (second order) Eulerian pressure. In order to do so, let us start by writing the dimensionless Lagrangian density perturbation to the mean fluid density, $\tilde{\rho}$, as a function of the Lagrangian displacement $\tilde{\xi}$. The continuity relation can be written in a dimensional form as[22]

$$\rho_0 + \tilde{\rho} = \rho_0(1 + \tilde{\xi}_x)^{-1}. \quad (15)$$

Using again ξ_0 (amplitude of the displacement), k_0^{-1}, and $\rho_0 M$ as typical quantities to make dimensionless the displacement $\tilde{\xi}$, the coordinate $\tilde{x}$, and the density difference $\tilde{\rho}$, leads to the following dimensionless form of Eq. (15):

$$\frac{1}{M} + \rho = \frac{1}{M}\frac{1}{1 + M\xi_x}. \quad (16)$$

Now, expanding the factor $1/(1 + M\xi_x)$ to order $O(M^2)$ [in which the expression $\xi_x = \xi_x^{(0)} + M\xi_x^{(1)}$ to order $O(M)$ has been first introduced] leads to the dimensionless form of Eq. (15) as

$$\rho = -\xi_x^{(0)} + M[\xi_x^{(0)} - \xi_x^{(1)}]. \quad (17)$$

This relation enables us to get the expression of the Eulerian density as

$$\rho = -\xi_x^{(0)} + M[\xi_x^{(0)2} + \xi^{(0)}\xi_{xx}^{(0)} - \xi_x^{(1)}]. \quad (18)$$

Since the sound speed, c_0, is a constant, the pressure difference $\tilde{p}$ and the density difference $\tilde{\rho}$ are related to each other by the simple relation $\tilde{p} = \tilde{\rho}c_0^2$. Using the natural quantity $\rho_0 c_0^2 M$ to non-dimensionalize the pressure difference, one obtains the expression of the Eulerian pressure to order $O(M)$ as

$$p = -\xi_x^{(0)} + M[\xi_x^{(0)2} + \xi^{(0)}\xi_{xx}^{(0)} - \xi_x^{(1)}]. \quad (19)$$

Again, using Fubini's solution for a plane progressive wave in the previous equation and taking the average in time, leads to

$$\langle p \rangle_{\text{pw}} = M\langle p^{(1)} \rangle = -\frac{1}{4}M. \quad (20)$$

Thus, the mean Eulerian pressure in a plane progressive wave is uniform (and negative) throughout the beam.

For a plane standing wave of the form

$$\xi^{(0)} = \sin x \ \cos t, \quad (21)$$

it can be shown that the steady part of the second-order solution to the nonlinear wave equation is (see again Appendix A)

$$\langle \xi^{(1)} \rangle = \frac{1}{8}\sin 2x. \quad (22)$$

Taking the time average of Eq. (19), using Eq. (21) and the previous expression for $\langle \xi^{(1)} \rangle$ leads to the expression of the mean Eulerian pressure in a plane standing wave as

$$\langle p \rangle_{\text{sw}} = M\langle p^{(1)} \rangle = \frac{1}{4}M \ \cos 2x. \quad (23)$$

Even if they are expressed here in a dimensionless form, the results presented in Sec. II—specifically expressions (13), (14), (20), and (23)—are all classical. The goal of Sec. II was to remind to the reader of the expressions of the rectified (stationary) terms involved in the expression of the total force derived below for small spheres and more complicated bodies.

III. TOTAL FORCE ON A SMALL SPHERE

We now consider the case of a small rigid sphere of radius a free to move in a viscous Newtonian fluid with density ρ_0, dynamic viscosity η, and kinematic viscosity $\nu = \eta/\rho_0$ (see Fig. 2). The density of the sphere is denoted ρ_s. Here, and in the rest of the paper, the typical distance $(\nu/\omega)^{1/2}$ over which the vorticity diffuses through the action of viscous stresses is assumed to be much larger than the particle size. The dimensionless ratio $\epsilon = (a^2\omega/\nu)^{1/2}$ will thus be assumed to be small compared to one.

For the incompressible limit to be valid at order $O(1)$, the wavelength of the acoustic radiation is assumed to be much greater than the sphere radius, which can be written as $k = k_0 a \ll 1$. The location of the particle, with regard to its equilibrium position $\tilde{x}_0$, is denoted $\tilde{r}$ and we write its velocity $\tilde{q}$. The amplitude of the fundamental harmonic response of the particle to the acoustic field, $\tilde{r}^{(0)}$, is also assumed to be small compared to the particle radius. This condition is automatically satisfied provided that the amplitude of the acoustic wave, ξ_0, is also smaller than the particle radius. At order $O(1)$, the sphere will have no net motion (i.e., $\langle q^{(0)} \rangle = 0$),

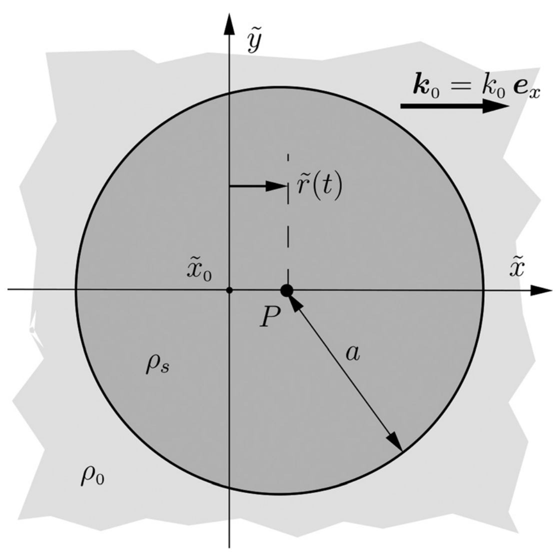

FIG. 2. Local geometry in the case of a sphere of radius a and definition of the displacement variable $\tilde{r}$. The densities of the particle and the fluid are denoted by ρ_s and ρ_0, respectively.

and its net dynamics will result from the balance between the total force and steady viscous drag at order $O(M)$.

