DISCRETE AND CONTINUUM MODELING OF BIOLOGICAL NETWORK FORMATION

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Summary
Network formation and transportation networks are fundamental processes in living systems. A new dynamic modeling approach to describe the formation of biological transport networks has recently been introduced by Hu and Cai. They propose a continuum model, based on macroscopic laws, as well as a discrete, purely local dynamic adaptation model. We present an overview of recent analytical and numerical results for these models.

Introduction
Transportation networks are ubiquitous in living systems such as leaf venation in plants, angiogenesis of blood vessels and neural networks which transport electric charge. Biologists, engineers, physicists and computer scientists have expressed great interest in understanding natural networks. One of the main research questions is what are the structural and topological properties of optimal networks, in particular the existence of loops.

Description of the model
Traditionally most of the methodological tools use discrete models, based on mathematical graph theory and discrete energy optimization, where the energy consumption within the network is minimized under the constraint of constant total cost. However, biological systems are continuously adapting their structures to meet the changing metabolic demand. Hu and Cai have recently introduced a new dynamic modeling approach [6] accounting for adaptation of networks to fluctuations in the flow in contrast to considering optimization as a global effect. Central to their discrete model is a purely local dynamic adaptation mechanism based on mechanical rules. In particular, this dynamic adaptation model responds only to local information and can naturally incorporate fluctuations in the flow.

To formulate the discrete model we consider a connected graph and associate each vessel of the graph with a non-negative conductivity. Assuming that the material cost for the vessel $i$ of the network is proportional to the power $C_i^\gamma$ of its conductivity $C_i$ for a parameter $\gamma > 0$ we consider the energy consumption as the sum of the kinetic energy of the material flow through the vessels and the metabolic cost of maintaining the network. This energy is constrained by the Kirchhoff law which expresses the conservation of mass.

Besides, one can formulate a continuum model based on macroscopic physical laws. This model was introduced in [5], studied in [1, 3, 4, 2] and consists of a very complex system of nonlinear partial differential equations. Because of its highly unusual coupling this model is also of mathematical interest.

Analysis of the model
Using methods from mathematical and numerical analysis we study the discrete and the macroscopic model and investigate the qualitative properties of network structures. Experimental studies of scaling relations of conductivities of parent and daughter edges in real networks suggest that the choice of the parameter $\gamma$ is crucial for the resulting network formation [3]. This is also underlined by the analytical and numerical results we have obtained indicating a phase transition behavior at $\gamma = 1$ with a uniform sheet, i.e. the network is tiled with loops, for $\gamma > 1$ and a loopless tree for $\gamma < 1$.

References