Department of Applied Mathematics and Theoretical Physics
Martin Benning

Inverse Problems

Example sheet **1** Presentation **23. October 2015**, 3 p.m.

Exercise 1 (Heat equation)

The heat equation is defined as the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u \, , \tag{1}$$

where Δ denotes the Laplace operator $\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$.

a) Compute the solution u of the heat equation for the initial value $u(x, 0) = f(x)$ with

$$f(x_1, x_2) = \sum_{p=0}^{m_1-1} \sum_{q=0}^{m_2-1} c_{pq} \, \varphi_{pq}(x_1, x_2) \, , \tag{2}$$

for $x = (x_1, x_2) \in \Omega = [0, 1]^2$, coefficients $c_{pq} \in \mathbb{R}$ and functions $\varphi_{pq} \in C^2(\Omega)$ that are Eigenfunctions of the (negative) Laplace operator with Eigenvalue λ_{pq}, i.e. $-\lambda_{pq}\varphi_{pq} = \Delta\varphi_{pq}$.

b) Show that the functions $\varphi_{pq} \in C^\infty(\Omega)$ with

$$\varphi_{pq}(x_1, x_2) := v_p w_q \cos(\pi q x_1) \cos(\pi p x_2)$$

and $p \in \{0, \dots, m_1 - 1\}$, $q \in \{0, \dots, m_2 - 1\}$, are Eigenfunctions of the (negative) Laplace operator, and compute the corresponding Eigenvalues λ_{pq}. Here the weights v_p and w_q are defined as

$$v_p = \begin{cases} \frac{1}{\sqrt{m_1}} & p = 1 \\ \sqrt{\frac{2}{m_1}} & 2 \leq p \leq m_1 \end{cases} \quad \text{and } w_q = \begin{cases} \frac{1}{\sqrt{m_2}} & q = 1 \\ \sqrt{\frac{2}{m_2}} & 2 \leq q \leq m_2 \end{cases} . \tag{3}$$

c) Use MATLAB and the results of Exercise 1 a) and b) to compute the solution u of the heat equation for f given as the discrete image `'trees.tif'`, evaluated at the points $x_1 = (2i + 1)/(700)$, $i \in \{0, \dots, 349\}$, and $x_2 = (2j + 1)/(516)$, $j \in \{0, \dots, 257\}$. Visualise your results for suitable choices of t.
Hint: Make use of the MATLAB commands `imread`, `dct2` and `idct2`.

Please turn over!

Exercise 2 (Inverse heat equation)
We now want to consider the inverse problem of (1). Instead of an initial value $u(x, 0)$ we are given the accumulated value $f(x)$ at time $t = T$ with $T > 0$, i.e. $u(x, T) = f(x)$.

a) Compute the solution u of the heat equation for $t \in [0, T]$ and $u(x, T) = f(x)$, with f being defined as in Exercise 1 a), Equation (2).

b) Show that for $f \in L^\infty(\Omega)$ the inverse problem of the heat equation is ill-posed in the sense of Hadamard.

c) Use Matlab and the results of Exercise 2 a) and 1 b) to compute the solution of the inverse problem of the heat equation for f given as the discrete image `'moon.tif'`, evaluated at the points $x_1 = (2i + 1)/(716)$, $i \in \{0, \dots, 357\}$, and $x_2 = (2j + 1)/(1074)$, $j \in \{0, \dots, 536\}$. Visualise your results for suitable choices of t.

Exercise 3 (Integral operators)
For $\Omega = [0, 1]$ we consider the integral operator $K : L^2(\Omega) \to L^2(\Omega)$ with

$$(Ku)(y) := \int_0^1 k(x, y)u(x)\,dx\,,$$

for $k \in L^2(\Omega \times \Omega)$. Show that

a) K is linear with respect to u.

b) K is a bounded linear operator, i.e. $\|Ku\|_{L^2(\Omega)} \leq \|K\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\|u\|_{L^2(\Omega)}$. Give also an estimate for $\|K\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}$.

c) the adjoint K^* is given via

$$(K^*v)(y) = \int_0^1 k(y, x)v(x)\,dx$$