A. Steady pressure-velocity field past the sphere

The total force exerted on small bodies results from two main contributions. First, the particle experiences the stresses exerted by the streaming flow. This flow arises from the coupling of the leading order flow—which results from the superimposition of the incident and perturbative (or scattered) flows—with itself through the nonlinear term of the Navier-Stokes equation. The steady streaming flow created by an oscillating sphere in an infinite medium is a classical problem.[23–25] As shown in Appendix B in the case of symmetric bodies which are symmetrically oriented in an acoustic field, the contribution of the steady streaming stress to the final evaluation of the total force can be neglected in the limit $\epsilon \ll 1$. The second contribution comes from to the oscillatory displacement of the particle in the pressure/velocity gradient field. This nonlinear Lagrangian effect, which generally yields steady terms in the expression of the total force, is what could be properly named "acoustic force" since it is intimately linked to the acoustic nature of the incident radiation. Note that the acoustic force (as defined above) has a non-zero contribution in both limits $\epsilon \ll 1$ and $\epsilon \gg 1$.[6]

Here, we focus on the pressure/velocity field experienced by the particle in its own frame of reference, i.e., the dimensionless fields $u(x_0+r,\,t)$ and $p_x(x_0+r,\,t)$, which would be measured in time at the particle center P (see Fig. 2). So, let us consider the dimensional velocity $\tilde{u}(\tilde{x}_0+\tilde{r},\,t)$ "seen" by the particle in its own frame of reference. The previous expression can be expanded to first order in $\tilde{r}$ and one gets

$$\tilde{u}[\tilde{x}_0+\tilde{r}(t),t] = \tilde{u}(\tilde{x}_0,t)+\tilde{r}(t)\tilde{u}_{\tilde{x}}(\tilde{x}_0,t). \tag{24}$$

As the velocity field, the distance $\tilde{x}$ and the displacement $\tilde{r}$ are made dimensionless using $\xi_0\omega$, k_0^{-1} and ξ_0, respectively, Eq. (25) can be written in the following dimensionless form:

$$u[x_0+r(t),t] = u(x_0,t)+M\,r(t)u_x(x_0,t), \tag{25}$$

since $M=\xi_0 k_0$. Thus, expanding u and u_x to first order in Mach number leads to the following expression, valid to first order in M:

$$u(x_0+r,t) = u^{(0)}(x_0,t)+M[u^{(1)}(x_0,t)+r^{(0)}(t)u_x^{(0)}(x_0,t)]. \tag{26}$$

The order $O(M)$ expansion for the pressure gradient can be derived in the same way and one obtains

$$p_x(x_0+r,t) = p_x^{(0)}(x_0,t)+M[p_x^{(1)}(x_0,t)+r^{(0)}(t)p_{xx}^{(0)}(x_0,t)]. \tag{27}$$

In Eqs. (26) and (27), the $O(1)$ terms are purely harmonic of pulsation 1 (i.e., ω if we use dimensional quantities) while all $O(M)$ quadratic terms contain a double harmonic and a mean contribution. So, the $O(1)$ harmonic term will determine the leading order dynamical response of the particle, $r^{(0)}$, whilst steady components of each $O(M)$ term, namely, $\langle u^{(1)}\rangle$, $\langle p_x^{(1)}\rangle$, $\langle r^{(0)}u_x^{(0)}\rangle$, and $\langle r^{(0)}p_{xx}^{(0)}\rangle$ will be involved in the calculation of the total force.

B. Dynamical response at leading order

First, let us derive the dimensionless form of relationship between velocity and pressure for the incident field. Whereas the relevant quantity for pressure and distance were $\rho_0 c_0^2 M$ and k_0^{-1} in Sec. II, here we choose instead the viscous steady stress $\eta\xi_0\omega a^{-1}$ and the size a of the particle, since we perform a local analysis (i.e., at the particle scale). Using the same typical velocity as the one used in Sec. II (for the particle displacement is *a priori* of the same order as the fluid displacement), namely, $\xi_0\omega$ leads to

$$p_x^{(0)} = -\epsilon^2 u_t^{(0)}, \tag{28}$$

which is the dimensionless form of the linearized Euler equation

$$\rho_0\tilde{u}_t^{(0)} = -p_x^{(0)}. \tag{29}$$

Now, in order to derive the dynamic response of the particle at leading order, let us first consider the force experienced by a particle oscillating in a uniform oscillating Stokes flow field. If we neglect the compressibility of the fluid, the perturbed flow resulting from the presence of the sphere is ruled by the unsteady Stokes's equations. Choosing a, ω^{-1}, $\xi_0\omega$ as typical length, time, and velocity scales, the dimensionless unsteady Stokes's equations are

$$\epsilon^2\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\varpi+\nabla^2\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{v}=0, \tag{30}$$

where $(\boldsymbol{v},\varpi)$ denote the velocity and pressure disturbance field in the reference frame of the laboratory. For a particle moving at velocity $q\,e_x$ in a uniform field $u(t)e_x$, the previous set of equations is completed by the boundary conditions

$$\boldsymbol{v} = u\,e_x \text{ at infinity, and } \boldsymbol{v} = q\,e_x \text{ on } \mathcal{S}. \tag{31}$$

The problem can be formulated in a similar fashion in the frame of reference of the surrounding fluid (i.e., where the fluid is motionless at infinity). In such a frame, the Stokes equation keep their original form but an extra inertial force density, $-\epsilon^2\dot{u}e_x$, must be added on the right-hand side of the first equation. This force density can be written as the gradient of an additional inertial pressure field, $p_i=\epsilon^2\dot{u}\,x\,\boldsymbol{e}_x$, and can be integrated in the global pressure gradient term. If then $(\boldsymbol{v}_u,\varpi_u)$ denotes the global disturbed field in the frame of reference of the surrounding fluid, we get

$$\epsilon^2\frac{\partial \boldsymbol{v}_u}{\partial t} = -\nabla\varpi_u+\nabla^2\boldsymbol{v}_u, \ \nabla\cdot\boldsymbol{v}_u=0, \tag{32}$$

with the new set of boundary conditions

$$\boldsymbol{v}_u = 0 \text{ at infinity, and } \boldsymbol{v}_u = (q-u)e_x \text{ on } \mathcal{S}. \tag{33}$$

In the following, we use the subscript u to refer to a situation where the fluid velocity vanishes at infinity.

If we assume that both the sphere and the surrounding fluid oscillate in the laboratory frame at the same pulsation, the second problem is formally the same as the one of a sphere oscillating in a quiescent fluid, the solution to which is given in number of works (see, e.g., Ref. 26). The surrounding uniform velocity field and the particle velocity take the form $\hat{u}\mathrm{e}^{-\mathrm{i}t}\boldsymbol{e}_x$ and $\hat{q}\mathrm{e}^{-\mathrm{i}t}e_x$. The integration of the stress tensor corresponding to the field $(\boldsymbol{v}_u, \varpi_u)$ over the particle surface leads to the classical expression of the hydrodynamic force exerted on the particle in the reference frame of the surrounding fluid, $\boldsymbol{F}_u$, non-dimensionalized by $\eta a \xi_0 \omega$, as[26,27]

$$\hat{\boldsymbol{F}}_u = 6\pi\Omega^s(\epsilon)(\hat{u} - \hat{q})\boldsymbol{e}_x, \quad \text{with } \Omega^s(\epsilon) = 1 + \mathrm{e}^{-\pi/4}\epsilon - \frac{\mathrm{i}}{9}\epsilon^2. \tag{34}$$

