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Inverse Problems

Example sheet **3** Presentation **27. November 2015**, 3 p.m.

Exercise 1 (Error Estimates)
For a proper, lower semi-continuous and convex functional $J : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ with non-empty subdifferential ∂J, the generalised Bregman distance is defined as

$$D_J^p(u, v) := J(u) - J(v) - \langle p, u - v\rangle \, , \, p \in \partial J(v) \, . \tag{1}$$

Furthermore we assume that for the J-minimising solution $u^\dagger$ (as defined in the lecture) the source condition

$$\exists \, w \in \mathcal{V} : \quad K^* w \in \partial J(u^\dagger) \tag{2}$$

is satisfied.

a) Show that $D_J^p(u, v) \geq 0$ for all $u, v \in \mathcal{X}$ and $p \in \partial J(v)$.

b) Show that for $f^\delta \in \mathcal{V}$ and $f = Ku^\dagger$, satisfying $\|f - f^\delta\|_\mathcal{V} \leq \delta$, there exists a p such that the estimate

$$D_J^p(u_\alpha, u^\dagger) \leq \frac{\delta^2}{2\alpha} + \frac{\alpha \|w\|_\mathcal{V}^2}{2}$$

holds true. Here u_α is a minimiser of $\tilde{T}_\alpha$ as introduced in the lecture.

Exercise 2 (Exact recovery)
Let $u_\lambda \neq 0$ be a generalised Eigenfunction satisfying $\lambda K^* K u_\lambda \in \partial J(u_\lambda)$ and $\|Ku_\lambda\|_\mathcal{V} = 1$. Further let J be absolutely one-homogeneous, i.e. $J(cu) = |c|J(u)$ for all constants $c \in \mathbb{R}$.

a) Compute λ.

b) Show that $u_\alpha = \max\{\gamma - \lambda\alpha, 0\} u_\lambda$ is a global minimiser of $\tilde{T}_\alpha$ as introduced in the lecture, for $f = \gamma K u_\lambda$.

c) Let $\lambda_0 := \min_{\|Ku\|_\mathcal{V}=1} J(u)$. Prove that minimisers u_α of $\tilde{T}_\alpha$ have a systematic bias of the form

$$\|Ku_\alpha - f\|_\mathcal{V} \geq \alpha\lambda_0$$

if $\|f\|_\mathcal{V} \geq \alpha\lambda_0$.

Exercise 3 (Deconvolution)

The discrete two-dimensional convolution of a signal $u \in \mathbb{R}^{n_1 \times n_2}$ with a kernel $g \in \mathbb{R}^{(2r_1+1)\times(2r_2+1)}$ can be described via

$$f_{i_1,i_2} = (Cu)_{i_1,i_2} := \sum_{k_1=-r_1}^{r_1} \sum_{k_2=-r_2}^{r_2} u_{k_1,k_2}\, g_{i_1-k_1,i_2-k_2}\,, \tag{3}$$

for $i_1, r_i \in \{1, \ldots, n_1\}$, $i_2, r_2 \in \{1, \ldots, n_2\}$.

a) **Forward problem**: Implement the discrete two-dimensional convolution operator (with periodic boundary conditions) as a MATLAB©-class named `convolution`. An object `C` of this class should be initialised via the command `C = convolution(imsize, kernel);`. Here `imsize` denotes the image dimensions of the two-dimensional image to be convolved (n_1 and n_2 in (3)), and `kernel` is the convolution kernel. Moreover, the object should be able to compute the convolution via the call `f = C*u;`, where `f` and `u` are column vector representations of the convolved image f and the original image u. Further should it be possible to compute the adjoint (transpose) convolution via `u = C'*f;`.
Hint: Make use of the discrete convolution theorem that allows to diagonalise the convolution

operator in the discrete Fourier transform domain.

b) **Regularised inverse problem**: Implement the following discrete deconvolution model

$$u_\alpha \in \operatorname*{arg\,min}_{u \in \mathbb{R}^n} \left\{ \frac{1}{2}\|Cu - f\|_2^2 + \alpha \max_{\substack{p \in \mathbb{R}^n \times \mathbb{R}^n \\ \||p\|_2\|_\infty \leq 1}} \langle \nabla u, p \rangle \right\} \tag{4}$$

via the primal-dual hybrid gradient method introduced in the lecture. Here C denotes the discrete convolution (3), $n = n_1 n_2$, and ∇ is a forward finite-difference approximation of the gradient.
Hint: The class `C` of Exercise 3 a) is not necessarily required for this task.

c) Take your favourite picture, load it into MATLAB©via `imread` and convert it into a gray-value image with `double`-precision intensity values. Create an object of the class you have implemented in Exercise 3 a) to convolve this image with a convolution kernel `kernel`. Use the MATLAB©-command `fspecial` to create a convolution kernel of your choice. Subsequently, create different noisy versions of your blurry image for different noise levels.

d) Use the column-vector representations of your noisy images of Exercise 3 c) as the input for your deconvolution algorithm of Exercise 3 b). Compute reconstructions of (4) for different choices of α and visualise your results.

Exercise 4 (Bregman iteration)

With the help of the generalised Bregman distance (1) we can define the following iterative regularisation scheme (also known as Bregman iteration)

$$\begin{aligned} u_\alpha^{k+1} &\in \arg\min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \|Ku - f\|_{\mathcal{V}}^2 + \alpha J(u) - \langle p_\alpha^k, u \rangle \right\} \\ p_\alpha^{k+1} &= p_\alpha^k + \frac{1}{\alpha} K^*(f - Ku_\alpha^{k+1}) \end{aligned}, \tag{5}$$

for $\alpha > 0$ and with $u_\alpha^0 = p_\alpha^0 = 0$ and $p_\alpha^k \in \partial J(u_\alpha^k)$ for all $k \in \mathbb{N}$.

a) Show that the iterates of (5) satisfy $\|Ku_\alpha^{k+1} - f\|_{\mathcal{V}} \leq \|Ku_\alpha^k - f\|_{\mathcal{V}}$.

b) Argue why the Bregman iteration together with Morozov's discrepancy principle as a stopping criterion is a useful strategy to find J-minimising solutions. Give also an estimate for the parameter η in the discrepancy principle.

c) Verify the estimate

$$D_J^{p_\alpha^k}(u^\dagger, u_\alpha^k) \leq \frac{\alpha \|w\|_{\mathcal{V}}^2}{2k}$$

for $k \in \mathbb{N} \setminus \{1\}$, exact data ($\delta = 0$) and the source condition (2).

d) Show that iteration (5) can also be written as

$$\begin{aligned} u_\alpha^{k+1} &\in \arg\min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \|Ku - f_\alpha^k\|_{\mathcal{V}}^2 + \alpha J(u) \right\} ; \\ f_\alpha^{k+1} &= f_\alpha^k + f - Ku_\alpha^{k+1} , \end{aligned}$$

for $f_\alpha^0 = f$.

e) Implement the Bregman iteration (5) for the deconvolution problem (4) of Exercise 3. Use the discrepancy principle as a stopping criterion, and compute reconstructions for different choices of α.