In order to obtain the force experienced by the particle in the reference frame of the laboratory, $\boldsymbol{F}$, one must add to the right-hand side of the previous equation an extra term corresponding to the integration of the inertial pressure field, $p_i = \epsilon^2 \dot{u} x \boldsymbol{e}_x$, over the surface of the particle. Since the pressure field p is linear, this term can be written in a convenient form $-\mathcal{V} dp_i/dx$, where $\mathcal{V}$ is the volume of the particle. For the sphere, $\mathcal{V} = (4/3)\pi$, so the additional term leads to the force

$$\hat{\boldsymbol{F}} = 6\pi\Omega^s(\epsilon)(\hat{u} - \hat{q})\boldsymbol{e}_x + \mathrm{i}\frac{4}{3}\pi\epsilon^2\hat{u}\boldsymbol{e}_x, \tag{35}$$

which can be re-written in the condensed form

$$\hat{\boldsymbol{F}} = 6\pi\left[\Lambda^s(\epsilon)\,\hat{u} - \Omega^s(\epsilon)\hat{q}\right]\boldsymbol{e}_x, \quad \text{with } \Lambda^s(\epsilon) = 1 + \mathrm{e}^{-i\pi/4}\epsilon - \frac{\mathrm{i}}{3}\epsilon^2. \tag{36}$$

This relation was first given in a more general framework by Mazur and Bedeaux[28] and re-derived by Kim and Karrila.[26] The method above demonstrates a straightforward route to obtaining the total force on a particle oscillating in a uniform Stokes flow, itself oscillating at the same frequency. Note that this method still holds for any oscillating flow (uniform or not), provided the terms quadratic in space can be neglected in the surrounding flow—in other words, as long as the Faxen's terms can be neglected in the expression of the force. The final formula makes clear the double origin of the force exerted on the particle. The first term comes from the velocity of the particle relative to the fluid. The second one is related to the pressure field that makes the surrounding fluid flow (whether the particle is present or not). If the particle is following the oscillating fluid and $\hat{q} = \hat{u}$, the only force comes from the external pressure field (i.e., the term $-\mathcal{V} dp_i/dx$) since the particle does not disturb the flow.

We now return to our main problem of interest, namely, the spherical particle moving under the effect of the acoustic field. The $O(1)$ displacement of the spherical particle moving in an acoustic field must satisfy the Newton's law as

$$\frac{4}{3}\pi a^3 \rho_s \xi_0 \omega^2 \dot{q} = 6\pi\eta a \xi_0 \omega\left[\Lambda^s(\epsilon)\hat{u}^{(0)} - \Omega^s(\epsilon)\hat{q}^{(0)}\right], \tag{37}$$

where $\dot{q}$ is the derivative of the velocity with respect to the dimensionless time. This can be rewritten in the more condensed form

$$\hat{\dot{q}}^{(0)} = \frac{9}{2}\beta\,\epsilon^{-2}\left[\Lambda^s(\epsilon)\hat{u}^{(0)} - \Omega^s(\epsilon)\hat{q}^{(0)}\right], \tag{38}$$

where $\beta = \rho_0/\rho_s$ is the ratio between fluid and particle densities. In Eq. (38) the velocity field $\hat{u}$ and the particle velocity $\hat{q}$ involved in Eq. (36) have been replaced by the $O(1)$ quantities $\hat{u}^{(0)}$ and $\hat{q}^{(0)}$, which simply means that Faxen's term are neglected at this order, which is justified as long as the Mach number is small as compared to 1.

From Eq. (38), the displacement $\hat{r}^{(0)} = -\mathrm{i}\hat{q}^{(0)}$ is given by

$$\hat{r}^{(0)} = \Gamma^s(\epsilon)\hat{u}^{(0)}, \tag{39}$$

where

$$\Gamma^s(\epsilon) = -\frac{9\beta\Omega_u(\epsilon)}{2\epsilon^2 + 9\mathrm{i}\beta\Omega(\epsilon)}. \tag{40}$$

Our result must be compared to the solution proposed by Doinikov[11] in the general framework of arbitrary ϵ and Mach number. The difference between our results and Doinikov's solution is shown in Fig. 3 with the parameters $c_0 = 10^3$ m s^{-1} and $a = 10^{-6}$ m. Specifically, in Figs. 3(a) and 3(b) we plot the quantities Δ_R and Δ_I defined as

$$\Delta_R = 2\frac{\mathrm{Re}\left[\hat{r}^{(0)}\right] - \mathrm{Re}\left[\hat{r}_D^{(0)}\right]}{\mathrm{Re}\left[\hat{r}^{(0)}\right] + \mathrm{Re}\left[\hat{r}_D^{(0)}\right]} \tag{41}$$

and

$$\Delta_I = 2\frac{\mathrm{Im}\left[\hat{r}^{(0)}\right] - \mathrm{Im}\left[\hat{r}_D^{(0)}\right]}{\mathrm{Im}\left[\hat{r}^{(0)}\right] + \mathrm{Im}\left[\hat{r}_D^{(0)}\right]}, \tag{42}$$

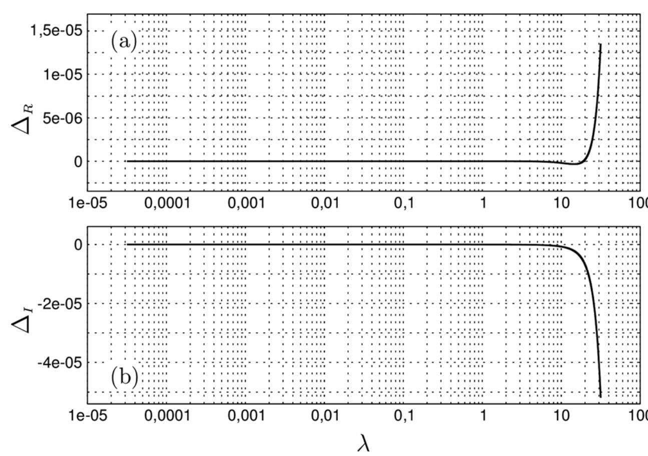

FIG. 3. Comparison between expression (40) and Doinikov's solution. The difference is small provided compressibility effects are negligible. In the present case, Δ_R and Δ_I have been plotted for $c = 10^3$ ms^{-1} and $a = 10^{-6}$ m.

where $\hat{r}_D^{(0)}$ is the displacement computed from Doinikov's article. The divergence between the two solutions occurs as soon as the particle does probe the compressibility of the fluid, namely, beyond $\omega a/c_0 \sim 1$, and the difference between the solutions becomes large beyond $\epsilon \simeq (ac_0/\nu)^{1/2} \simeq 30$. To conclude, we note that we can derive the expressions of the real and imaginary parts of Γ^s in the limit $\epsilon \ll 1$, as we need them to calculate the asymptotic expansion of the total force. They are

$$\mathrm{Re}[\Gamma^s] = \frac{2}{9}\frac{\beta-1}{\beta}\epsilon^2 - \frac{2}{9\sqrt{2}}\frac{\beta-1}{\beta}\epsilon^3 + O(\epsilon^5), \tag{43}$$

$$\mathrm{Im}[\Gamma^s] = 1 + \frac{2}{9\sqrt{2}}\frac{\beta-1}{\beta}\epsilon^3 + O(\epsilon^4). \tag{44}$$

C. Expression of the total force

Considering the analysis of the local steady field experience by the sphere presented in Sec. III A, the total dimensional force experienced by the particle balancing the steady drag at quadratic order in ξ_0 is given by

$$\tilde{F}_{\mathrm{tot}} = 6\pi\eta a(\langle \tilde{r}^{(0)}\tilde{u}_x^{(0)}\rangle + \langle \tilde{u}^{(1)}\rangle) - \tilde{\mathcal{V}}(\langle \tilde{p}_x^{(1)}\rangle - \langle \tilde{r}^{(0)}\tilde{p}_{xx}^{(0)}\rangle), \tag{45}$$

where the subscript tot stands for *total*. By choosing k_0^{-1}, ω^{-1}, $\xi_0\omega$, and $\rho_0 c_0^2 M$, $F^\star = \rho_0(\xi_0 c_0)^2(k_0 a)^3$ as typical length, time, velocity, pressure, and force scales, the previous expression now takes the form

$$F_{\mathrm{tot}} = \frac{4}{3}\pi\left[\frac{9}{2}\epsilon^{-2}\left(\langle r^{(0)}u_x^{(0)}\rangle + \langle u^{(1)}\rangle\right) - \langle p_x^{(1)}\rangle - \langle r^{(0)}p_{xx}^{(0)}\rangle\right]. \tag{46}$$

The latter expression is what is called *total force* in the present article, that is the sum of all the forces experienced by the particle which counter-balance the steady viscous drag at order $O(M)$.

1. Progressive wave

In the case of a plane progressive wave, the $O(1)$ quantities involved in expression (46) take the form

$$u_x^{(0)} = i\,\mathrm{e}^{\mathrm{i}(x-t)} \quad \text{and} \quad p_{xx}^{(0)} = -\mathrm{e}^{\mathrm{i}(x-t)}. \tag{47}$$

Using expressions (13), (20), (40), and (47) in the general expression for the force, Eq. (46), yields

$$F_{\mathrm{tot,pw}} = \frac{4}{3}\pi\left[\frac{9}{2}\epsilon^{-2}\left(\frac{1}{2}\mathrm{Im}[\Gamma^s] - \frac{1}{2}\right) - \frac{1}{2}\mathrm{Re}[\Gamma^s]\right], \tag{48}$$

which, using Eqs. (43) and (44), can finally be transformed into

$$F_{\mathrm{tot,pw}} = \frac{2}{3}\pi\frac{\beta-1}{\beta}\epsilon/\sqrt{2} + O(\epsilon^2), \tag{49}$$

valid in the limit $\epsilon \ll 1$. Its dimensional form is given by

$$\tilde{F}_{\mathrm{tot,pw}} = \frac{2}{3}\pi F^\star\frac{\beta-1}{\beta}\epsilon/\sqrt{2} + O(\epsilon^2). \tag{50}$$

This result has to be compared to the one given by Doinikov[11] in the relevant limit. Our result and the one in Ref. 11 are identical to within a prefactor of 11/5. We believe that this difference might be due to the truncations made in Ref. 11 in order to approximate the D_i functions in the original article. These truncations would give only an approximate result for the $O(\epsilon^3)$ terms in Γ^s which are required to obtain the final $O(\epsilon)$ result, Eq. (50).

The general trends of the present theory are also consistent with the result derived by Settnes and Bruus.[18] In particular, both expressions are proportional to $F^\star$. The main difference comes from the dependence of the prefactor with regard to $\chi(\beta,\epsilon) = [(1-\beta)/\beta]\epsilon$. Settnes and Bruus find a total force proportional to $[\chi(\beta,\epsilon)]^2$ whereas we/Doinikov find a linear dependence in $\chi(\beta,\epsilon)$. This has two mains implications. First, in Settnes and Bruus' article, no sign reversal of the force is observed when the particle density becomes lower than the fluid density. Second, the force goes faster to zero, as the frequency decreases. We think that the difference of scaling in $\chi(\beta,\epsilon)$ arises from the lack of Basset term in matching inner and outer solutions, forcing Settnes and Bruus to go further in the expansion to get a non-zero term.

2. Standing wave

In the case of a plane standing wave, the $O(1)$ quantities involved in expression (46) take the following form:

$$u_x^{(0)} = \sin x\,\mathrm{e}^{\mathrm{i}t} \quad \text{and} \quad p_{xx}^{(0)} = \mathrm{i}\cos x\,\mathrm{e}^{-\mathrm{i}t}. \tag{51}$$

Using expressions (14), (23), (40), and (51) in the general expression (46) yields a dimensionless force

$$F_{\mathrm{tot,sw}} = \frac{4}{3}\pi\sin 2x\left[\frac{9}{8}\epsilon^{-2}\mathrm{Re}[\Gamma^s] + \frac{1}{2} - \frac{1}{4}\mathrm{Im}[\Gamma^s]\right], \tag{52}$$

which, using Eqs. (43) and (44), can finally be transformed into

$$F_{\mathrm{tot,sw}} = \frac{1}{3\beta}\pi\sin 2x\left[(2\beta-1) + (1-\beta)\epsilon/\sqrt{2}\right] + O(\epsilon^2), \tag{53}$$

which is valid for small ϵ. Its dimensional form is

$$\tilde{F}_{\mathrm{tot,sw}} = \frac{1}{3\beta}\pi F^\star\sin 2k\tilde{x}_0\left[(2\beta-1) + (1-\beta)\epsilon/\sqrt{2}\right] + O(\epsilon^2). \tag{54}$$

Again, this result has to be compared to the one given by Doinikov[11] in the same limit, and we see that both expressions are equivalent at leading order. They also agree qualitatively to order $O(\epsilon)$ to within the same 11/5 prefactor as discussed above.

The result in Eq. (54) is also in perfect accordance with the expression of Settnes and Bruus at leading order.

However, the corrective term from Settnes and Bruus (not given in their paper) can be shown to be again proportional to $[\chi(\beta,\epsilon)]^2$ whereas we/Doinikov obtain a corrective term linear in $\chi(\beta,\epsilon)$. Note that it seems to be a general feature of the low ϵ theory that the force experienced in a progressive wave is of the same form as the corrective term in the expression of the force experienced in a standing wave. in the case of a standing wave, the three expressions (present approach−Doinikov−Settnes and Bruus) are identical at leading order but differ by the shape of the corrective term, which is not problematic except for $\beta \simeq 1/2$ (when the corrective term becomes dominant).

IV. TOTAL FORCE ON SYMMETRIC BODIES

We now show that the method outlined above can be generalized to non-spherical rigid bodies with certain symmetries in their shapes such that the forces arising from steady streaming can be neglected. We consider shapes which are instantaneously invariant under the transformation $\Pi_x\Pi_y\Pi_z$, where Π_x, Π_y, and Π_z are the reflections through the planes yz, xz, and xy including the origin, respectively. This invariance is automatically satisfied for bodies possessing three distinct planes of symmetry, including arbitrary ellipsoids, symmetric dumbells, cylinders, disks, etc. In what follows, we refer to these bodies as "symmetric."

A. Dynamical response at leading order

Consider a non-spherical symmetric body of volume $\tilde{\mathcal{V}}$. The typical length scale chosen to non-dimensionalize is the radius, a, of the equivalent sphere defined by $(4/3)\pi a^3 = \tilde{\mathcal{V}}$. The key parameter ϵ is again defined as $(a^2\omega/\nu)^{1/2}$, with a being the equivalent sphere radius.

The problem of computing the instantaneous drag force experienced by a (non-rotating) particle oscillating in a quiescent fluid has been addressed by several authors. Kanwal[29] first showed that the instantaneous drag force exerted on an axisymmetric body oscillating along its axis of symmetry with a dimensionless velocity $\boldsymbol{q} = \hat{\boldsymbol{q}}\,\mathrm{e}^{-\mathrm{i}t}$, could be written in the following form:

$$\hat{\boldsymbol{F}}_u = -6\pi A[1 + A\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon]\hat{\boldsymbol{q}} + O(\epsilon^2), \quad (55)$$

where A denotes the dimensionless steady Stokes drag coefficient. Here the force is made dimensionless using the scaling $\eta\, a\, \xi_0\omega$, so that for the sphere, $A = 1$ and Eq. (55) is equivalent to the Stokes formula in which only the first two terms are retained. Williams[30] noticed that Kanwal's expression was not valid for bodies of arbitrary shapes and derived the tensorial form of Eq. (55), which holds for all type of body at arbitrary fixed orientation,[26] as

$$\hat{\boldsymbol{F}}_u = -6\pi[\boldsymbol{A} + \boldsymbol{A}^2\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon]\hat{\boldsymbol{q}} + O(\epsilon^2), \quad (56)$$

where $\boldsymbol{A}$ is the steady Stokes tensor. Both expressions (55) and (56) are valid to order $O(\epsilon)$ only.

Lawrence and Weinbaum[31] calculated the expression of the instantaneous drag on prolate and oblate near-spheres oscillating along their axis of symmetry, deriving a result valid for any ϵ. That result was generalized to arbitrary near-spheres by Zhang and Stone.[32] In Zhang's work, the near-sphere is defined by its polar equation $r = 1 + \epsilon f(\theta,\phi)$, and the instantaneous drag takes the form

$$\hat{\boldsymbol{F}}_u = -6\pi\left[\boldsymbol{\Omega}^s(\epsilon)\,\boldsymbol{\delta} - \frac{3}{8\pi}\epsilon\left(\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon + 1\right)^2\int_{\mathcal{S}} f\,\boldsymbol{n}\,\boldsymbol{n}\,d\mathcal{S}\right]\hat{\boldsymbol{q}}, \quad (57)$$

valid at order $O(\epsilon)$ for any ϵ. Equation (57) and the result from Ref. 31 for a spheroid are equivalent at order $O(\epsilon)$.[33] The previous expressions of the drag are valid either for near-spheres at arbitrary ϵ or for arbitrary shapes (and aspect ratios) at small ϵ, and provided the steady drag is known. Lawrence and Weinbaum[34] proposed the following *ad hoc* composite formula in order to fill the gap

$$\hat{\boldsymbol{F}}_u \simeq -6\pi\left[\boldsymbol{A} + \boldsymbol{B}\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon - \mathrm{i}\boldsymbol{M}\epsilon^2 + (\boldsymbol{A}^2 - \boldsymbol{B})\frac{\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon}{1 + \mathrm{e}^{-\mathrm{i}\pi/4}\epsilon}\right]\hat{\boldsymbol{q}}, \quad (58)$$

where $\boldsymbol{B}$ is Basset's tensor and $\boldsymbol{M}$ is the added-mass tensor. Luckily, and as shown below, the expressions of $\boldsymbol{B}$ and $\boldsymbol{M}$ are in fact not required in order to calculate the total force at small ϵ.

For small values of ϵ, the previous expression can thus be approximated by

$$\hat{\boldsymbol{F}}_u = -6\pi\left[\boldsymbol{A} + \boldsymbol{A}^2\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon - \mathrm{i}\boldsymbol{A}'\epsilon^2 - \boldsymbol{A}''\sum_{n>2}(-\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon)^n\right]\hat{\boldsymbol{q}}, \quad (59)$$

with $\boldsymbol{A}' = \boldsymbol{M} + \boldsymbol{A}^2 - \boldsymbol{B}$ and $\boldsymbol{A}'' = \boldsymbol{A}^2 - \boldsymbol{B}$. Following the same steps as in Sec. III B, the force experienced by the rigid body oscillating at velocity $\boldsymbol{q} = \hat{\boldsymbol{q}}\mathrm{e}^{-\mathrm{i}t}$ in a uniform oscillating flow field $\boldsymbol{u} = \hat{\boldsymbol{u}}\,\mathrm{e}^{-\mathrm{i}t}$ can be written as

$$\hat{\boldsymbol{F}} = 6\pi[\boldsymbol{\Lambda}(\epsilon)\hat{\boldsymbol{u}} - \boldsymbol{\Omega}(\epsilon)\,\hat{\boldsymbol{q}}], \quad (60)$$

with

$$\boldsymbol{\Omega}(\epsilon) = \boldsymbol{A} + \mathrm{e}^{-\mathrm{i}\pi/4}\epsilon\boldsymbol{A}^2 - \mathrm{i}\boldsymbol{A}'\epsilon^2 - \boldsymbol{A}''\sum_{n>2}(-\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon)^n, \quad (61)$$

and

$$\boldsymbol{\Lambda}(\epsilon) = \boldsymbol{\Omega}(\epsilon) - \frac{2}{9}\mathrm{i}\epsilon^2\boldsymbol{\delta}. \quad (62)$$

For the sake of simplicity, we now assume that there is no hydrodynamic coupling between translation and rotation. This is strictly true at order $O(\epsilon)$ for a near sphere in an oscillating viscous flow[32] or for an arbitrary spheroid in a steady uniform creeping flow.[26] The expression for the displacement of a sphere at leading order had been obtained from Eq. (38). Likewise, the displacement of a symmetric body will obtained by solving the equation (Newton's law)

$$\hat{\boldsymbol{q}} = \frac{9}{2}\beta\epsilon^{-2}[\boldsymbol{\Lambda}(\epsilon)\hat{\boldsymbol{u}} - \boldsymbol{\Omega}(\epsilon)\hat{\boldsymbol{q}}], \quad (63)$$

which is the tensorial version of Eq. (38). In the natural reference system of the body $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, the symmetric tensors $\boldsymbol{A}$, $\boldsymbol{A}'$, and $\boldsymbol{A}''$ have diagonal representations $[A_1, A_2, A_3]$,

$[A'_1, A'_2, A'_3]$, and $[A''_1, A''_2, A''_3]$, respectively. Consequently, tensors $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$ also have diagonal representations in this basis, namely, $[\Omega_1, \Omega_2, \Omega_3]$ and $[\Lambda_1, \Lambda_2, \Lambda_3]$ with

$$\Omega_i = A_i + A_i^2 \mathrm{e}^{-\mathrm{i}\pi/4}\epsilon - iA'_i\epsilon^2 + A''_i \sum_{n>2} (-\mathrm{e}^{-\mathrm{i}\pi/4}\epsilon)^n \tag{64}$$

and

$$\Lambda_i = \Omega_i - \frac{2}{9}\mathrm{i}\epsilon^2. \tag{65}$$

Denoting $\boldsymbol{e}_i$ the axis of symmetry oriented along the direction of the wave vector, Eq. (63) reduces to

$$\hat{\hat{q}}_i = \frac{9}{2}\beta\epsilon^{-2}[\Lambda_i(\epsilon)\,\hat{u} - \Omega_i(\epsilon)\hat{q}_i], \tag{66}$$

the solution of which is formally identical to expression (40), that is,

$$\hat{r}_i^{(0)} = \Gamma_i(\epsilon)\hat{u}^{(0)}, \tag{67}$$

where

$$\Gamma_i(\epsilon) = -\frac{9\beta\Lambda_i(\epsilon)}{2\epsilon^2 + 9\mathrm{i}\beta\Omega_i(\epsilon)}. \tag{68}$$

As in the case of a sphere, the real and imaginary parts of Γ_i can be expanded in the small parameter ϵ, and one gets

$$\mathrm{Re}[\Gamma_i] = \frac{2}{9A_i}\frac{\beta-1}{\beta}\epsilon^2 - \frac{2}{9\sqrt{2}}\frac{\beta-1}{\beta}\epsilon^3 + O(\epsilon^4), \tag{69}$$

$$\mathrm{Im}[\Gamma_i] = 1 + \frac{2}{9\sqrt{2}}\frac{\beta-1}{\beta}\epsilon^3 + O(\epsilon^5). \tag{70}$$

Notably, the tensors $\boldsymbol{A}'$ and $\boldsymbol{A}''$ are not involved in the ϵ-expansion of Γ_i at order $O(\epsilon^3)$.

B. General expression for the total force

Expression (46) can be generalized to the case of a symmetric body arbitrarily oriented relative to the direction of the acoustic radiation and one obtains

$$\boldsymbol{F}_{\mathrm{tot}} = \frac{4}{3}\pi\left[\frac{9}{2}\epsilon^{-2}(\boldsymbol{A}\cdot\boldsymbol{e}_x)\left(\langle r^{(0)}u_x^{(0)}\rangle + \langle u^{(1)}\rangle\right) - \left(\langle p_x^{(1)}\rangle + \langle r^{(0)}p_{xx}^{(0)}\rangle\right)\boldsymbol{e}_x\right]. \tag{71}$$

Unlike the case of the sphere, the total force can now have components in the normal directions relative to the wave vector.

C. Progressive wave

In the case of a particle with its axis $\boldsymbol{e}_i$ oriented along x, the y and z components of the force vanish and the expression for F^x_{tot} simplifies to

$$F_{\mathrm{tot,pw}} = \frac{4}{3}\pi\left[\frac{9}{2}\epsilon^{-2}A_i\left(\frac{1}{2}\mathrm{Im}[\Gamma_i] - \frac{1}{2}\right) - \frac{1}{2}\mathrm{Re}[\Gamma_i]\right]. \tag{72}$$

Using the expressions of real and imaginary parts of Γ_i, the previous expression becomes

$$F_{\mathrm{tot,pw}} = \frac{2}{3}\pi A_i\frac{\beta-1}{\beta}\epsilon/\sqrt{2}, \tag{73}$$

which, in a dimensional form, becomes

$$\tilde{F}_{\mathrm{tot,pw}} = \frac{2}{3}\pi F^\star A_i\frac{\beta-1}{\beta}\epsilon/\sqrt{2}. \tag{74}$$

We therefore see that the effect of non-sphericity of the particle shows up at leading order, which, in the case of a progressive wave is $O(\epsilon)$. The non-sphericity of the particle does not modify the direction of the force but it affects its magnitude. For instance, for a prolate ellipsoid of revolution (axis $\boldsymbol{e}_1$) of aspect ratio 2, $A_1 = 0.38$ and $A_2 = 0.43$.

Therefore, the radiation force is less efficient on such a prolate ellipsoid (for any orientation) than on a sphere of same volume (and density). The $O(M)$ steady velocity resulting from the balance between total force and viscous drag will however not be affected since the latter is proportional to A_i as well.

D. Standing wave

Again, in the case of a particle with its axis $\boldsymbol{e}_i$ oriented along x, the y and z components of the force vanish and the expression for F^x_{tot} simplifies to

$$F_{\mathrm{tot,sw}} = \frac{4}{3}\pi\sin 2x_0\left[\frac{9}{8}\epsilon^{-2}A_i\mathrm{Re}(\Gamma_i) + \frac{1}{2} - \frac{1}{4}\mathrm{Im}(\Gamma_i)\right], \tag{75}$$

which, after using Eqs. (69) and (70), yields

$$F_{\mathrm{tot,sw}} = \frac{1}{3\beta}\pi\sin\,2x_0\left[(2\beta-1) + A_i(1-\beta)\epsilon/\sqrt{2}\right]. \tag{76}$$

The dimensional form of the previous equation is

$$F_{\mathrm{tot,sw}} = \frac{1}{3\beta}\pi F^\star\sin 2k_0\tilde{x}_0\left[(2\beta-1) + A_i(1-\beta)\epsilon/\sqrt{2}\right]. \tag{77}$$

A symmetric particle oscillating along one of its axis under the effect of a plane standing wave experiences almost the same force as the equivalent sphere (same volume). It is only the first correction at order ϵ which is affected by the difference in shape, as in the case of a progressive wave.

V. DISCUSSION

In the present paper, we derived an expression of the total force in the limit of small ϵ (i.e., large viscous diffusion length compared to the particle size), suitable for bodies possessing three planes of symmetry. The practical use of the expressions we proposed in the cases of progressive and standing waves only requires the knowledge of the steady (Stokes) drag tensor. After outlining a simple method applied to the classical case of a sphere, we then showed how to generalize the results to the case of symmetric bodies possessing three planes of symmetry. For a plane progressive wave, the radiation pressure is shown to be equal to the one experienced by a sphere of same volume and density multiplied by the dimensionless viscous drag of the particle. In the case of a standing wave, there is almost no effect of the shape and the radiation pressure at leading order is equal to the one experienced by the equivalent sphere, with a shape-dependent correction at order $O(\epsilon)$. It is notable that our derivations recover all scalings computed in Ref. 11 and thus agree with the assumptions therein. This is to be contrasted with the different approach, and scalings, proposed in Ref. 35. Generalizing the results presented in this paper to the case of large ratios a/δ would be important but difficult since this would require the calculation of the steady streaming generated by oscillating bodies of arbitrary shape.

For the derivations presented in this article to hold, we have to be in the asymptotic limit

$$a \ll \delta \ll k_0^{-1}. \tag{78}$$

Equivalently, given a typical size a, and a typical order of magnitude for the speed of sound c_0, the frequency of the acoustic field must be smaller than $\min[\nu/a^2, c_0^2/\nu]$, for the double inequality to be satisfied. As a practical example, consider glycerol for which the kinematic viscosity is $\nu \simeq 1.4 \times 10^{-3}$ m^{2}s^{-1}, the density $\rho \simeq 1.3$ kgm^{-3}, and the speed of sound is $c_0 \simeq 1.9 \times 10^3$ ms^{-1} at ambient temperature. For $\omega = 10^7$ s^{-1} [for the conditions in Eq. (78) to be satisfied] and an amplitude of displacement $\xi_0 = 10^{-8}$ m (so that $\xi_0/a \ll 1$), the maximum force experienced by a spherical particle of silica with 1 μm radius a (density $\rho_s = 2.2$ kgm^{-3}) in the case of a progressive wave is $F_{\text{tot,pw}} = -7.46 \times 10^{-15}$ N whereas it is $F_{\text{tot,sw}} = 1.75 \times 10^{-14}$ N in the case of a standing wave. The value of $F_{\text{tot,pw}}$, since it is proportional to A_i would be affected by a change of the particle shape. In the case of a progressive incident wave, a prolate spheroid of aspect ratio 2 (with $A_i = 0.38$) would-experience a total force almost three times as small as in the spherical case. Note that this difference would be of less practical importance when gravity plays a dominant role since the weight $W = (4/3)\pi a^3 \rho_s g(1-\beta) \approx -4.15 \times 10^{-14}$ N is greater than $F_{\text{tot,pw}}$ in magnitude by a factor of about 5.6. Note also that in a case where the buoyancy plays no role, the drift velocity induced by a progressive radiation would not change since the counteracting drag force is proportional to A_i as well. Conversely, the value of the total force in the case of a standing wave is very slightly affected by a change of a particle shape, since $F_{\text{tot,pw}}$ does not depend on A_i at leading order. Only the drift velocity would be altered since the drag is proportional to A_i. A change in shape will thus modify the value of the total force experienced by a particle in a progressive wave but will not affect its drift velocity. By contrast, such a change will have almost no effect on the total force generated by a standing wave, but will alter the drift velocity of the particle through its viscous drag coefficient. From a practical standpoint, the total force exerted by an acoustic radiation on small symmetric bodies, that is the ability for an acoustic radiation to levitate, and allow actuation of particle at the micron-scale, could be significantly affected by the shape of the body.

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APPENDIX A: ACOUSTIC FIELD AT ORDER $O(M)$

We review here Fubini's solution to the nonlinear wave equation in the case of a plane progressive wave.[21] The $O(1)$ and $O(M)$ solutions are the ones to be used in Eqs. (12) and (19) to get the results Eqs. (13) and (20). The second part of the appendix is devoted to the case of a plane standing wave and results Eqs. (13) and (20) are derived.

We consider first the situation of a semi-infinite pipe full of fluid closed by an oscillating wall. The boundary condition in displacement imposed at the wall is given by

$$\xi = 0, \quad \xi_t = 0 \text{ for } t = 0, \quad x = 0, \tag{A1}$$

$$\xi(t) = 1 - \cos t \text{ for } t > 0, \quad x = 0. \tag{A2}$$

The solution to system (4) and (5) can be shown to be of the following form:[21]

$$\xi^{(0)} = 1 - \cos(t - x), \tag{A3}$$

$$\xi^{(1)} = \frac{1}{4} x[1 - \cos 2(t - x)]. \tag{A4}$$

Using the previous forms of $\xi^{(0)}$ and $\xi^{(1)}$ in Eqs. (12) and (19) and taking the average in time leads to the expressions (13) and (20) of the mean velocity an pressure in the case of a plane progressive wave.

We consider now a plane standing wave of the form

$$\xi^{(0)} = \sin x \cos t. \tag{A5}$$

As already stated in Sec. II A, as long as the Lagrangian velocity is bounded (in time), the first two terms in Eq. (12) vanish when time averaged. The convective term $-\xi^{(0)}\xi^{(0)}_{tx}$ has also a zero time average, such that the mean Eulerian velocity is zero until order $O(M)$.

The Eulerian pressure can be calculated by introducing Eq. (A5) in the nonlinear wave Eq. (5). We consider, as previously done by Westervelt,[9] that the dissipation is large enough to keep the magnitude of the higher-order solutions smaller than the magnitude of the $O(1)$ solution. So, introducing Eq. (A5) in Eq. (2) and taking average in time leads to

$$\langle \xi^{(1)}_{xx} \rangle = -\frac{1}{2}\sin(2x). \tag{A6}$$

After integration, we get

$$\langle \xi^{(1)} \rangle = -\frac{1}{8}\sin(2x). \quad \text{(A7)}$$

Introducing Eqs. (A5) and (A7) in Eq. (19) then leads to the expression (23) of the Eulerian pressure in a standing wave.

APPENDIX B: ACOUSTIC STREAMING AT SMALL ϵ

The effects of the steady streaming induced by the oscillations of a symmetric particle of equivalent radius a under the effect of an acoustic field are investigated in the limit of small ϵ. The net force acting on the particle is shown to be negligible at order $O(\epsilon)$, as long as the required criteria of symmetry are satisfied. Here we address the situation of a particle in a plane standing wave of the form $\tilde{u}(x) = \xi_0\omega \sin(k_0\tilde{x})$, but the case a progressive wave can be treated similarly.

The incident velocity field can be expanded in the vicinity of the average position x_0 of the particle, which yields

$$\tilde{u}(x) = \xi_0\omega \sin k_0\tilde{x}_0 - k_0(\tilde{x} - \tilde{x}_0)\cos k_0\tilde{x}_0 + O(k_0^2). \quad \text{(B1)}$$

In the frame of reference of the particle, using Eq. (40) and neglecting the particle displacement relative to the particle radius, the previous expression transforms into

$$\begin{aligned}\tilde{u}(x) &= \xi_0\omega[1 + \mathrm{i}\Gamma^s(\epsilon)] \sin k_0\tilde{x}_0 \\ &\quad -\xi_0\omega\, k_0(\tilde{x} - \tilde{x}_0)\cos k_0\tilde{x}_0 + O(k_0^2).\end{aligned} \quad \text{(B2)}$$

In the small ϵ limit, the quantity $\alpha = \mathrm{i}(1 + \mathrm{i}\Gamma^s)$ is equivalent to

$$\alpha = \frac{2}{9}\frac{1-\beta}{\beta}\epsilon^2. \quad \text{(B3)}$$

By taking a, $\xi_0\omega$ as typical distance and velocity, the field $\tilde{u}$ can be written in the following dimensionless form:

$$u(x) = \mathrm{i}\alpha \sin kx_0 - k(x - x_0)\cos kx_0 + O(k^2), \quad \text{(B4)}$$

where $k = k_0 a$. In the reference frame of the particle, the incident field is the sum of a uniform field of order α (considering the amplitude $\xi_0\omega$ chosen for the non-dimensionalization), and a linear (compressible) component of amplitude k.

Choosing the quantities $\rho_0 M$ and $\rho_0\xi_0 a\omega^2$ as typical density perturbation and stress and defining the small parameter $\varepsilon = \xi_0/a$, the Navier-Stokes equations takes the form

$$(1 + \varepsilon k\rho)\left[\frac{\partial \boldsymbol{v}}{\partial t} + \varepsilon(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = \nabla\cdot\sigma, \quad \text{(B5)}$$

$$k\frac{\partial p}{\partial t} + \nabla\cdot\boldsymbol{v} + \varepsilon k\nabla\cdot(\rho v) = 0, \quad \text{(B6)}$$

$$p = \rho. \quad \text{(B7)}$$

To quantify the net effect of the steady streaming on the particle in the limit $\epsilon \ll 1$, we assume that ϵ is the small parameter of the problem. So, we seek the perturbation solution to the previous system by expanding the velocity pressure and density as powers of ϵ,

$$\boldsymbol{v} = \boldsymbol{v}^{(0)} + \varepsilon\boldsymbol{v}^{(1)} + O(\varepsilon^2), \quad \text{(B8)}$$

$$p = p^{(0)} + \varepsilon p^{(1)} + O(\varepsilon^2), \quad \text{(B9)}$$

$$\rho = \rho^{(0)} + \varepsilon\rho^{(1)} + O(\varepsilon^2), \quad \text{(B10)}$$

and, as a consequence of Eqs. (B8) and (B9), $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(0)} + \varepsilon\boldsymbol{\sigma}^{(1)} + O(\varepsilon^2)$.

At order $O(1)$, the system (B8) and (B9) yields

$$\frac{\partial \boldsymbol{v}^{(0)}}{\partial t} = \nabla\cdot\boldsymbol{\sigma}^{(0)}, \quad \text{(B11)}$$

$$k\frac{\partial \rho^{(0)}}{\partial t} + \nabla\cdot\boldsymbol{v}^{(0)} = 0, \quad \text{(B12)}$$

$$p^{(0)} = \rho^{(0)}. \quad \text{(B13)}$$

These equation are the dimensionless forms of those in Ref. 11. As suggested by Lamb, $\boldsymbol{v}^{(0)}$ can be written as the sum of an irrotationnal and a zero gradient term. We can further write $\boldsymbol{v}^{(0)}$ as the sum of two flows, each one corresponding to the symmetric and antisymmetric parts of the incident field (B4):

$$\boldsymbol{v}^{(0)} = \alpha\,\boldsymbol{v}_a^{(0)} + k\boldsymbol{v}_s^{(0)}. \quad \text{(B14)}$$

To order $O(\varepsilon)$, Eq. (B11) yields

$$\rho^{(0)}k\frac{\partial \boldsymbol{v}^{(0)}}{\partial t} + \frac{\partial \boldsymbol{v}^{(1)}}{\partial t} + \left(\boldsymbol{v}^{(0)}\cdot\nabla\right)\boldsymbol{v}^{(0)} = \nabla\cdot\boldsymbol{\sigma}^{(1)}, \quad \text{(B15)}$$

which, when using Eq. (B12) and taking the average in time, leads to

$$\nabla\cdot\langle\boldsymbol{\sigma}^{(1)}\rangle = \nabla\cdot\langle\boldsymbol{v}^{(0)}\boldsymbol{v}^{(0)}\rangle. \quad \text{(B16)}$$

Using Eq. (B14) in the previous equation, and considering that, due to the global symmetry of the system, only the crossed products $\boldsymbol{v}_a^{(0)}\boldsymbol{v}_s^{(0)}$ will lead to a non-zero net force on the particle, one can deduce the order of magnitude $\langle\boldsymbol{\sigma}^{(1)}\rangle$ of the steady part of the antisymetric stress tensor, namely, $\langle\boldsymbol{\sigma}^{(1)}\rangle \sim \varepsilon k\alpha$. Coming back to dimensional quantities, one gets

$$\langle\tilde{\sigma}^{(1)}\rangle \sim \varepsilon\,\rho_0\xi_0 a\omega^2\,k\alpha. \quad \text{(B17)}$$

Multiplying the previous expression by the typical surface a^2 provides the order of magnitude of the dimensional steady streaming force $\tilde{F}_{SS}$:

$$\tilde{F}_{SS} \sim F^\star\frac{1-\beta}{\beta}\epsilon^2, \quad \text{(B18)}$$

where $F^\star = \rho_0(\xi_0 c_0)^2(k_0 a)^3$. Therefore, for small values of ϵ, the steady streaming term steps in the expression of the total force at order ϵ^2. This is why expressions (49) and (53) are correct at order $O(\epsilon)$.

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